Unit 1: Whole Numbers

1.1.1 Place Value and Names for Whole Numbers

Learning Objective(s)

1 Find the place value of a digit in a whole number.
2 Write a whole number in words and in standard form.
3 Write a whole number in expanded form.

Introduction

Mathematics involves solving problems that involve numbers. We will work with **whole numbers**, which are any of the numbers 0, 1, 2, 3, and so on. We first need to have a thorough understanding of the number system we use. Suppose the scientists preparing a lunar command module know it has to travel 382,564 kilometers to get to the moon. How well would they do if they didn't understand this number? Would it make more of a difference if the 8 was off by 1 or if the 4 was off by 1?

In this section, you will take a look at digits and place value. You will also learn how to write whole numbers in words, standard form, and expanded form based on the place values of their digits.

The Number System

Objective 1

A **digit** is one of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. All numbers are made up of one or more digits. Numbers such as 2 have one digit, whereas numbers such as 89 have two digits. To understand what a number really means, you need to understand what the digits represent in a given number.

The position of each digit in a number tells its value, or **place value**. We can use a **place-value chart** like the one below to easily see the place value for each digit. The place values for the digits in 1,456 are shown in this chart.

Place-Value Chart														
Trillions			Billions			Millions			Thousands			Ones		
											1	4	5	6
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

In the number 1,456, the digit 1 is in the thousands place. The digit 4 is in the hundreds place. The digit 5 is in the tens place, and the digit 6 is in the ones place.

As you see above, you can tell a digit's value by looking at its position. Look at the number of digits to the right of the digit, or write your number into a place-value chart, with the last digit in the ones column. Both these methods are shown in the example below.

Example

Problem **The development of a city over the past twenty years cost $962,234,532,274,312. What is the value of the digit 6 in this number?**

Place-Value Chart														
Trillions			Billions			Millions			Thousands			Ones		
9	**6**	2	2	3	4	5	3	2	2	7	4	3	1	2
Hundreds	**Tens**	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

Write the number in the place-value chart. Read the value of the 6 from the chart.

$962,234,532,274,312
60,000,000,000,000

Answer The value of the digit 6 is 60 trillion.

Self Check A

In a far away galaxy, there are 2,968,351,472 stars. What does the digit 3 represent in this problem?

Periods and Standard Form

Objective **2**

The **standard form** of a number refers to a type of notation in which digits are separated into groups of three by commas. These groups of three digits are known as **periods**. For example, 893,450,243 has three periods with three digits in each period, as shown below.

Place-Value Chart														
Trillions			Billions			Millions			Thousands			Ones		
						8	9	3	4	5	0	2	4	3
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

Let’s examine the number of digits and periods in a greater number. The number of body cells in an average adult human is about one hundred trillion. This number is written as 100,000,000,000,000. Notice that there are 15 digits and 5 periods. Here is how the number would look in a place-value chart.

Place-Value Chart														
Trillions			Billions			Millions			Thousands			Ones		
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

You are now familiar with the place values of greater numbers, so let’s examine a problem that involves converting from standard form to a word name.

Converting Standard Form to Word Names

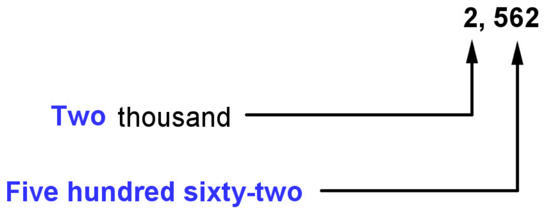
We often use word names to write numbers. A word name for 42 is “forty-two.” The total number of weeks in a year, 52, is written as “fifty-two.”

For whole numbers with three digits, use the word “hundred” to describe how many hundreds there are in the number. For example, for the number of days in a normal year, 365, the digit 3 is in the hundreds place. The word name for the number is “three hundred sixty-five.”

For whole numbers with four digits, begin the name with the number of thousands, followed by the period name, as in the example below.

Example

Problem **A man owes $2,562 on a car. Write the word name for this.**

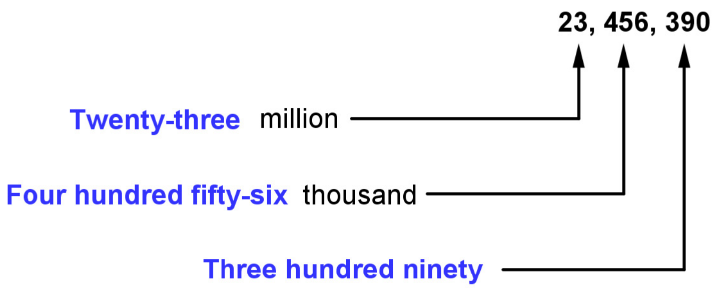


Answer The word name is two thousand, five hundred sixty-two.

For word names of greater numbers, begin at the left with the greatest period. For each period, write the one- to three-digit number in the period, and then the period name. See the example below.

Example

Problem **The construction of a new athletic center cost $23,456,390. Write the word name for this number.**



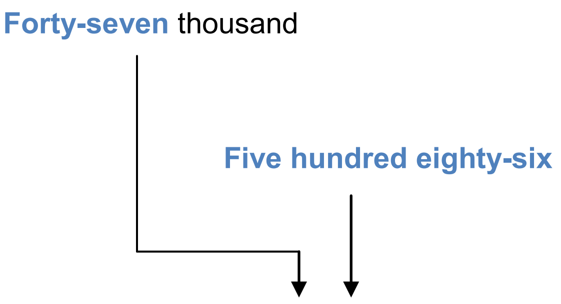
Answer The word name is twenty-three million, four hundred fifty-six thousand, three hundred ninety.

Converting Word Names to Standard Form

When converting word names to standard form, the word “thousand” tells you which period the digits are in. See the example below.

Example

Problem **Forty-seven thousand, five hundred eighty-six blueberries are produced on a farm over the course of three years. Write this number in standard form.**



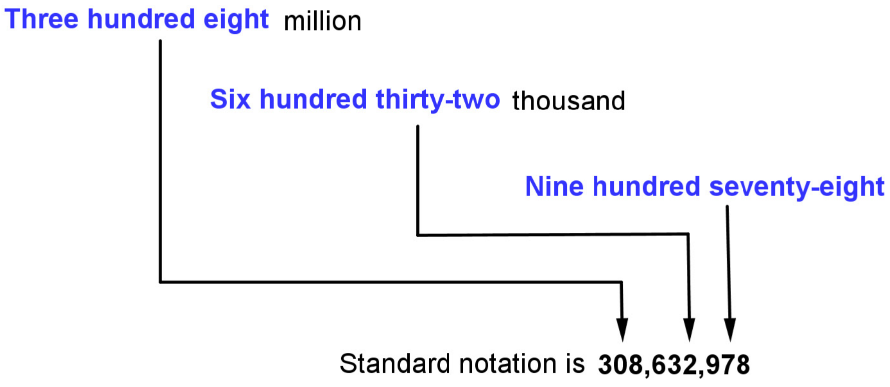
Standard Notation is 47,586

Answer The number in standard form is 47,586.

Below is an example with a number containing more digits. The words “million” and “thousand” tell you which periods the digits are in. The periods are separated by commas.

Example

Problem **There are three hundred eight million, six hundred thirty-two thousand, nine hundred seventy-eight bacteria in a sample of soil. Write this number in standard form.**



Answer The number in standard form is 308,632,978.

Some numbers in word form may not mention a specific period. For example, three million, one hundred twelve written in standard form is 3,000,112. Because the thousands period is not mentioned, you would write three zeros in the thousands period. You can use a place-value chart to make it easier to see the values of the digits. See the example below.

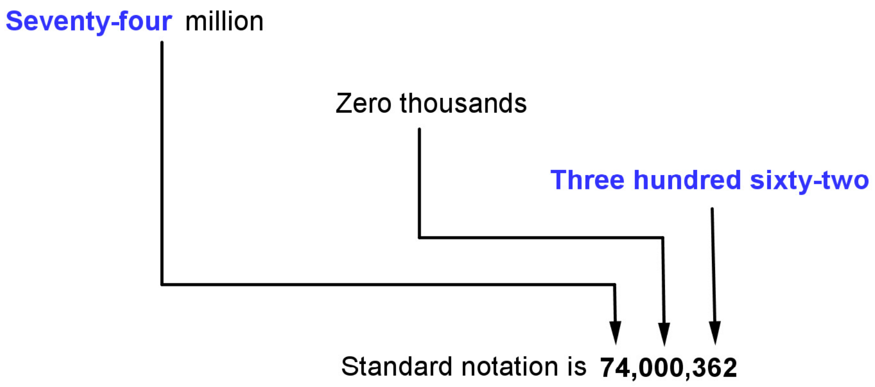
Example

Problem **A company had a new office building constructed. The final cost was seventy-four million, three hundred sixty-two dollars. Write this number in standard form.**

Place-Value Chart

Trillions			Billions			Millions			Thousands			Ones		
							7	4	0	0	0	3	6	2
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

Placing this number in a place- value chart shows that the thousands period is zero.



Remember to separate each period with a comma.

Answer The number written in standard form is $74,000,362.

Writing Numbers in Expanded Form

Objective **3**

Sometimes it is useful to write numbers in **expanded form**. In expanded form, the number is written as a sum of the value of each digit.

Example

Problem **During the week, Mike drives a total of 264 miles. Write 264 in expanded form.**

First, identify the value of each digit.

In numerical form:

264	**200**
2**6**4	**60**
26**4**	**4**

In word form:

264	**2 hundreds**
2**6**4	**6 tens**
26**4**	**4 ones**

Then, write the numbers as a sum.

Answer 264 written in expanded form is
200 + 60 + 4, or
2 hundreds + 6 tens + 4 ones, or
(2 • 100) + (6 • 10) + (4 • 1)

You can also use a place-value chart to help write a number in expanded form. Suppose the number of cars and pick-up trucks in the U.S. at this very moment is 251,834,697. Place this number in a place-value chart.

Place-Value Chart														
Trillions			Billions			Millions			Thousands			Ones		
						2	5	1	8	3	4	6	9	7
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

2 hundred millions	200,000,000
+ 5 ten millions	+50,000,000
+ 1 million	+1,000,000
+ 8 hundred thousands	+800,000
+ 3 ten thousands	+30,000
+ 4 thousands	+4,000
+ 6 hundreds	+600
+ 9 tens	+90
+ 7 ones	+7

Summary

Whole numbers that are greater than 9 consist of multiple digits. Each digit in a given number has a place value. To better understand place value, numbers can be put in a place-value chart so that the value of each digit can be identified. Numbers with more than three digits can be separated into groups of three digits, known as periods. Any whole number can be expressed in standard form, expanded form, or as a word name.

1.1.1 Self Check Solutions

Self Check A

three hundred thousands
The digit 3 is in the hundred thousands place.

1.1.2 Rounding Whole Numbers

Learning Objective(s)

1 Learn the rules for rounding.

2 Round whole numbers to specific place values, including tens, hundreds, and thousands.

Introduction

In some situations, you don’t need an exact answer. In these cases, **rounding** the number to a specific **place value** is possible. For example, if you travelled 973 miles, you might want to round the distance to 1,000 miles, which is easier to think about. Rounding also comes in handy to see if a calculation is reasonable.

Rounding Whole Numbers

Objective 1

Objective 2

These are the rules for rounding **whole numbers**:

First, identify the digit with the place value to which you are rounding. You might circle or highlight the digit so you can focus on it better.

Then, determine the possible numbers that you would obtain by rounding. These possible numbers are close to the number that you’re rounding to, but have zeros in the digits to the right.

If you are rounding 186 to the nearest ten, then180 and 190 are the two possible numbers to round to, as 186 is between 180 and 190. But how do you know whether to round to180 or 190?

Usually, round a number to the number that is closest to the original number.

When a number is halfway between the two possible numbers, round up to the greater number.

Since 186 is between 180 and 190, and 186 is closer to 190, you round up to 190.

You can use a number line to help you round numbers.

Example

Problem **A camera is dropped out of a boat, and sinks to the bottom of a pond that is 37 feet deep. Round 37 to the nearest ten.**

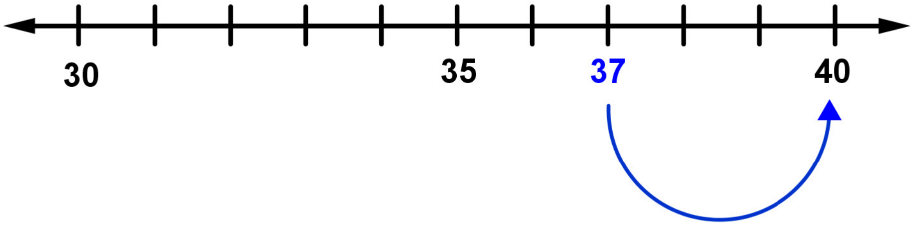
37

The digit you're rounding to is the tens digit, 3.

30 37 40

37 is between 30 and 40.

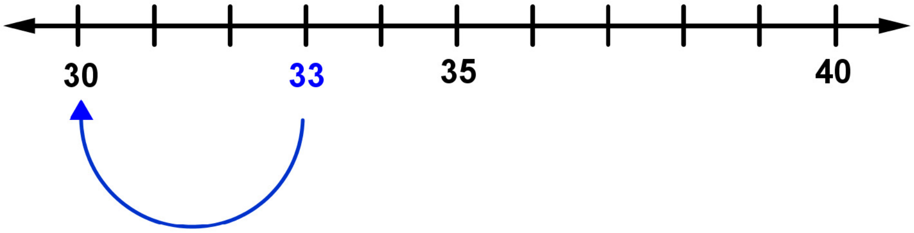
37 is only 3 away from 40, but it's 7 away from 30. So, 37 is closer to 40.



Answer To the nearest ten, 37 rounds to 40.

Example

Problem **Round 33 to the nearest ten.**



33 → 30, because 33 is closer to 30.

Answer To the nearest ten, 33 rounds to 30.

You can determine where to round without using a number line by looking at the digit to the right of the one you're rounding to. If that digit is less than 5, round down. If it's 5 or greater, round up. In the example above, you can see without a number line that 33 is rounded to 30 because the ones digit, 3, is less than 5.

Example

Problem ***Round 77 to the nearest ten.***

77 → 80, because the ones digit, 7, is 5 or greater.

Answer 77 rounded to the nearest ten is 80.

Example

Problem **There are 576 jellybeans in a jar. Round this number to the nearest ten.**

576 → 580, because the ones digit, 6, is 5 or greater.

Answer 576 rounded to the nearest ten is 580.

In the previous examples, you rounded to the tens place. The rounded numbers had a 0 in the ones place. If you round to the nearest hundred, the rounded number will have zeros in the tens and ones places. The rounded number will resemble 100, 500, or 1200.

Example

Problem **A runner ran 1,539 meters, but describes the distance he ran with a rounded number. Round 1,539 to the nearest hundred.**

1,539 → 1,500, because the next digit is less than 5.

Answer 1,539 rounded to the nearest hundred is 1,500.

If you round to the nearest thousand, the rounded number will have zeros in the hundreds, tens, and ones places. The rounded number will resemble 1,000, 2,000, or 14,000.

Example

Problem **A plane’s altitude increased by 2,721 feet. Round this number to the nearest thousand.**

2,721 → 3,000, because the next digit, 7, is 5 or greater.

Answer 2,721 rounded to the nearest thousand is 3,000.

Now that you know how to round to the nearest ten, hundred, and thousand, try rounding to the nearest ten thousand.

Example

Problem **Round 326,749 to the nearest ten thousand.**

326,749 → 330,000, because the next digit, 6, is 5 or greater.

Answer 326,749 rounded to the nearest ten thousand is 330,000.

Self Check A

A record number of 23,386 people voted in a city election. Round this number to the nearest hundred.

Summary

In situations when you don’t need an exact answer, you can round numbers. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. Whether you round up or round down usually depends on which number is closest to your original number. When a number is halfway between the two possible numbers, round up to the larger number.

1.1.2 Self Check Solutions

Self Check A

A record number of 23,386 people voted in a city election. Round this number to the nearest hundred.

23,400

The two possible numbers are 23,300 and 23,400, and 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater so you should round up.

1.1.3 Comparing Whole Numbers

Learning Objective(s)

1 Use > or < to compare whole numbers.

Introduction

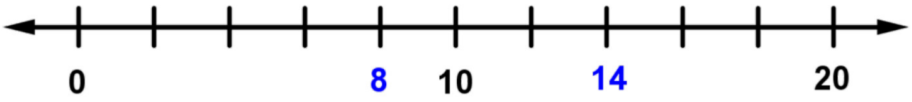
There will be times when it’s helpful to compare two numbers and determine which number is greater, and which one is less. This is a useful way to compare quantities such as travel time, income, or expenses. The symbols < and > are used to indicate which number is greater, and which is less than the other.

Comparing Whole Numbers

When comparing the values of two numbers, you can use a number line to determine which number is greater. The number on the right is always greater than the number on the left. In the example below, you can see that 14 is greater than 8 because 14 is to the right of 8 on the number line.

Example

Problem **Which number is greater, 8 or 14?**



14 is to the right of 8, so 14 is greater than 8.

Answer 14 is greater than 8.

In the example below, you can determine which number is greater by comparing the digits in the ones place value.

Example

Problem **Which number is greater, 33 or 38?**

In both **3**3 and **3**8, the digit in the tens place is **3.**

Because they have the same number in the tens place, you can determine which one is greater by comparing the digits in the ones place.

In the number 3**8**, the digit in the ones place is **8**.

In the number 3**3**, the digit in the ones place is **3**.

Because 8 is greater than 3, 38 is greater than 33.

Answer 38 is greater than 33. This answer was obtained from comparing their digits in the ones place value, which are 8 and 3, respectively.

Self Check A

Which number is greater, 17 or 11?

If one number is significantly greater than another number, it may be difficult to compare the numbers effectively with a number line. In general, **whole numbers** with more digits are greater than whole numbers with fewer digits. For example, 542 is greater than 84 because 542 has the digit 5 in the hundreds place. There are no hundreds in 84.

Self Check B

Which number is greater, 71 or 710?

Inequalities

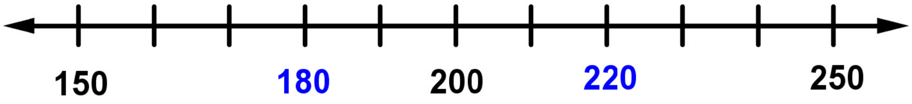
Objective **1**

An **inequality** is a mathematical sentence that compares two numbers that aren't equal. Instead of an equal sign (=), inequalities use greater than (>) or less than (<) symbols. The important thing to remember about these symbols is that the small end points towards the lesser number, and the larger (open) end is always on the side of the greater number.

There are other ways to remember this. For example, the wider part of the symbol represents the jaws of an alligator, which "gobbles up" the greater number. So "35 is greater than 28" can be written as 35 > 28, and "52 is less than 109" can be written as 52 < 109.

Example

Problem **Replace ? with < or > to make a true sentence: 180 ? 220.**



180 is to the left of 220, so 180 < 220. The symbol points at 180, which is the lesser number.

Answer 180 < 220

Self Check C

Which expression correctly compares the numbers 85 and 19?

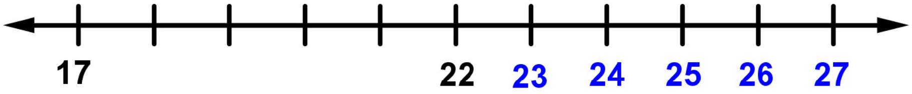
A) 85 < 19 B) 19 = 85 C) 85 > 19 D) 19 > 85

Many times an answer needs to be a range of values rather than just a single value. For example, you want to make more than $22 an hour. This can be expressed as all numbers greater than 22. See the example below.

Example

Problem **? > 22. What whole number(s) will make this statement true?**

The symbol points at 22, so the numbers you want to put in the brackets are *greater* than 22. There are many numbers that work.



Answer 23, 24, 25, 26, and any additional whole numbers that are greater than 26 make this statement true.

Self Check D

A farmer has produced 230 pumpkins for the autumn harvest. Last year, he produced 198. Write an expression that compares these two numbers.

Summary

To compare two values that are not the same, you can write an inequality. You can use a number line or place value to determine which number is greater than another number. Inequalities can be expressed using greater than (>) or less than (<) symbols.

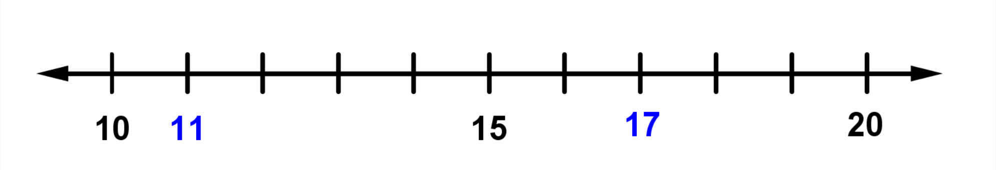
1.1.3 Self Check Solutions

Self Check A

Which number is greater, 17 or 11?

17

The number 17 is 6 units to the right of 11 on the number line.



Number line for Self Check A.

Self Check B

Which number is greater, 71 or 710?

710

The number 710 has 7 hundreds, but 71 has no hundreds.

Self Check C

Which expression correctly compares the numbers 85 and 19?

85 > 19

The open part of the symbol faces the larger number, 85, and the symbol points at the smaller number, 19.

Self Check D

A farmer has produced 230 pumpkins for the autumn harvest. Last year, he produced 198. Write an expression that compares these two numbers.

230 > 198

230 is greater than 198, and this is reflected in the symbol because the open part of the symbol faces 230. The expression 198 < 230 would also be correct.

1.2.1 Adding Whole Numbers and Applications

Learning Objective(s)

1 Add whole numbers without regrouping.
2 Add whole numbers with regrouping.
3 Find the perimeter of a polygon.
4 Solve application problems using addition.

Introduction

Adding is used to find the total number of two or more quantities. The total is called the **sum**, or the number that results from the addition. You use addition to find the total distance that you travel if the first distance is 1,240 miles and the second distance is 530 miles. The two numbers to be added, 1,240 and 530, are called the **addends**. The total distance, 1,770 miles, is the sum.

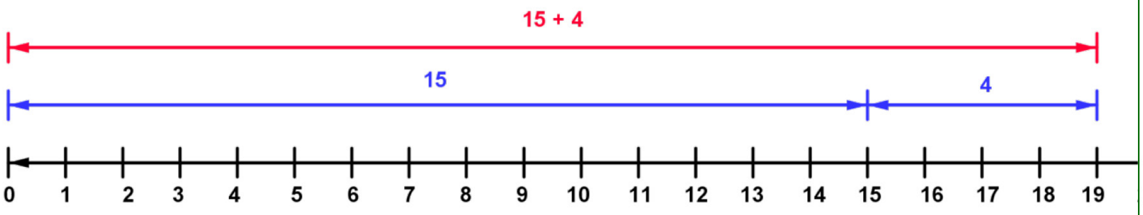
Adding Whole Numbers, without Regrouping

Objective 1

Adding numbers with more than one digit requires an understanding of **place value**. The place value of a digit is the value based on its position within the number. In the number 492, the 4 is in the hundreds place, the 9 is in the tens place, and the 2 is in the ones place. You can use a number line to add. In the example below, the blue lines represent the two quantities, 15 and 4, that are being added together. The red line represents the resulting quantity.

Example

Problem **15 + 4 = ?**



On the number line, the blue line segment stretches across 15 units, representing the number 15. The second blue segment shows that if you add 4 more units, the resulting number is 19.

Answer *15 + 4 = 19*

You can solve the same problem without a number line, by adding vertically. When adding numbers with more than 1 digit, it is important to line up your numbers by place value, as in the example below. You must add ones to ones, tens to tens, hundreds to hundreds, and so on.

Example

Problem **15 + 4 = ?**

$\begin{array}{r} 1\ 5 \\ +\ \ 4 \\ \hline \end{array}$	Because 5 and 4 have the same place value, make sure they are aligned when you add.
$\begin{array}{r} 1\ 5 \\ +\ \ 4 \\ \hline 9 \end{array}$	First, add the ones digits (the numbers on the right). The result goes in the ones place for the answer.
$\begin{array}{r} 1\ 5 \\ +\ \ 4 \\ \hline 1\ 9 \end{array}$	Then, add the tens digits and put the result in the tens place of the answer. In this case, there is no tens digit in the second number, so the result is the same as the tens digit of the first number (1).

Answer 15 + 4 = 19

This strategy of lining up the numbers is effective for adding a series of numbers as well.

Example

Problem **1 + 2 + 3 + 2 = ?**

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ +\ 2 \\ \hline 8 \end{array}$$

Answer 1 + 2 + 3 + 2 = 8

Adding Whole Numbers, with Regrouping

When adding whole numbers, a place-value position can have only one digit in it. If the sum of digits in a place value position is more that 10, you have to **regroup** the number of tens to the next greater place value position.

When you add, make sure you line up the digits according to their place values, as in the example below. As you regroup, place the regrouped digit above the appropriate digit in the next higher place value position and add it to the numbers below it.

Example

Problem **45 + 15 = ?**

$\begin{array}{r} 1 \\ 45 \\ +\ 15 \\ \hline 0 \end{array}$	Add the ones. Regroup as needed. The sum of 5 and 5 is 10. This is 1 ten and 0 ones. Write the number of ones (0) in the ones place and the 1 ten in the tens place above the 4.
$\begin{array}{r} 1 \\ 45 \\ +\ 15 \\ \hline 60 \end{array}$	Add the tens, 1 + 4 + 1 is 6 tens. The final sum is 60.

Answer 45 + 15 = 60

You must add digits in the ones place first, the digits in the tens place next, and so on. Go from right to left.

Example

Problem **4,576 + 698 = ?**

$\begin{array}{r} 4{,}576 \\ +\quad 698 \\ \hline \end{array}$	First, write the problem with one addend on top of the other. Be sure you line up the place values!
$\begin{array}{r} 1 \\ 4{,}576 \\ +\quad 698 \\ \hline 4 \end{array}$	Add the numbers in the ones place. Since the sum is 14, write the ones value (4) in the ones place of the answer. Write the 1 ten in the tens place above the 7.
$\begin{array}{r} 11 \\ 4{,}576 \\ +\quad 698 \\ \hline 74 \end{array}$	Add the numbers in the tens place. Since the sum is 17 tens, regroup17 tens as 1 hundred, 7 tens. Write 7 in the tens place in the answer and write the 1 hundred in the hundreds place above the 5.
$\begin{array}{r} 1\ 11 \\ 4{,}576 \\ +\quad 698 \\ \hline 274 \end{array}$	Add the numbers in the hundreds place, including the 1. Again, the sum is more than one digit. Rename 12 hundreds as 2 hundreds and 1 thousand. Write the 2 in the hundreds place and the 1 above the 4 in the thousands place.
$\begin{array}{r} 1\ 11 \\ 4{,}576 \\ +\quad 698 \\ \hline 5{,}274 \end{array}$	Add the numbers in the thousands place, including the 1. The final sum is 5,274.

Answer 4,576 + 698 = 5,274

Adding Numbers Using the Partial Sums Method

Another way to add is the partial sums method. In the example below, the sum of 23 + 46 is found using the partial sums method. In this method, you add together all the numbers with the same place value and record their *values* (not just a single digit). Once you have done this for each place value, add their sums together.

Example	
Problem **23 + 46 = ?**	
Step 1: Add Tens 23…………… 20 46…………… 40 60	Let's begin by adding the values in the tens position. Notice that the digits in the tens place are highlighted, and on the right, the values are written as 20 and 40.
Step 2: Add Ones 23……………… 3 46……………… 6 9	Add the values in the ones place.
Step 3: Add Parts 60 + 9 69	Finally, add the two sums together.
Answer 23 + 46 = 69	

The next example adds a series of three numbers. Notice that hundreds is the greatest place value now, so hundreds are added before the tens. (You can add in any order that you prefer.) Also notice that in Step 3, the value in the ones column for 350 is zero, but you still add that in to make sure everything is accounted for.

Example	
Problem **225 + 169 + 350 = ?**	
Step 1: Add Hundreds 225………. 200 169………. 100 350………. 300 600	Add the values represented by the digits in the hundreds place first. This gives a sum of 600.
Step 2: Add Tens 225………… 20 169………… 60 350………… 50 130	Next, add the values from the tens place. The sum is 130.

Step 3: Add Ones

225…...……. 5
169………….. 9
350………….. 0
14

Add the values from the ones place. The sum is 14.

Step 4: Add Parts

$$\begin{array}{r} 600 \\ 130 \\ \underline{+\ 14} \\ 744 \end{array}$$

At this point, you have a sum for each place value. Add together these three sums, which gives a final value of 744.

Answer 225 + 169 + 350 = 744

Self Check A

A local company built a playground at a park. It took the company 124 hours to plan out the playground, 243 hours to prepare the site, and 575 hours to build the playground. Find the total number of hours the company spent on the project.

When adding multi-digit numbers, use the partial sums method or any method that works best for you.

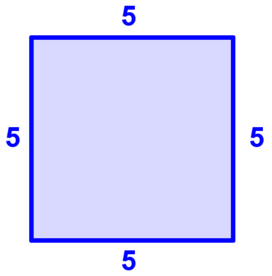
Finding the Perimeter of a Polygon

Objective 3

A **polygon** is a many-sided closed figure with sides that are straight line segments. Triangles, rectangles, and pentagons (five-sided figures) are polygons, but a circle or semicircle is not. The **perimeter** of a polygon is the distance around the polygon. To find the perimeter of a polygon, add the lengths of its sides, as in the example below.

Example

Problem **One side of a square has a length of 5 cm. Find the perimeter.**



Draw the polygon and label the lengths of the sides. Since the side lengths of a square are equal, each side is 5 cm.

$$\begin{array}{r} 5 \\ 5 \\ 5 \\ +\ \ 5 \\ \hline 20 \end{array}$$

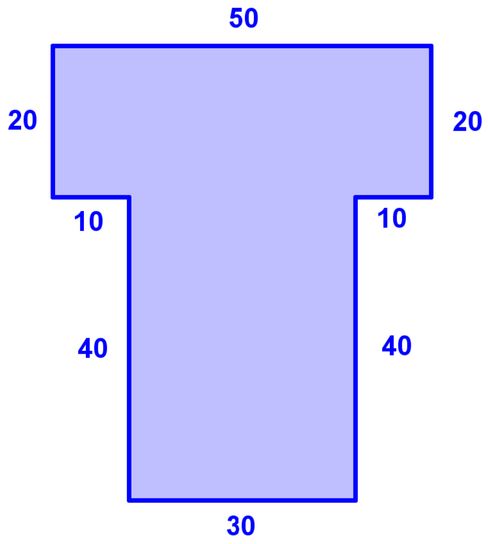
Add the lengths of each side.

Answer The perimeter is 20 cm.

The key part of completing a polygon problem is correctly identifying the side lengths. Once you know the side lengths, you add them as you would in any other addition problem.

Example

Problem **A company is planning to construct a building. Below is a diagram illustrating the shape of the building's floor plan. The length of each side is given in the diagram. Measurements for each side are in feet. Find the perimeter of the building.**



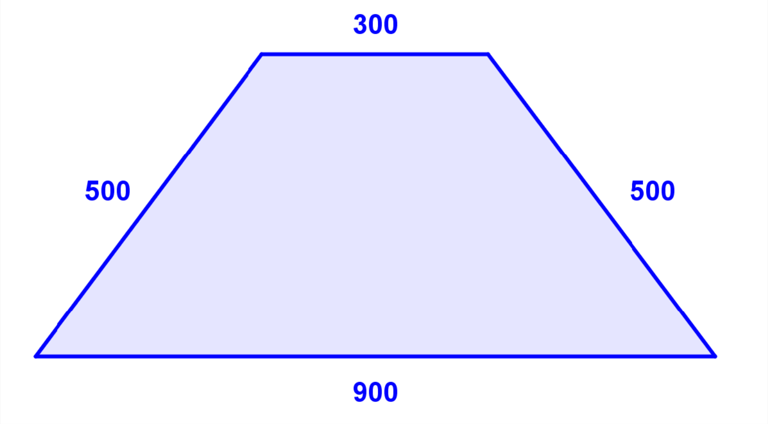
$$\begin{array}{r} 50 \\ 20 \\ 20 \\ 10 \\ 10 \\ 40 \\ 40 \\ +\ 30 \\ \hline 220 \end{array}$$

Add the lengths of each side, making sure to align all numbers according to place value.

Answer The perimeter is 220 ft.

Self Check B

Find the perimeter of the trapezoid in feet.



Objective **4**

Solving Application Problems

Addition is useful for many kinds of problems. When you see a problem written in words, look for key words that let you know you need to *add* numbers.

Example	
Problem	**A woman preparing an outdoor market is setting up a stand with 321 papayas, 45 peaches, and 213 mangos. How many pieces of fruit in total does the woman have on her stand?**
321 45 + 213	The words “how many… in total” suggest that you need to add the numbers of the different kinds of fruits. Use any method you like to add the numbers. Below, the partial sums method is used.
Step 1: Add Hundreds 321………. **300** 045………. **0** 213………. **200** **500**	Add the numbers represented by the digits in the hundreds place first. This gives a sum of 600.

Step 2: Add Tens

Next, add the numbers from the tens place. The sum is 70.

321………… 20
145………… 40
213………… 10
70

Step 3: Add Ones

Add the numbers from the ones.

321…..…….. 1
145………….. 5
213………….. 3
9

Step 4: Add Parts

Add together the three previous sums. The final sum is 579.

500
70
+ 9
579

Answer The woman has 579 pieces of fruit on her stand.

Example

Problem **Lynn has 23 rock CDs, 14 classical music CDs, 8 country and western CDs, and 6 movie soundtracks. How many CDs does she have in all?**

23
14
8
+ 6

The words "how many… in all" suggest that addition is the way to solve this problem.

To find how many CDs Lynn has, you need to add the number of CDs she has for each music style.

2
23
14
8
+ 6
51

Use whatever method you prefer to find the sum of the numbers.

Answer Lynn has 51 CDs.

The following phrases also appear in problem situations that require addition.

Phrase	**Example problem**
Add to	Jonah was planning a trip from Boston to New York City. The distance is 218 miles. His sister wanted him to visit her in Springfield, Massachusetts, on his way. Jonah knew this would **add** 17 miles **to** his trip. How long is his trip if he visits his sister?
Plus	Carrie rented a DVD and returned it one day late. The store charged $5 for a two-day rental, **plus** a $3 late fee. How much did Carrie pay for the rental?
Increased by	One statistic that is important for football players in offensive positions is *rushing*. After four games, one player had rushed 736 yards. After two more games, the number of yards rushed by this player **increased by** 352 yards. How many yards had he rushed after the six games?
More than	Lavonda posted 38 photos to her social network profile. Chris posted 27 **more** photos to his **than** Lavonda. How many photos did Chris post?

Example

Problem **Lena was planning a trip from her home in Amherst to the Museum of Science in Boston. The trip is 91 miles. She had to take a detour on the way, which added 13 miles to her trip. What is the total distance she traveled?**

The word “added” suggests that addition is the way to solve this problem.

To find the total distance, you need to add the two distances.

$$\begin{array}{r} 91 \\ \underline{+\,13} \\ 104 \end{array}$$

Answer The total distance is 104 miles.

It can help to seek out words in a problem that imply what operation to use. See if you can find the key word(s) in the following problem that provide you clues on how to solve it.

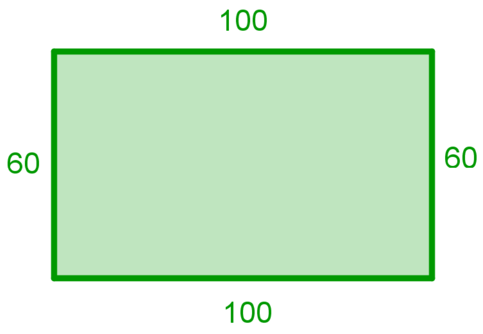
Self Check C

A city was struck by an outbreak of a new flu strain in December. To prevent another outbreak, 3,462 people were vaccinated against the new strain in January. In February, 1,298 additional people were vaccinated. How many people in total received vaccinations over these two months?

Drawing a diagram to solve problems is very useful in fields such as engineering, sports, and architecture.

Example

Problem **A coach tells her athletes to run one lap around a soccer field. The length of the soccer field is 100 yards, while the width of the field is 60 yards. Find the total distance that each athlete will have run after completing one lap around the perimeter of the field.**



The words “total distance” and “perimeter” both tell you to add.

Draw the soccer field and label the various sides so you can see the numbers you are working with to find the perimeter.

$$\begin{array}{r} 1 \\ 100 \\ 100 \\ 60 \\ +\ \ 60 \\ \hline 20 \end{array}$$

There is a zero in the ones place, and the sum of 6 and 6 in the tens place is 12 tens. Place 2 tens in the tens place in the answer, and regroup 10 tens as 1 hundred.

$$\begin{array}{r} 1 \\ 100 \\ 100 \\ 60 \\ +\ \ 60 \\ \hline 320 \end{array}$$

By adding the 1 hundred to the other digits in the hundreds place, you end up with a 3 in the hundreds place of the answer.

Answer Each athlete will have run 320 yards.

Summary

You can add numbers with more than one digit using any method, including the partial sums method. Sometimes when adding, you may need to regroup to the next greater place value position. Regrouping involves grouping ones into groups of tens, grouping tens into groups of hundreds, and so on. The perimeter of a polygon is found by adding the lengths of each of its sides.

1.2.1 Self Check Solutions

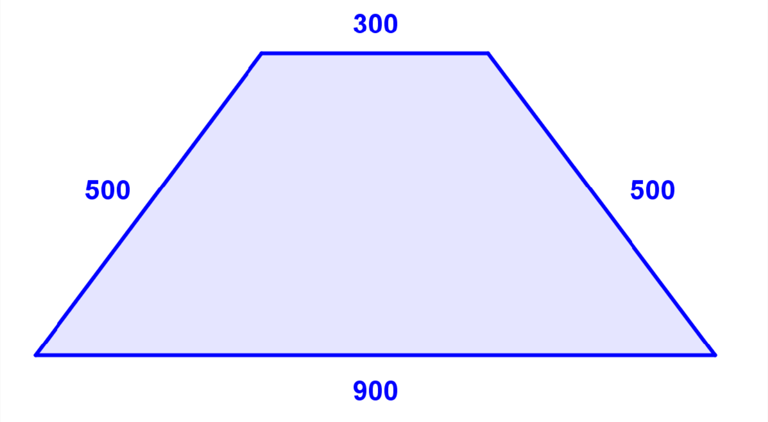
Self Check A

A local company built a playground at a park. It took the company 124 hours to plan out the playground, 243 hours to prepare the site, and 575 hours to build the playground. Find the total number of hours the company spent on the project.

800 + 130 + 12 = 942 hours

Self Check B

Find the perimeter of the trapezoid in feet.



300+ 500 + 500 + 900 = 2,200 ft

Self Check C

A city was struck by an outbreak of a new flu strain in December. To prevent another outbreak, 3,462 people were vaccinated against the new strain in January. In February, 1,298 additional people were vaccinated. How many people in total received vaccinations over these two months?

3462 + 1298 = 4,760

1.2.2 Subtracting Whole Numbers and Applications

Learning Objective(s)
1 Subtract whole numbers without regrouping.
2 Subtract whole numbers with regrouping.
3 Solve application problems using subtraction.

Introduction

Subtracting involves finding the difference between two or more numbers. It is a method that can be used for a variety of applications, such as balancing a checkbook, planning a schedule, cooking, or travel. Suppose a government official is out of the U.S. on business for 142 days a year, including travel time. The number of days per year she is in the U.S. is the difference of 365 days and 142 days. Subtraction is one way of calculating the number of days she would be in the U.S. during the year.

When subtracting numbers, it is important to line up your numbers, just as with addition. The **minuend** is the greater number from which the lesser number is subtracted. The **subtrahend** is the number that is subtracted from the minuend. A good way to keep minuend and subtrahend straight is that since subtrahend has “subtra” in its beginning, it goes next to the subtraction sign and is the number being subtracted. The **difference** is the quantity that results from subtracting the subtrahend from the minuend. In 86 – 52 = 34, 86 is the *minuend*, 52 is the *subtrahend*, and 34 is the *difference*.

Subtracting Whole Numbers

Objective **1**

When writing a subtraction problem, the minuend is placed above the subtrahend. This can be seen in the example below, where the minuend is 10 and the subtrahend is 7.

Example

Problem **10 – 7 = ?**

$$\begin{array}{r} 10 \\ -\underline{7} \\ 3 \end{array}$$

Answer 10 – 7 = 3

When both numbers have more than one digit, be sure to work with one place value at a time, as in the example below.

Example

Problem **689 – 353 = ?**

$$\begin{array}{r} 6\;8\;9 \\ -\underline{3\;5\;3} \end{array}$$

First, set up the problem and align the numbers by place value.

$$\begin{array}{r} 6\;8\;9 \\ -\underline{3\;5\;3} \\ 6 \end{array}$$

Then, subtract the ones.

$$\begin{array}{r} 6\;8\;9 \\ -\underline{3\;5\;3} \\ 3\;6 \end{array}$$

Next, subtract the tens.

$$\begin{array}{r} 6\;8\;9 \\ -\underline{3\;5\;3} \\ 3\;3\;6 \end{array}$$

Finally, subtract the hundreds.

Answer 689 – 353 = 336

Lining up numbers by place value becomes especially important when you are working with larger numbers that have more digits, as in the example below.

Example

Problem **9,864 – 743 =?**

$$\begin{array}{r} 9\;8\;6\;4 \\ -\underline{\;\;7\;4\;3} \end{array}$$

First, set up the problem and align the numbers by place value.

$$\begin{array}{r} 9\;8\;6\;4 \\ -\underline{\;\;7\;4\;3} \\ 1 \end{array}$$

Then, subtract the ones.

$$\begin{array}{r} 9\;8\;6\;4 \\ -\underline{\;\;7\;4\;3} \\ 2\;1 \end{array}$$

Next, subtract the tens.

$$\begin{array}{rrrrr} & 9 & 8 & 6 & 4 \\ - & & 7 & 4 & 3 \\ \hline & & 1 & 2 & 1 \end{array}$$

Now, subtract the hundreds.

$$\begin{array}{rrrrr} & 9 & 8 & 6 & 4 \\ - & & 7 & 4 & 3 \\ \hline & 9 & 1 & 2 & 1 \end{array}$$

There is no digit to subtract in the thousands place, so keep the 9.

Answer 9,864 – 743 = 9,121

Self Check A

Subtract: 2,489 – 345.

Subtracting Whole Numbers, with Regrouping

Objective **2**

You may need to regroup when you subtract. When you **regroup**, you rewrite the number so you can subtract a greater digit from a lesser one.
When you're subtracting, just regroup to the next greater place-value position in the minuend and add 10 to the digit you're working with. As you regroup, cross out the regrouped digit in the minuend and place the new digit above it. This method is demonstrated in the example below.

Example

Problem **3,225 – 476 = ?**

$$\begin{array}{rrrrr} & 3 & 2 & 2 & 5 \\ - & & 4 & 7 & 6 \\ \hline \end{array}$$

First, set up the problem and align the digits by place value.

$$\begin{array}{rrrrr} & & & 1 & 15 \\ & 3 & 2 & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & & & & 9 \end{array}$$

Since you can't subtract 6 from 5, regroup, so 2 tens and 5 ones become 1 ten and 15 ones. Now you can subtract 6 from 15 to get 9.

$$\begin{array}{rrrrr} & & 1 & 11 & 15 \\ & 3 & \cancel{2} & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & & & 4 & 9 \end{array}$$

Next, you need to subtract 7 tens from 1 ten. Regroup 2 hundreds as 1 hundred, 10 tens and add the 10 tens to 1 ten to get 11 tens. Now you can subtract 7 from 11 to get 4.

$$\begin{array}{rrrrr} & 2 & 11 & 11 & 15 \\ & \cancel{3} & \cancel{2} & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & & 7 & 4 & 9 \end{array}$$

To subtract the digits in the hundreds place, regroup 3 thousands as 2 thousands, 10 hundreds and add the 10 hundreds to the 1 hundred that is already in the hundreds place. Now, subtract 4 from 11 to get 7.

$$\begin{array}{r} 2\ 11\ 11\ 15 \\ \cancel{3}\ \cancel{2}\ \cancel{2}\ \cancel{5} \\ -\quad 4\ \ 7\ \ 6 \\ \hline 2\ \ 7\ \ 4\ \ 9 \end{array}$$

Since there is no digit in the thousands place of the subtrahend, bring down the 2 in the thousands place into the answer.

Answer 3,225 – 476 = 2,749

Self Check B

Subtract: 1,610 – 880.

Checking Your Work

You can check subtraction by adding the difference and the subtrahend. The sum should be the same as the minuend.

Example

Problem **Check to make sure that 7 subtracted from 12 is equal to 5.**

12 – 7 = 5

Here, write out the original equation. The minuend is 12, the subtrahend is 7, and the difference is 5.

$$\begin{array}{r} 5 \\ +\ 7 \\ \hline 12 \end{array}$$

Here, add the difference to the subtrahend, which results in the number 12. This confirms that your answer is correct.

Answer The answer of 5 is correct.

Checking your work is very important and should always be performed when time permits.

Subtracting Numbers, Using the Expanded Form

An alternative method to subtract involves writing numbers in expanded form, as shown in the examples below. If you have 4 tens and want to subtract 1 ten, you can just think (4 – 1) tens and get 3 tens. Let's see how that works.

Example	
Problem	**45 – 12 = ?**
45 = 40 + 5 12 = 10 + 2	Let's write the numbers in expanded form so you can see what they really mean.
45 = 40 + 5 12 = 10 + 2 30	Look at the tens. The minuend is 40, or 4 tens. The subtrahend is 10, or 1 ten. Since 4 – 1 = 3, 4 tens – 1 ten = 3 tens, or 30.
45 = 40 + 5 –12 = 10 + 2 30 + 3	Look at the ones. 5 – 2 = 3. So, 30 + 3 = 33.
Answer 45 – 12 = 33	

Now let's use this method in the example below, which asks for the difference of 467 and 284. In the tens place of this problem, you need to subtract 8 from 6. What can you do?

Example	
Problem	**467 – 284 = ?**
Step 1: Separate by place value 4 hundreds + 6 tens + 7 ones 2 hundreds + 8 tens + 4 ones	Write both the minuend and the subtrahend in expanded form.
Step 2: Identify impossible differences 6 – 8 = [ ]	Here, we identify differences that are not whole numbers. Since 8 is greater than 6, you won't get a whole number difference.
Step 3: Regroup 3 hundreds + 16 tens + 7 ones – 2 hundreds + 8 tens + 4 ones 1 hundred + 8 tens + 3 ones	Regroup one of the hundreds from the 4 hundreds into 10 tens and add it to the 6 tens. Now you have 16 tens. Subtracting 8 tens from 16 tens yields a difference of 8 tens.
Step 4: Combine the parts 1 hundred + 8 tens + 3 ones = 183	Combining the resulting differences for each place value yields a final answer of 183.
Answer 467 – 284 = 183	

Self Check C

A woman who owns a music store starts her week with 965 CDs. She sells 452 by the end of the week. How many CDs does she have remaining?

Example

Problem	**45 – 17 = ?**
	When you try to subtract 17 from 45, you would first try to subtract 7 from 5. But 5 is less than 7.
45 = 40 + 5 17 = 10 + 7	Let's write the numbers in expanded form so you can see what they really mean.
45 = 30 + 15 17 = 10 + 7	Now, regroup 4 tens as 3 tens and 10 ones. Add the 10 ones to 5 ones to get 15 ones, which is greater than 7 ones, so you can subtract.
45 = 30 + 15 – 17 = 10 + 7 20 + 8	Finally, subtract 7 from 15, and 10 from 30 and add the results: 20 + 8 = 28.
Answer 45 – 17 = 28	

Solve Application Problems Using Subtraction

Objective 3

You are likely to run into subtraction problems in every day life, and it helps to identify key phrases in a problem that indicate that subtraction is either used or required. The following phrases appear in problem situations that require subtraction.

Phrase or word	**Example problem**
Less than	The cost of gas is 42 cents per gallon **less than** it was last month. The cost last month was 280 cents per gallon. How much is the cost of gas this month?
Take away	Howard made 84 cupcakes for a neighborhood picnic. People **took away** 67 cupcakes. How many did Howard have left?
Decreased by	The temperature was 84°F in the early evening. It **decreased by** 15° overnight. What was the temperature in the morning?

Subtracted from	Jeannie works in a specialty store on commission. When she sells something for $75, she **subtracts** $15 **from** the $75 and gives the rest to the store. How much of the sale goes to the store?
The difference	What is **the difference** between this year's rent of $1,530 and last year's rent of $1,450?
Fewer than	The number of pies sold at this year's bake sale was 15 **fewer** than the number sold at the same event last year. Last year, 32 pies were sold. How many pies were sold this year?

When translating a phrase such as "5 fewer than 39" into a mathematical expression, the order in which the numbers appears is critical. Writing 5 – 39 would not be the correct translation. The correct way to write the expression is 39 – 5. This results in the number 34, which is 5 fewer than 39. The chart below shows how phrases with the key words above can be written as mathematical expressions.

Phrase	Expression
three subtracted from six	6 – 3
the difference of ten and eight	10 – 8
Nine fewer than 40	40 – 9
Thirty-nine decreased by fourteen	39 – 14
Eighty-five take away twelve	85 – 12
Four less than one hundred eight	108 – 4

Example	
Problem	**Each year, John is out of the U.S. on business for 142 days, including travel time. The number of days per year he is in the U.S. is the difference of 365 days and 142 days. How many days during the year is John in the U.S.?**
	The words "the difference of" suggest that you need to subtract to answer the problem.
3 6 5 - 1 4 2	First, write out the problem based on the information given and align numbers by place value.

$\begin{array}{r} 365 \\ -\underline{142} \\ 3 \end{array}$	Then, subtract numbers in the ones place.
$\begin{array}{r} 365 \\ -\underline{142} \\ 23 \end{array}$	Subtract numbers in the tens place.
$\begin{array}{r} 365 \\ -\underline{142} \\ 223 \end{array}$	Finally, subtract numbers in the hundreds place.

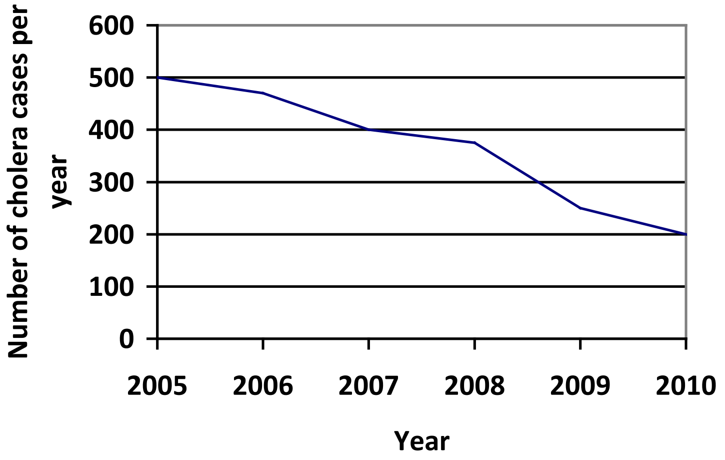
Answer John is in the U.S. 223 days during the year.

Self Check D

To make sure he was paid up for the month on his car insurance, Dave had to pay the difference of the amount on his monthly bill, which was $289, and what he had paid earlier this month, which was $132. Write the difference of $289 and $132 as a mathematical expression.

Example

Problem **An African village is now getting cleaner water than it used to get. The number of cholera cases in the village has declined over the past five years. Using the graph below, determine the difference between the number of cholera cases in 2005 and the number of cases in 2010.**



The words "the difference" suggest that you need to subtract to answer the problem.

First, use the graph to find the number of cholera cases per year for the two years: 500 in 2005 and 200 in 2010.

$$\begin{array}{r} 500 \\ -\,200 \\ \hline 300 \end{array}$$

Then write the subtraction problem and align numbers by place value. Subtract the numbers as you usually would.

Answer 500 – 200 = 300 cases

Summary

Subtraction is used in countless areas of life, such as finances, sports, statistics, and travel. You can identify situations that require subtraction by looking for key phrases, such as *difference* and *fewer than*. Some subtraction problems require regrouping to the next greater place value, so that the digit in the minuend becomes greater than the corresponding digit in the subtrahend. Subtraction problems can be solved without regrouping, if each digit in the minuend is greater than the corresponding digit in the subtrahend.

In addition to subtracting using the standard algorithm, subtraction also can be accomplished by writing the numbers in expanded form so that both the minuend and the subtrahend are written as the sums of their place values.

1.2.2 Self Check Solutions

Self Check A

Subtract: 2,489 – 345.

2,144

Self Check B

Subtract: 1,610 – 880.

730

Self Check C

A woman who owns a music store starts her week with 965 CDs. She sells 452 by the end of the week. How many CDs does she have remaining?

965 – 452 = 513

Self Check D

To make sure he was paid up for the month on his car insurance, Dave had to pay the difference of the amount on his monthly bill, which was $289, and what he had paid earlier this month, which was $132. Write the difference of $289 and $132 as a mathematical expression.

The difference of 289 and 132 can be written as 289 – 132.

1.2.3 Estimation

Learning Objective(s)

1 Use rounding to estimate sums and differences.

2 Use rounding to estimate the solutions for application problems.

Introduction

An **estimate** is an answer to a problem that is close to the solution, but not necessarily exact. Estimating can come in handy in a variety of situations, such as buying a computer. You may have to purchase numerous devices: a computer tower and keyboard for $1,295, a monitor for $679, the printer for $486, the warranty for $196, and software for $374. Estimating can help you know *about* how much you'll spend without actually adding those numbers exactly.

Estimation usually requires **rounding**. When you round a number, you find a new number that's close to the original one. A rounded number uses zeros for some of the place values. If you round to the nearest ten, you will have a zero in the ones place. If you round to the nearest hundred, you will have zeros in the ones and tens places. Because these place values are zero, adding or subtracting is easier, so you can find an estimate to an exact answer quickly.

It is often helpful to estimate answers before calculating them. Then if your answer is not close to your estimate, you know something in your problem-solving process is wrong.

Using Rounding to Estimate Sums and Differences

Objective 1

Suppose you must add a series of numbers. You can round each addend to the nearest hundred to estimate the sum.

Example	
Problem	**Estimate the sum 1,472 + 398 + 772 + 164 by rounding each number to the nearest hundred.**
1,472….1,500 398……. 400 772……..800 164……..200	First, round each number to the nearest hundred.

$$\begin{array}{r} 1{,}5\,0\,0 \\ 4\,0\,0 \\ 8\,0\,0 \\ +\quad \underline{2\,0\,0} \\ 2{,}9\,0\,0 \end{array}$$

Then, add the rounded numbers together.

Answer The estimate is 2,900.

In the example above, the exact sum is 2,806. Note how close this is to the estimate, which is 94 greater.

In the example below, notice that rounding to the nearest ten produces a far more accurate estimate than rounding to the nearest hundred. In general, rounding to the lesser place value is more accurate, but it takes more steps.

Example

Problem **Estimate the sum 1,472 + 398 + 772 + 164 by first rounding each number to the nearest ten.**

First, round each number to the nearest ten.

1,472….1,470

398…….. 400

772……...770

164……...160

Next, add the ones and then the tens. Here, the sum of 7, 7, and 6 is 20. Regroup.

$$\begin{array}{r} 1\,2 \\ 1\,4\,7\,0 \\ 4\,0\,0 \\ 7\,7\,0 \\ +\quad \underline{1\,6\,0} \\ 0\,0 \end{array}$$

Now, add the hundreds. The sum of the digits in the hundreds place is 18. Regroup.

$$\begin{array}{r} 1\,2 \\ 1\,4\,7\,0 \\ 4\,0\,0 \\ 7\,7\,0 \\ +\quad \underline{1\,6\,0} \\ 8\,0\,0 \end{array}$$

Finally, add the thousands. The sum in the thousands place is 2.

$$\begin{array}{r} 1\,2 \\ 1\,4\,7\,0 \\ 4\,0\,0 \\ 7\,7\,0 \\ +\quad \underline{1\,6\,0} \\ 2\,8\,0\,0 \end{array}$$

Answer The estimate is 2,800.

Note that the estimate is 2,800, which is only 6 less than the actual sum of 2,806.

Self Check A

In three months, a freelance graphic artist earns $1,290 for illustrating comic books, $2,612 for designing logos, and $4,175 for designing web sites. Estimate how much she earned in total by first rounding each number to the nearest hundred.

You can also estimate when you subtract, as in the example below. Because you round, you do not need to subtract in the tens or hundreds places.

Example

Problem **Estimate the difference of 5,876 and 4,792 by first rounding each number to the nearest hundred.**

5,876….5,900 4,792….4,800	First, round each number to the nearest hundred.
$\begin{array}{r} 5{,}9\,0\,0 \\ -\ \underline{4{,}8\,0\,0} \\ 1{,}1\,0\,0 \end{array}$	Subtract. No regrouping is needed since each number in the minuend is greater than or equal to the corresponding number in the subtrahend.

Answer The estimate is 1,100.

The estimate is 1,100, which is 16 greater than the actual difference of 1,084.

Self Check B

Estimate the difference of 474,128 and 262,767 by rounding to the nearest thousand.

Solving Application Problems by Estimating

Objective **2**

Estimating is handy when you want to be sure you have enough money to buy several things.

Example

Problem **When buying a new computer, you find that the computer tower and keyboard cost $1,295, the monitor costs $679, the printer costs $486, the 2-year warranty costs $196, and a software package costs $374. Estimate the total cost by first rounding each number to the nearest hundred.**

1,295….1,300

679…….. 700

486……...500

196……...200

374……...400

First, round each number to the nearest hundred.

$$\begin{array}{r} 2 \\ 1\,3\,0\,0 \\ 7\,0\,0 \\ 5\,0\,0 \\ 2\,0\,0 \\ +\ \underline{4\,0\,0} \\ 3{,}1\,0\,0 \end{array}$$

Add.

After adding all of the rounded values, the estimated answer is $3,100.

Answer The total cost is approximately $3,100.

Estimating can also be useful when calculating the total distance one travels over several trips.

Example

Problem **James travels 3,247 m to the park, then 582 m to the store. He then travels 1,634 m back to his house. Find the total distance traveled by first rounding each number to the nearest ten.**

3247…..3,250

582…….580

1634…. 1,630

First, round each number to the nearest ten.

$$\begin{array}{r} 1 \\ 3\,2\,5\,0 \\ 5\,8\,0 \\ +\ \underline{1\,6\,3\,0} \\ 6\,0 \end{array}$$

Adding the numbers in the tens place gives 16, so you need to regroup.

$$\begin{array}{r} 1\,1 \\ 3\,2\,5\,0 \\ 5\,8\,0 \\ +\ \underline{1\,6\,3\,0} \\ 4\,6\,0 \end{array}$$

Adding the numbers in the hundreds place gives 14, so regroup.

$$\begin{array}{r} 1\,1 \\ 3\,2\ 5\,0 \\ 5\,8\,0 \\ +\ \underline{1\,6\,3\,0} \\ 5{,}4\,6\,0 \end{array}$$

Adding the numbers in the thousands place gives 5.

Answer The total distance traveled was approximately 5,460 meters.

In the example above, the final estimate is 5,460 meters, which is 3 less than the actual sum of 5,463 meters.

Estimating is also effective when you are trying to find the difference between two numbers. Problems dealing with mountains like the example below may be important to a meteorologist, a pilot, or someone who is creating a map of a given region. As in other problems, estimating beforehand can help you find an answer that is close to the exact value, preventing potential errors in your calculations.

Example

Problem **One mountain is 10,496 feet high and another mountain is 7,421 feet high. Find the difference in height by first rounding each number to the nearest 100.**

10,496….10,500

7,421… ..7,400

First, round each number to the nearest hundred.

$$\begin{array}{r} 1\,0\,5\,0\,0 \\ -\ \ \underline{7\,4\,0\,0} \\ 3{,}1\,0\,0 \end{array}$$

Then, align the numbers and subtract.

The final estimate is 3,100, which is 25 greater than the actual value of 3,075.

Answer The estimated difference in height between the two mountains is 3,100 feet.

Self Check C

A space shuttle traveling at 17,581 miles per hour decreases its speed by 7,412 miles per hour. Estimate the speed of the space shuttle after it has slowed down by rounding each number to the nearest hundred.

Summary

Estimation is very useful when an exact answer is not required. You can use estimation for problems related to travel, finances, and data analysis. Estimating is often done before adding or subtracting by rounding to numbers that are easier to think about. Following the rules of rounding is essential to the practice of accurate estimation.

1.2.3 Self Check Solutions

Self Check A

In three months, a freelance graphic artist earns \$1,290 for illustrating comic books, \$2,612 for designing logos, and \$4,175 for designing web sites. Estimate how much she earned in total by first rounding each number to the nearest hundred.

Round the numbers to \$1,300, \$2,600, and \$4,200 and added them together to get the estimate: \$8,100

Self Check B

Estimate the difference of 474,128 and 262,767 by rounding to the nearest thousand.

Round to 474,000 and 263,000 and subtract to get 211,000

Self Check C

A space shuttle traveling at 17,581 miles per hour decreases its speed by 7,412 miles per hour. Estimate the speed of the space shuttle after it has slowed down by rounding each number to the nearest hundred.

17,600 – 7,400 = 10,200 mi/h

1.3.1 Multiplying Whole Numbers and Applications

Learning Objective(s)

1 Use three different ways to represent multiplication.
2 Multiply whole numbers.
3 Multiply whole numbers by a power of 10.
4 Use rounding to estimate products.
5 Solve application problems using multiplication.

Introduction

Multiplication is one of the four basic arithmetic operations (along with addition, subtraction, and division.) People use multiplication for a variety of everyday tasks including calculating the cost to purchase multiple items that are the same price, determining sales tax, and finding area and other geometric measures. If you wanted to calculate the cost of 6 baseball caps at $14.00 each, you could add 14.00 + 14.00 + 14.00 + 14.00 + 14.00 + 14.00, or use multiplication, which is a shortcut for repeated addition.

Ways to Represent Multiplication

Objective 1

Instead of adding the same number over and over again, an easier way to reach an answer is to use multiplication. Suppose you want to find the value in pennies of 9 nickels. You can use addition to figure this out. Since a nickel is worth 5 pennies, or 5 cents, you can find the value of 9 nickels by adding 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5. This repeated addition shows that 9 nickels have a value of 45 cents.

All of this addition can become very tiring. So, the math operation called *multiplication* can help perform repeated addition of whole numbers much more quickly. To find the value of these nickels you could write a multiplication equation: 5 • 9 = 45.

5 • 9 = 45 is read "five times nine equals 45" or "five multiplied by 9 is equal to 45." The numbers that are being multiplied are called **factors**. The factors in this example are 5 and 9. The result of the multiplication (or the answer) is called the **product**. The product for 5 • 9 is 45.

In addition to showing multiplication as 2 • 3 = 6, you can show multiplication by using the × sign, 2 × 3 = 6, and also with parentheses: (2)(3) = 6, or 2(3) = 6.

3 Ways to Write Multiplication

Using a multiplication or times sign: 2 × 3 = 6
Using a dot: 2 • 3 = 6 (this dot is NOT a decimal point)
Using parentheses: (2)(3) = 6 or 2(3) = 6

When you are adding the same number over and over again, you can use multiplication. You take the number that you are adding and rewrite it as a multiplication problem, multiplying it by the number of times you are adding it. For example, if you were serving 2 cookies each to 13 children, you could add 2 thirteen times or you could use multiplication to find the answer.

$2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 2 \cdot 13 = 26$

You could also write this using parentheses: $2(13) = 26$

What is Multiplication?

Objective **2**

In order to understand what multiplication is, consider three different ways to think about multiplication of whole numbers.

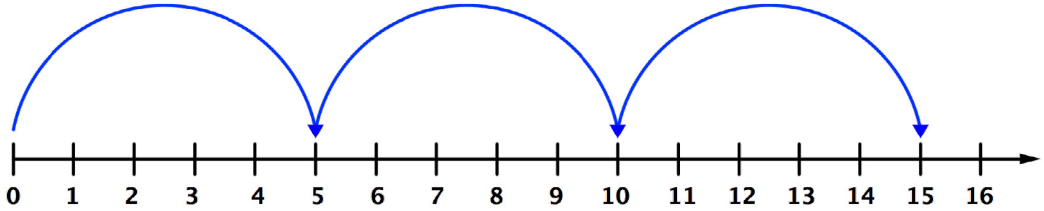
Approach 1: Set Model

Multiplication is a way of writing repeated addition. When you read the problem $3 \cdot 5$ you could think of this as 3 groups of 5 things—3 plates with 5 cookies on each plate; 3 baskets, each with 5 oranges in it; or 3 piles with 5 coins in each pile. We could show this as a picture:

$3 \cdot 5 = 3$ groups of $5 = 15$

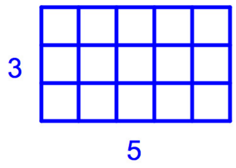
Approach 2: Number line Model

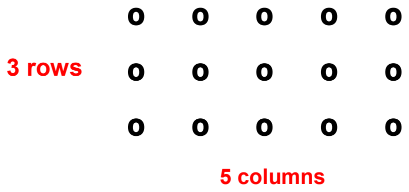
Multiplication can also be shown on a number line. The problem, $3 \cdot 5$ is modeled on the number line below. You can see that the arrows cover a distance of 5 units at a time. After 3 "jumps" on the number line, the arrow ends at 15.



Approach 3: Area Model

Another way of thinking about multiplication is to think about an array or area model to represent multiplication. You could think of $3 \cdot 5$ as 3 rows of 5 things. This might be a box of chocolates that has 3 rows of 5 chocolates, or a meeting room that is set up with 3 rows of 5 chairs. The pictures below show two rectangular arrangements of $3 \cdot 5$.



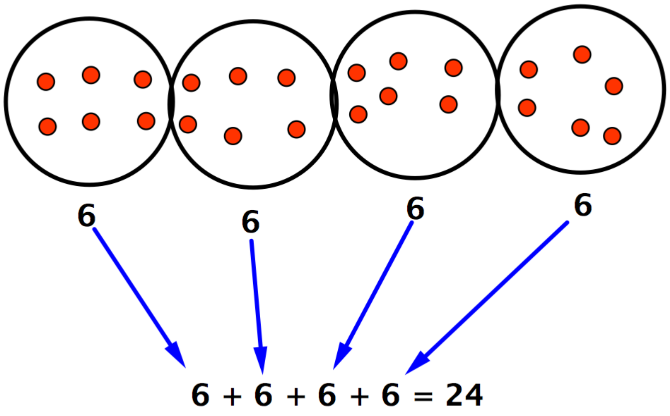


Do you see how both pictures represent the product 15? The picture on the left shows an area of 3 by 5. If you count the small squares that make up the rectangle, they total 15. Similarly, in the picture on the right, you see that 3 rows of 5 circles is equal to 15 circles.

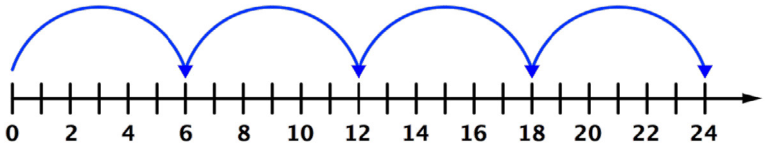
Example

Problem **What is the product of 4 • 6? Use the set model, number line model and area model to represent the multiplication problem.**

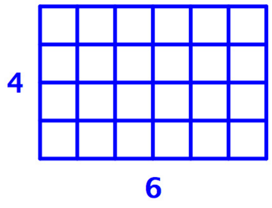
Set Model:



Number line model:



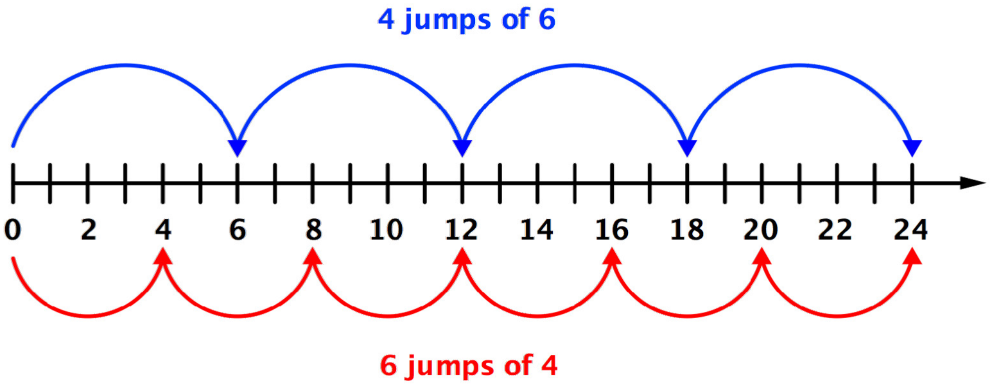
Area Model:



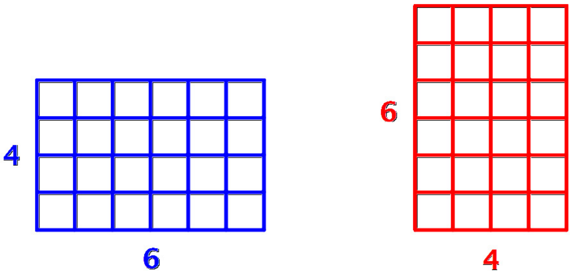
Answer 4 • 6 = 24

If you switch the order in which you multiply two numbers, the product will not change. This is true for any two numbers you multiply. Think about the problem shown above.

You could make 6 jumps of 4 or 4 jumps of 6 on the number line and end up at 24.



Or, you could make 6 rows of 4 or 4 rows of 6 and still have 24 squares.

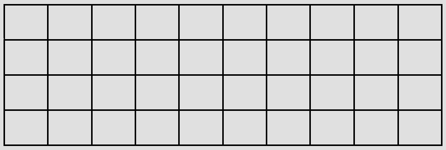


6 • 4 = 24 and 4 • 6 = 24.

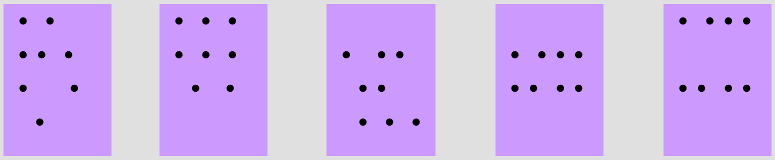
Self Check A

Tanisha modeled 5 • 8 using the following models. Which models are accurate representations of the multiplication of these two factors?

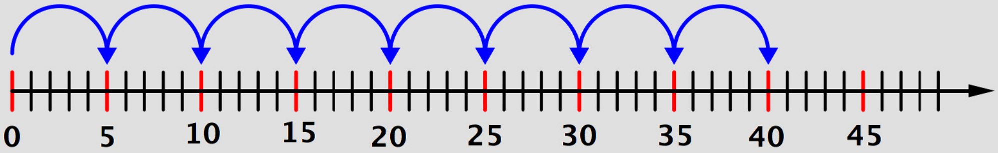
 # 1



#2



#3



Multiplying Greater Numbers

Let’s go back to the question posed at the opening of this topic of study. How can you use multiplication to figure out the total cost of 6 baseball caps that cost $14 each? (You do not have to pay sales tax). You can figure out the cost by multiplying 14 • 6.

One way to do this computation is to break 14 down into parts and multiply each part by 6.

$$14 = 10 + 4 \quad \text{So,}$$

$$\begin{aligned} 14 \bullet 6 &= 10 \bullet 6 + 4 \bullet 6 \\ &= 60 + 24 \\ &= 84 \end{aligned}$$

You may recall seeing the multiplication computed as follows:

$$\begin{array}{r} {}^{2}14 \\ \times\ \underline{6} \\ 84 \end{array}$$

In this notation, some of the steps are written down with special notation. You may be able to see that the product of 6 and 4 (24) is written by putting the 4 in the ones place and writing a small 2 up above the 1. This 2 actually stands for 20. Then, 6 is multiplied by 1. We are actually multiplying 6 by 1 ten and adding 20 to get the 80 in 84.

An example of multiplying two two-digit numbers is shown below. When performing this multiplication, each part of each number is multiplied by the other number. The numeric notation and an accompanying description are provided.

Example

Problem **47 • 52**

$$\begin{array}{r} 47 \\ \underline{\times\ \ 52} \end{array}$$

Stack the numbers with the place values aligned.

$$\begin{array}{r} 1 \\ 47 \\ \underline{\times\ \ 52} \\ 4 \end{array}$$

Multiply the ones. 2 x 7 = 14 ones. Write 4 ones in the ones place and regroup 10 ones into the tens place.

$$\begin{array}{r} 1 \\ 47 \\ \underline{\times\ \ 52} \\ 94 \end{array}$$

Multiply 2 ones by 4 tens, and add the regrouped 1 ten, 2 ones x 4 tens + 1 ten = 9 tens.

$$\begin{array}{r} {}^{3}47 \\ \underline{\times\ \ 52} \\ 94 \\ 50 \end{array}$$

Multiply the tens. 5 x 7 = 35 tens. Write 5 tens in the tens place and regroup.

$$\begin{array}{r} {}^{3}47 \\ \underline{\times\ \ 52} \\ 94 \\ 2350 \end{array}$$

Multiply 5 tens x 4 tens = 20 hundreds.
Add the regrouped 3, which is 3 hundred.

$$\begin{array}{r} {}^{3}47 \\ \underline{\times\ \ 52} \\ {}^{1}94 \\ \underline{+\ 2350} \\ 2{,}444 \end{array}$$

Add the two lines, 94 + 2350.

Answer 47 • 52 = 2,444

Notice that you are multiplying each of the parts of each number by the parts of the other number. You are doing so in a systematic way, from ones places to tens places. You are also using notation to keep track of what you have regrouped. These are shown with small raised numbers.

To keep your columns straight and your work organized, consider using grid paper or lined paper turned sideways so the lines form columns. Here is an example of a problem written on grid paper:

		2	3
	×	1	2
		4	**6**
+	**2**	**3**	**0**
	2	**7**	**6**

When you are multiplying whole numbers, be sure to line up the digits by their place values. In the example above, the digits in the ones place are lined up: the 2 in 12 is directly below the 3 in 23.

Multiplying Whole Numbers by 10

Objective **3**

When you multiply numbers by 10 or powers of 10 (100; 1,000; 10,000; 100,000), you'll discover some interesting patterns. These patterns occur because our number system is based on ten: ten *ones* equal ten; ten *tens* equal one hundred; ten *hundreds* equal one thousand. Learning about these patterns can help you compute easily and quickly.

Consider the example of 25 • 100. First, let's use the standard algorithm method to multiply these numbers.

Example

Problem **25 • 100**

$$\begin{array}{r} 100 \\ \underline{\times\ 25} \\ 500 \\ \underline{2000} \\ 2{,}500 \end{array}$$

Answer 25 • 100 = 2,500

Using the standard algorithm we calculated 25 • 100 = 2,500.

Look at the table below to find a pattern in the factors and products. See how the number of zeros in the power of 10 (10, 100, 1,000, etc.) relates to the number of zeros in the product.

Factors	Product
5 • 10	= 50
5 • 100	= 500
5 • 1,000	= 5,000
5 • 10,000	= 50,000

You can see that the number of zeros in the product matches the number of zeros in the power of 10 (10, 100, 1,000, etc.). Will this always be true or is it true only in certain situations? Look at two more patterns:

Factors	Product
10 • 10	= 100
10 • 100	= 1,000
10 • 1,000	= 10,000
10 • 10,000	= 100,000

Factors	Product
120 • 10	= 1,200
120 • 100	= 12,000
120 • 1,000	= 120,000
120 • 10,000	= 1,200,000

Notice that in these last two examples both factors had zeros in them. The number of zeros in the product is equal to the sum of the number of zeros at the end of each of the factors.

The example below illustrates how to multiply 140 • 3000.

Example		
Problem	**140 • 3000**	
	$\begin{array}{r} {}^{1}14 \\ \underline{x\ \ 3} \\ \mathbf{42} \end{array}$	Identify the non-zero parts of the factors and multiply these parts. Multiply 3 ones by 4 ones. 4 • 3 = 12. Write 2 in the ones place and regroup the 1 ten.
	420,000	Count the number of zeros in each factor. 140 has **one** zero; 3,000 has **three** zeros. 1 + 3 = 4 Write another 4 zeros after the 42.
Answer	140 • 3000 = 420,000	

Multiplying by Ten

When you multiply a whole number by 10 or a power of 10, first multiply the nonzero parts of the numbers. Then include a number of zeros at the end of the product equal to the total number of zeros at the end of the factors.

13 • 100 = 1,300
180 • 2,000 = 360,000

Self Check B

An apple orchard sold 100 bags of apples. If there are 30 apples in each bag, how many apples did the orchard sell?

Using Rounding to Estimate Products

Objective 4

Sometimes you don't need an exact product because an estimate is enough. If you're shopping, stopping to make a calculation with pencil and paper, or even a calculator is inconvenient. Usually, shoppers will round numbers up so they will be sure that they have enough money for their purchases.

Estimating products is also helpful for checking an answer to a multiplication problem. If your actual calculation is quite different from your estimate, there is a good chance you have made a place value and/or regrouping mistake.

To estimate a product, you often round the numbers first. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. If you are rounding a number to the nearest ten, you round it to the ten that is closest to the original number. An example of this is rounding 317 to the nearest ten. In this case, you round 317 to 320. If the number is half way in between (315), generally round up to 320.

Rounding factors can make it easy to multiply in your head. Let's consider the multiplication problem 145 • 29. To estimate this product by rounding, you can round to the nearest ten.

Example		
Problem	**Use rounding to estimate the product of 145 • 29.**	
	150 • 30	Round numbers to the nearest ten.
	15 • 3 = 45	Multiply the non-zero numbers.
	4,500	Count the zeros in the factors and include that many zeros after the 45.
Answer	The estimate of 145 • 29 is 4,500.	

You can use a calculator to see if your estimate seems reasonable. Or you can use estimation to make sure that the answer that you got on a calculator is reasonable. (Have you ever input the wrong numbers?)

	Key entries: 145 × 29 = Result: 4,205 The exact product and the estimate are close enough to give you confidence in your calculations.

Self Check C

A factory produces 58 packages of cookies in one hour. There are 32 cookies in each package. Which is the best estimate of the number of cookies the factory produces in one hour?

A) 1,800　　B) 1,500　　C) 18,000　　D) 180

Using Multiplication in Problem Solving

Objective 5

Multiplication is used in solving many types of problems. Below are two examples that use multiplication in their solutions to the problem.

	Example
Problem	**A case of cat food has two layers. Each layer has 4 rows of 6 cans. How many cans are in a case of cat food?**
$\begin{array}{r} 6 \\ \underline{x\ \ 4} \\ 24 \end{array}$	First find the number of cans in one layer. You can multiply to figure this out.
$\begin{array}{r} 24 \\ \underline{x\ \ 2} \\ 48 \end{array}$	Since there are two layers, you multiply the number of cans in one layer by 2.
Answer	There are 48 cans in a case of cat food.

Example

Problem **A theater has 45 rows with 40 seats in each row. How many seats are there in the theater?**

$$\begin{array}{r} {}^{2}45 \\ \underline{\text{x } 40} \\ 00 \\ \underline{+1800} \\ \mathbf{1{,}800} \end{array}$$

You can solve this problem by adding 40, 45 times, but that would take a lot of work. Multiplication is the way to go.

Answer There are 1,800 seats in the theater.

Self Check D

A lawn care company charges $35 to mow a lawn. If the company mows 32 lawns, how much money will it make?

Summary

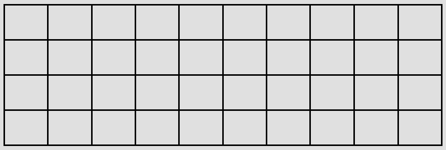
Multiplication can make repeated addition easier to compute in calculations and problem solving. Multiplication can be written using three symbols: parentheses, a times sign, or a multiplication dot. To perform multiplication with two-digit factors or greater, you can use the standard algorithm where you multiply each of the numbers in each factor by the numbers in the other factor. Using strategies such as short cuts for multiplying by powers of 10 and estimation to check your answers can make multiplication easier as well as reduce errors.

1.3.1 Self Check Solutions

Self Check A

Tanisha modeled 5 • 8 using the following models. Which models are accurate representations of the multiplication of these two factors?

 # 1



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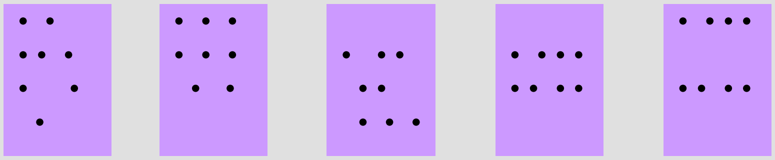

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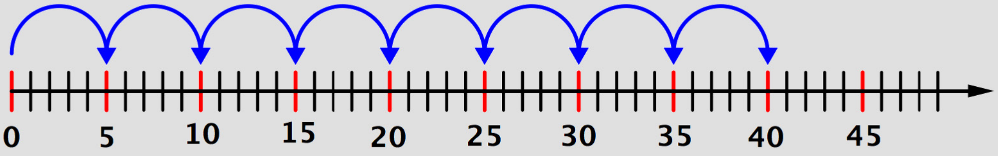


Only models #2 and #3 accurately represent 5 • 8.
Model #2 shows 5 groups of 8, which equals 40.
Model #3 shows skip counting 8 times by 5 to get to 40.
Model #1 shows 4 rows of 10, which is not a model for 5 • 8.

Self Check B

An apple orchard sold 100 bags of apples. If there are 30 apples in each bag, how many apples did the orchard sell?

30 • 100 = 3,000. 3 • 1= 3 and add 3 zeros since there is one zero in 30 and 2 in 100.

Self Check C

A factory produces 58 packages of cookies in one hour. There are 32 cookies in each package. Which is the best estimate of the number of cookies the factory produces in one hour?

Multiplying 60 • 30 would give a good estimate. 60 • 30 =1,800

Self Check D

A lawn care company charges $35 to mow a lawn. If the company mows 32 lawns, how much money will it make?

$35 • 32 = $1,120

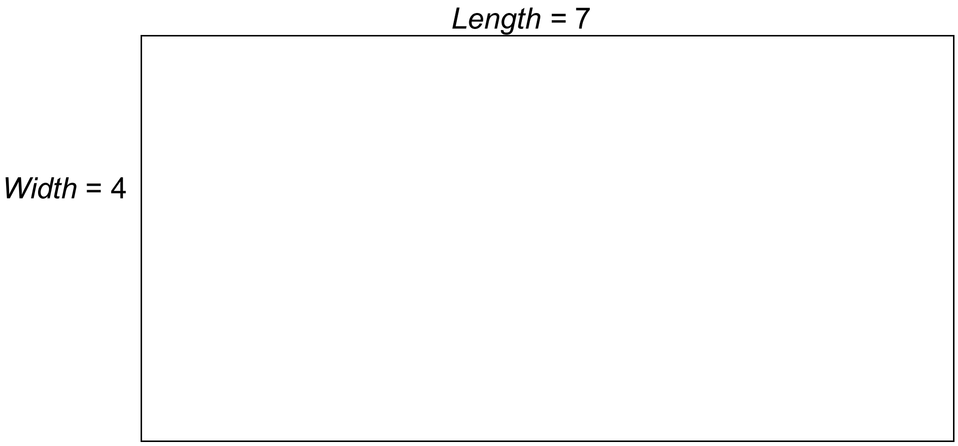
1.3.2 Areas of Rectangles

Learning Objective(s)
1 Find the area of a rectangle.
2 Find the area of shapes that are combinations of rectangles.

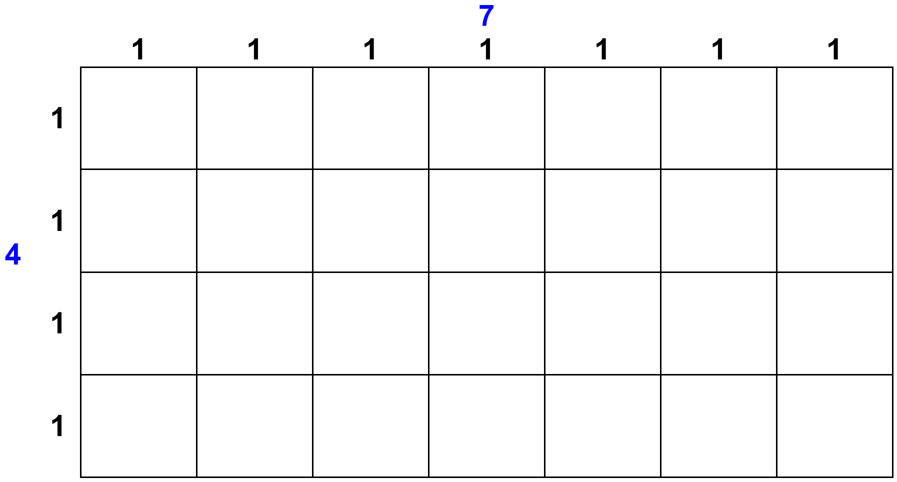
Finding the Area of a Rectangle

Objective 1

The formula for the area of a rectangle uses multiplication: *length* • *width* = *area.* Applying what you know about multiplication, you can find the area of any rectangle if you know its dimensions (length and width). Consider the rectangle that is 4 by 7 shown below. Its length is 7 and its width is 4.



You can divide the rectangle into units by making 7 columns and 4 rows.



You can see that dividing the rectangle in this way results in 28 squares. You could say that the area of the rectangle is 28 square units. You could also find the area by multiplying 7 • 4. (Note: Area is always measured in square units: square inches, square centimeters, square feet, etc.)

Consider an example of a larger rectangle, like one that is found on a soccer field. At each end of a soccer field, centered at the goal, is a large rectangle. This rectangle is called the *penalty box* because fouls committed within the lines of this rectangle may result in a penalty kick. On a regulation soccer field, the penalty box is 44 yards by 18 yards. What is the area of a penalty box?

Example

Problem **44 yards • 18 yards**

$$\begin{array}{r} {}^{3}44 \\ \times\ \underline{\ \ 18} \\ 352 \\ \underline{440} \\ 792 \end{array}$$

Answer The area of the rectangle is 792 square yards.

Self Check A

What is the area of a rectangle whose length is 23 feet and whose width is 7 feet?

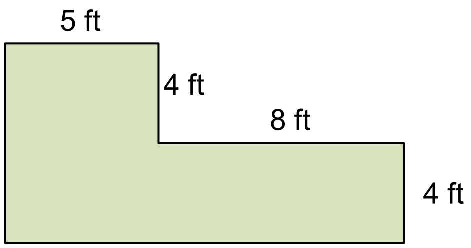
Finding the Area of a Shapes Build from Rectangles

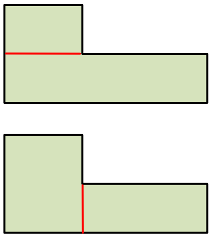
Objective **2**

Many shapes, while not rectangles, can be thought of as combinations of rectangles. By breaking a shape up into simpler pieces, we can use what we already know to find the areas of these shapes.

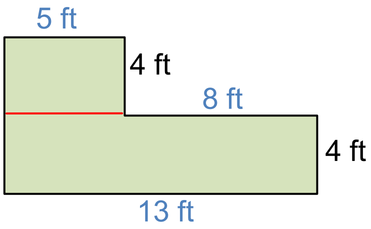
Example

Problem **Find the area of the shape below.**





Notice that we can divide this L-shaped object up into two separate rectangles. There are two ways we could separate them.



If we use the first approach, we'll have to add the two horizontal pieces together to find the total width of the bottom rectangle:
5 ft + 8 ft = 13ft

5 ft • 4 ft = 20 square feet
13 ft • 4 ft = 52 square feet

Now we can find the areas of the two rectangles

20 square feet + 52 square feet
= 72 square feet

We add these together to get the combined area of the original shape.

Answer The area of the shape is 72 ft^2.

1.3.2 Self Check Solutions

Self Check A
What is the area of a rectangle whose length is 23 feet and whose width is 7 feet?

23 • 7 = 161 square feet.

1.4 Dividing Whole Numbers and Applications

Learning Objective(s)

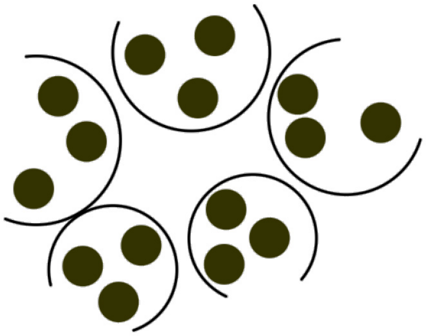
1 Use three different ways to represent division.
2 Divide whole numbers.
3 Perform long division.
4 Divide whole numbers by a power of 10.
5 Recognize that division by 0 is not defined.
6 Solve application problems using division.

Introduction

Some people think about division as "fair sharing" because when you divide a number you are trying to create equal parts. Division is also the **inverse operation** of multiplication because it "undoes" multiplication. In multiplication, you combine equal sets to create a total. In division, you separate a whole group into sets that have the same amount. For example, you could use division to determine how to share 40 empanadas among 12 guests at a party.

What is Division?

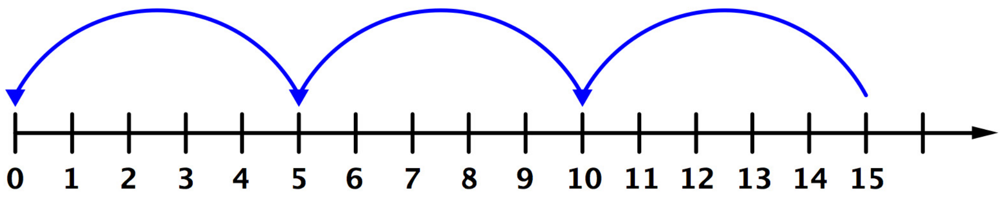
Division is splitting into equal parts or groups. For example, one might use division to determine how to share a plate of cookies evenly among a group. If there are 15 cookies to be shared among five people, you could divide 15 by 5 to find the "fair share" that each person would get. Consider the picture below.



15 cookies split evenly across 5 plates results in 3 cookies on each plate. You could represent this situation with the equation:

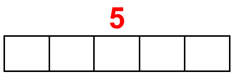
$15 \div 5 = 3$

You could also use a number line to model this division. Just as you can think of multiplication as repeated addition, you can think of division as repeated subtraction. Consider how many jumps you take by 5s as you move from 15 back to 0 on the number line.

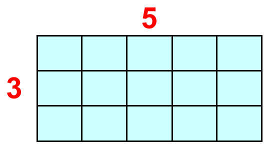


Notice that there are 3 jumps that you make when you skip count by 5 from 15 back to 0 on the number line. This is like subtracting 5 from 15 three times. This repeated subtraction can be represented by the equation: 15 ÷ 5 = 3.

Finally, consider how an area model can show this division. Ask yourself, if you were to make a rectangle that contained 15 squares with 5 squares in a row, how many rows would there be in the rectangle? You can start by making a row of 5:



Then keep adding rows until you get to 15 small squares.

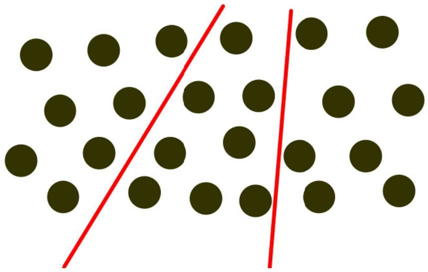


The number of rows is 3. So, 15 divided by 5 is equal to 3.

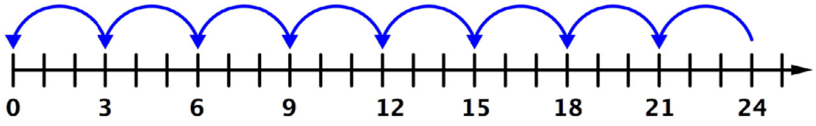
Example

Problem **Find 24 ÷ 3 using a set model and a number line model.**

Set Model:



Number line model:



Answer 24 ÷ 3 = 8

Ways to Represent Division

Objective **1**

As with multiplication, division can be written using a few different symbols. We showed this division written as 15 ÷ 5 = 3, but it can also be written two other ways:

$$\begin{array}{r} 3 \\ 5\overline{)15} \end{array}$$

$\frac{15}{5} = 3$ which can also be written more compactly as 15/5 = 3

Each part of a division problem has a name. The number that is being divided up, that is the total, is called the **dividend**. In the work in this topic, this number will be the larger number, but that is not always true in mathematics. The number that is dividing the dividend is called the **divisor**. The answer to a division problem is called the **quotient**.

The blue box summarizes the terminology and common ways to represent division.

3 Ways to Represent Division

12 ÷ 3 = 4	**with a division symbol**; this equation is read “12 divided by 3 equals 4.”
$\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$	**with a division or long division symbol**; this expression is read “12 divided by 3 equals 4.” Notice here, though, that you have to start with what is underneath the symbol. This may take some getting used to since you are reading from right to left and bottom to top!
$\frac{12}{3} = 4$	**with a fraction bar**; this expression can also be read “12 divided by 3 equals 4.” In this format, you read from top to bottom.
	In the examples above, *12* is the **dividend**, *3* is the **divisor** and *4* is the **quotient.**
	Dividend ÷ Divisor = Quotient $\qquad \begin{array}{r} \textit{Quotient} \\ \textit{Divisor}\overline{)\textit{Dividend}} \end{array}$

Self Check A

Which of the following expressions represent dividing $56 equally among 7 people?

1. $\frac{7}{56}$ **2.** 56 ÷ 7 **3.** $56\overline{)7}$

Dividing Whole Numbers

Objective **2**

Once you understand how division is written, you are on your way to solving simple division problems. You will need your multiplication facts to perform division. If you do not have them memorized, you can guess and check or use a calculator.

Consider the following problems:
10 ÷ 5 = ?
48 ÷ 2 = ?
30 ÷ 5 = ?

In the first problem, 10 ÷ 5, you could ask yourself, “how many fives are there in ten?” You can probably answer this easily. Another way to think of this is to consider breaking up 10 into 5 groups and picturing how many would be in each group.

10 ÷ 5 = **2**

To solve 48 ÷ 2, you might realize that dividing by 2 is like splitting into two groups or splitting the total in half. What number could you double to get 48?

48 ÷ 2 = **24**

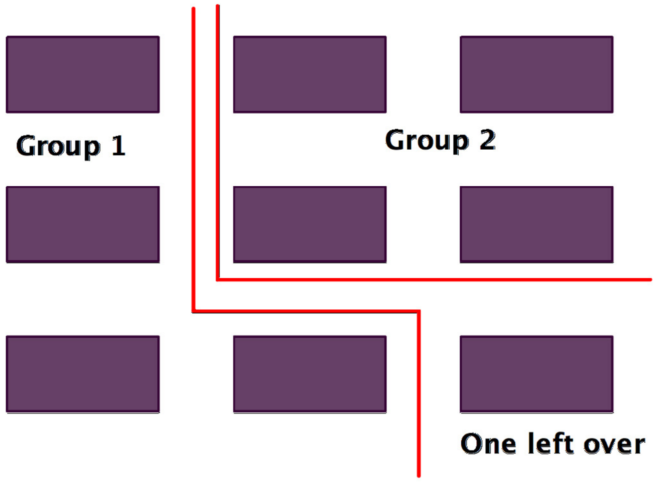
To figure out 30 ÷ 5, you could ask yourself, how many times do I have to skip count by 5 to get from 0 to 30? “5, 10, 15, 20, 25, 30. I have to skip count 6 times to get to 30.”

30 ÷ 5 = **6**

Example	
Problem	**35 ÷ 5**
Answer	7

Example	
Problem	**32 ÷ 4**
Answer	8

Sometimes when you are dividing, you cannot easily share the number equally. Think about the division problem 9 ÷ 2. You could think of this problem as 9 pieces of chocolate being split between 2 people. You could make two groups of 4 chocolates, and you would have one chocolate left over.



In mathematics, this left over part is called the **remainder**. It is the part that *remains* after performing the division. In the example above, the remainder is 1. We can write this as:

9 ÷ 2 = 4 R1

We read this equation: “Nine divided by two equals four with a remainder of 1.”

You might be thinking you could split that extra piece of chocolate in parts to share it. This is great thinking! If you split the chocolate in half, you could give each person another half of a piece of chocolate. They would each get $4\frac{1}{2}$ pieces of chocolate. We are not going to worry about expressing remainders as fractions or decimals right now. We are going to use the remainder notation with the letter R. Here’s an example:

Example		
Problem	**45 ÷ 6**	
	6 • 7 = 42	How many sixes are there in 45? Try 7.
	45 – 42 = 3	3 is not enough for another 6. So, 3 is the remainder.
Answer	45 ÷ 6 = 7 R3	

Since multiplication is the inverse of division, you can check your answer to a division problem with multiplication. To check the answer 7 R3, first multiply 6 by 7 and then add 3.

6 • 7 = 42
42 + 3 = 45, so the quotient 7 R3 is correct.

Self Check B

Compute 67 ÷ 7.

Performing Long Division

Objective 3

Long division is a method that is helpful when you are performing division that you cannot do easily in your head, such as division involving larger numbers. Below is an example of a way to write out the division steps.

Example

Problem **68 ÷ 4**

$4\overline{)68}$	Rewrite the division.
$\begin{array}{r} 1 \\ 4\overline{)68} \\ -4 \\ \hline 28 \end{array}$	Divide the tens. What is 6 divided by 4? Subtract 4 from 6 and bring down the next digit of the dividend, 8. What is 28 divided by 4?
$\begin{array}{r} 17 \\ 4\overline{)68} \\ -4 \\ \hline 28 \\ -28 \\ \hline 0 \end{array}$	7 • 4 = 28, so write a 7 above the 8. There is no remainder.
17 • 4 $\begin{array}{r} {}^{2}17 \\ \times\ 4 \\ \hline 68 \end{array}$	Check your answer using multiplication.

Answer 68 ÷ 4 = 17

Example

Problem **6,707 ÷ 233**

$\begin{array}{r} 2 \\ 233\overline{)6707} \\ \underline{466} \end{array}$	Examine the first 3 digits of the dividend and determine how many 233s are in it. Use guess and check. Try: 2 • 233 = 466 Try: 3 • 233 = 699 (too large)

$$\begin{array}{r} 2\\ 233\overline{)6707}\\ -\underline{466}\\ 2047 \end{array}$$

Subtract 466 from 670 and bring down the next digit of the dividend,7.

$$\begin{array}{r} 28\\ 233\overline{)6707}\\ -\underline{466}\\ 2047\\ -\underline{1864}\\ 183 \end{array}$$

How many 233s are in 2,047. It looks like close to 10 because 233 • 10 = 2,330.
Try 9, 233 • 9

$$\begin{array}{r} {}^{2}2{}^{2}33\\ \underline{\text{x}\quad 9}\\ 2{,}097 \end{array}$$

(Too large)

Must be 8!

$$\begin{array}{r} {}^{2}2{}^{2}33\\ \underline{\text{x}\quad 8}\\ 1{,}864 \end{array}$$

233 • 28 = 6,524
6,524 + 183 = 6,707

Check your answer using multiplication.
First, multiply 233 • 28.
Then, add the remainder.

Answer 6,707 ÷ 233 = 28 R183

Self Check C
Compute 417 ÷ 34.

Dividing Whole Numbers by a Power of 10

Objective **4**

Just as multiplication by powers of 10 results in a pattern, there is a pattern with division by powers of 10. Consider three quotients: 20 ÷ 10; 200 ÷ 10; and 2,000 ÷ 10.

Think about 20 ÷ 10. There are 2 tens in twenty, so 20 ÷ 10 = 2. The computations for 200 ÷ 10 and 2,000 ÷ 10 are shown below.

Example

Problem **200 ÷ 10**

$10\overline{)200}$ Rewrite the problem.

	$10\overline{)200}$	Divide the first digit of the dividend, 2, by the divisor. Since 2 ÷ 10 does not give a whole number, go to the next digit, 0.
	$\begin{array}{r} 2 \\ 10\overline{)200} \end{array}$	20 ÷ 10 = 2
	$\begin{array}{r} 2 \\ 10\overline{)200} \\ -\underline{20} \\ 0 \end{array}$	2 • 10 = 20 20 – 20 = 0
	$\begin{array}{r} 2 \\ 10\overline{)200} \\ -\underline{20} \\ 00 \end{array}$	Bring down the next digit, 0, of the dividend.
	$\begin{array}{r} 20 \\ 10\overline{)200} \\ -\underline{20} \\ 00 \\ \underline{0} \\ 0 \end{array}$	Since 10 still does not go into 00 and we have nothing left to bring down, multiply the 0 by 10. 0 • 10 = 0 0 – 0 = 0 We have no remainder.
Answer	200 ÷ 10 = 20	

Example		
Problem	**2000 ÷ 10**	
	$10\overline{)2000}$	Rewrite the problem.
	$10\overline{)2000}$	Divide the first digit of the dividend, 2, by the divisor. Since 2 ÷ 10 does not give a whole number, go to the next digit, 0.

$$\begin{array}{r} 2 \\ 10\overline{)2000} \end{array}$$

20 ÷ 10 = 2

$$\begin{array}{r} 2 \\ 10\overline{)2000} \\ -\underline{20} \\ 0 \end{array}$$

2 • 10 = 20
20 – 20 = 0

$$\begin{array}{r} 2 \\ 10\overline{)2000} \\ -\underline{20} \\ 00 \end{array}$$

Bring down the next digit, 0, of the dividend.

$$\begin{array}{r} 20 \\ 10\overline{)2000} \\ -\underline{20} \\ 000 \end{array}$$

Since 10 does not go into 00, add a 0 to the quotient and bring down the next digit, 0.

$$\begin{array}{r} 200 \\ 10\overline{)2000} \\ -\underline{20} \\ 000 \\ \underline{0} \\ 0 \end{array}$$

Since 10 still does not go into 000 and we have nothing left to bring down, add a 0 to the quotient, multiply the 0 by 10.
0 • 10 = 0
0 – 0 = 0
We have no remainder.

Answer 2,000 ÷ 10 = 200

Examine the results of these three problems to try to determine a pattern in division by 10.

20 ÷ 10 = **2**
200 ÷ 10 = **20**
2000 ÷ 10 = **200**

Notice that the number of zeros in the quotient decreases when a dividend is divided by 10: 20 becomes 2; 200 becomes 20 and 2,000 become 200. In each of the examples above, you can see that there is one fewer 0 in the quotient than there was in the dividend.

Continue another example of division by a power of 10.

Example

Problem **2,000 ÷ 100**

$100\overline{)2000}$	Rewrite the problem.
$100\overline{)2000}$	Divide the first digit of the dividend, 2, by the divisor. Since 2 ÷ 100 does not give a whole number, go to the next digit, 0.
$100\overline{)2000}$	Divide the first two digits of the dividend, 20, by the divisor. Since 20 ÷ 100 does not give a whole number, go to the next digit, 0.
$\begin{array}{r} 2 \\ 100\overline{)2000} \end{array}$	200 ÷ 100 = 2
$\begin{array}{r} 2 \\ 100\overline{)2000} \\ \underline{-200} \\ 0 \end{array}$	2 • 100 = 200 200 – 200 = 0
$\begin{array}{r} 2 \\ 100\overline{)2000} \\ \underline{-200} \\ 00 \end{array}$	Bring down the next digit, 0, of the dividend.
$\begin{array}{r} 20 \\ 100\overline{)2000} \\ \underline{-200} \\ 00 \\ \underline{0} \\ 0 \end{array}$	Since 100 still does not go into 00 and we have nothing left to bring down, add a 0 to the quotient, multiply the 0 by 10. 0 • 10 = 0 0 – 0 = 0 We have no remainder.

Answer 2,000 ÷ 100 = 20

Consider this set of examples of division by powers of 10. What pattern do you see?

20 ÷ 10 = **2**
200 ÷ 10 = **20**
2,000 ÷ 10 = **200**
2,000 ÷ 100 = **20**
2,000 ÷ 1000 = **2**

Notice that when you divide a number by a power of 10, the quotient has fewer zeros. This is because division by a power of 10 has an effect on the place value. For example, when you perform the division 18,000 ÷ 100 = 180, the quotient, 180, has two fewer zeros than the dividend, 18,000. This is because the power of 10 divisor, 100, has two zeros.

Self Check D
Compute 135,000 ÷ 100.

Division by Zero

Objective **5**

You know what it means to divide by 2 or divide by 10, but what does it mean to divide a quantity by 0? Is this even possible? Can you divide 0 by a number? Consider the two problems written below.

$\frac{0}{8}$ and $\frac{8}{0}$

We can read the first expression, "zero divided by eight" and the second expression, "eight divided by zero." Since multiplication is the inverse of division, we could rewrite these as multiplication problems.

0 ÷ 8 = ?
? • 8 = 0
The quotient must be 0 because 0 • 8 = 0
$\frac{0}{8} = 0$

Now let's consider $\frac{8}{0}$.

8 ÷ 0 = ?
? • 0 = 8

This is not possible. There is no number that you could multiply by zero and get eight. Any number multiplied by zero is always zero. There is no quotient for $\frac{8}{0}$. There is no quotient for any number when it is divided by zero.

Division by zero is an operation for which you cannot find an answer, so it is not allowed. We say that division by 0 is undefined.

Objective **6**

Using Division in Problem Solving

Division is used in solving many types of problems. Below are three examples from real life that use division in their solutions.

Example		
Problem	**Luana made 40 empanadas for a party. If the empanadas are divided equally among 12 guests, how many will each guest have? Will there be any leftover empanadas?**	
	40 ÷ 12	Since each guest will have an equal share, we can use division.
	$\begin{array}{r} 3 \\ 12\overline{)40} \\ -\ \underline{36} \\ 4 \end{array}$	Use trial and error. Try 3. 12 • 3 = 36 When 40 empanadas are divided equally among 12 people, there are 4 left over.
Answer	Each guest will have 3 empanadas. There will be 4 empanadas left over.	

Example		
Problem	**A case of floor tiles has 12 boxes in it. If the case costs $384, how much does one box cost?**	
	384 ÷ 12	Since the boxes each cost the same amount, you want to divide $384 into 12 equal parts.
	$12\overline{)384}$	Perform the division. Try to divide the first digit in the dividend by the divisor. 12 will not divide into 3, so go to the next digit.
	$\begin{array}{r} 3 \\ 12\overline{)384} \\ -\ \underline{36} \end{array}$	Perform 38 ÷ 12. Pick a quotient and test it. Try 3. 3 • 12 = 36.
	3	Subtract 36 from 38.

$$\begin{array}{r} 12\overline{)384} \\ \underline{-36} \\ 2 \end{array}$$

$$\begin{array}{r} 32 \\ 12\overline{)384} \\ \underline{-36} \\ 24 \\ -\underline{24} \\ 0 \end{array}$$

Bring the next digit of the dividend down and perform division.

12 • 2 = 24
24 – 24 = 0

Does 32 • 12 equal 384?

Check your answer by multiplying.

$$\begin{array}{r} 32 \\ \underline{\times\ 12} \\ 64 \\ +\underline{320} \\ 384 \end{array}$$

Yes! My answer is correct!

Answer Each box of tiles costs $32.

Example

Problem **A banana grower is shipping 4,644 bananas. There are 86 crates, each containing the same number of bananas. How many bananas are in each crate?**

4,644 ÷ 86

Since each crate, or box, has the same number of bananas, you can take the total number of bananas and divide by the number of crates.

Rewrite the division.

$$86\overline{)4644}$$

Use trial and error to determine what 464 ÷ 86 equals.
Try 5:

$$\begin{array}{r} {}^{3}86 \\ \underline{\times\ 5} \\ 430 \end{array}$$

5

$\begin{array}{r} 86\overline{)4644} \\ -\ \underline{430} \\ 344 \end{array}$	464 – 430 = 34 Then, bring down the next digit of the dividend, 4.
$\begin{array}{r} 54 \\ 86\overline{)4644} \\ -\underline{430} \\ 344 \\ -\ \underline{344} \\ 0 \end{array}$	Use trial and error to determine the quotient of 344 and 86. Try 4: $\begin{array}{r} {}^{2}86 \\ \underline{\times\ 4} \\ 344 \end{array}$
$\begin{array}{r} 86 \\ \underline{\times\ 54} \\ 344 \\ +\ \ 4\underline{300} \\ 4644 \end{array}$	Check your answer by multiplying.
	Yes! My answer is correct!

Answer Each crate contains 54 bananas.

Self Check E

A theater has 1,440 seats. The theater has 30 rows of seats. How many seats are in each row?

Summary

Division is the inverse operation of multiplication, and can be used to determine how to evenly share a quantity among a group. Division can be written in three different ways, using a fraction bar, ÷, and $\overline{)}$. Division can be represented as splitting a total quantity into sets of equal quantities, as skip subtracting on the number line, and as a dimension with an area model. Remainders may result when performing division and they can be represented with the letter R, followed by the number remaining. Since division is the inverse operation of multiplication, you need to know your multiplication facts in order to do division. For larger numbers, you can use long division to find the quotient.

1.4 Self Check Solutions

Self Check A

Which of the following expressions represent dividing $56 equally among 7 people?

1. $\frac{7}{56}$ **2.** $56 \div 7$ **3.** $56\overline{)7}$

#2 is the only expression that represents 56 divided by 7.

Self Check B

Compute 67 ÷ 7.

9 R4
9 • 7 = 63 and there are 4 left over.

Self Check C

Compute 417 ÷ 34.

12 R9
12 • 34 = 408 and 408 + 9 =417

Self Check D

Compute 135,000 ÷ 100.

1,350
1350 • 100 = 135,000.

Self Check E

A theater has 1,440 seats. The theater has 30 rows of seats. How many seats are in each row?

1440 ÷ 30 = 48.

1.5.1 Understanding Exponents and Square Roots

Learning Objective(s)

1 Evaluate expressions containing exponents.
2 Write repeated factors using exponential notation.
3 Find a square root of a perfect square.

Introduction

Exponents provide a special way of writing repeated multiplication. Numbers written in this way have a specific form, with each part providing important information about the number. Writing numbers using exponents can save a lot of space, too. The inverse operation of multiplication of a number by itself is called finding the square root of a number. This **operation** is helpful for problems about the area of a square.

Understanding Exponential Notation

Objective 1

Exponential notation is a special way of writing repeated factors, for example 7 • 7. Exponential notation has two parts. One part of the notation is called the **base**. The base is the number that is being multiplied by itself. The other part of the notation is the **exponent**, or power. This is the small number written up high to the right of the base. The exponent, or power, tells how many times to use the base as a **factor** in the multiplication. In the example, 7 • 7 can be written as 7^2, 7 is the base and 2 is the exponent. The exponent 2 means there are two factors.

$$7^2 = 7 \cdot 7 = 49$$

You can read 7^2 as “seven squared.” This is because multiplying a number by itself is called “**squaring** a number.” Similarly, raising a number to a power of 3 is called “**cubing** the number.” You can read 7^3 as “seven cubed.”

You can read 2^5 as “two to the fifth power” or “two to the power of five.” Read 8^4 as “eight to the fourth power” or “eight to the power of four.” This format can be used to read *any* number written in exponential notation. In fact, while 6^3 is most commonly read “six cubed,” it can also be read “six to the third power” or “six to the power of three.”

To find the value of a number written in exponential form, rewrite the number as repeated multiplication and perform the multiplication. Two examples are shown below.

Example

Problem **Find the value of 4^2.**

4 is the base. 2 is the exponent.	An exponent means repeated multiplication. The base is 4; 4 is the number being multiplied. The exponent is 2; This means to use two factors of 4 in the multiplication.
$4^2 = 4 \cdot 4$	Rewrite as repeated multiplication.
$4 \cdot 4 = 16$	Multiply.

Answer $4^2 = 16$

Example

Problem **Find the value of 2^5.**

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	Rewrite 2^5 as repeated multiplication. The base is 2, the number being multiplied. The exponent is 5, the number of times to use 2 in the multiplication.
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ $4 \cdot 2 \cdot 2 \cdot 2$ $8 \cdot 2 \cdot 2$ $16 \cdot 2$ 32	Perform multiplication.

Answer $2^5 = 32$

Self Check A

Find the value of 4^3.

Writing Repeated Multiplication Using Exponents

Objective **2**

Writing repeated multiplication in exponential notation can save time and space. Consider the example 5 • 5 • 5 • 5. We can use exponential notation to write this repeated multiplication as 5^4. Since 5 is being multiplied, it is written as the base. Since the base is used 4 times in the multiplication, the exponent is 4. The expression 5 • 5 • 5 • 5 can be rewritten in shorthand exponential notation as 5^4 and is read, "five to the fourth power" or "five to the power of 4."

To write repeated multiplication of the same number in exponential notation, first write the number being multiplied as the base. Then count how many times that number is used in the multiplication, and write that number as the exponent. Be sure to count the numbers, *not* the multiplication signs, to determine the exponent.

Example

Problem **Write 7 • 7 • 7 in exponential notation.**

7 is the base. The base is the number being multiplied, 7.

Since 7 is used 3 times, 3 is the exponent. The exponent tells the number of times the base is multiplied.

Answer *$7 \cdot 7 \cdot 7 = 7^3$* *This is read "seven cubed."*

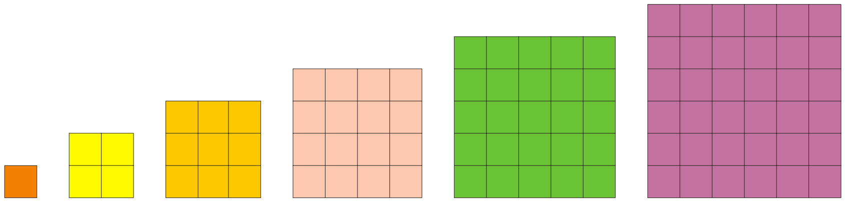
Self Check B

Write 10 • 10 • 10 • 10 • 10 • 10 in exponential notation.

Understanding and Computing Square Roots

Objective **3**

As you saw earlier, 5^2 is called "five squared." "Five squared" means to multiply five by itself. In mathematics, we call multiplying a number by itself "squaring" the number. We call the result of squaring a whole number a square or a **perfect square**. A perfect square is any number that can be written as a whole number **raised to the power** of 2. For example, 9 is a perfect square. A perfect square number can be represented as a square shape, as shown below. We see that 1, 4, 9, 16, 25, and 36 are examples of perfect squares.



Shape number	1	2	3	4	5	6
Number of small squares	1x1=1	2x2=4	3x3=9	4x4=16	5x5=25	6x6=36

To square a number, multiply the number by itself. 3 squared = 3^2 = 3 • 3 = 9.

Below are some more examples of perfect squares.

1 squared	1^2	1 • 1	1
2 squared	2^2	2 • 2	4
3 squared	3^2	3 • 3	9
4 squared	4^2	4 • 4	16
5 squared	5^2	5 • 5	25
6 squared	6^2	6 • 6	36
7 squared	7^2	7 • 7	49
8 squared	8^2	8 • 8	64
9 squared	9^2	9 • 9	81
10 squared	10^2	10 • 10	100

The **inverse operation** of squaring a number is called finding the **square root** of a number. Finding a square root is like asking, "what number multiplied by itself will give me this number?" The square root of 25 is 5, because 5 multiplied by itself is equal to 25. Square roots are written with the mathematical symbol, called a **radical sign**, that looks like this: $\sqrt{\ }$. The "square root of 25" is written $\sqrt{25}$.

Example	
Problem	**Find $\sqrt{81}$.**
	$\sqrt{81} = 9$
Answer	$\sqrt{81} = 9$

Self Check C

Find $\sqrt{36}$.

Summary

Exponential notation is a shorthand way of writing repeated multiplication of the same number. A number written in exponential notation has a base and an exponent, and each of these parts provides information for finding the value of the expression. The base tells what number is being repeatedly multiplied, and the exponent tells how many times the base is used in the multiplication. Exponents 2 and 3 have special names. Raising a base to a power of 2 is called "squaring" a number. Raising a base to a power of 3 is called "cubing" a number. The inverse of squaring a number is finding the square root of a number. To find the square root of a number, ask yourself, "What number can I multiply by itself to get this number?"

1.5.1 Self Check Solutions

Self Check A

Find the value of 4^3.

$4 \cdot 4 \cdot 4 = 64$.

Self Check B

Write $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ in exponential notation.

10^6

The base is 10 since that is the number that is being multiplied by itself. The exponent is 6 since there are six 10s in the multiplication.

Self Check C

Find $\sqrt{36}$.

Since $6 \cdot 6 = 36$, $\sqrt{36} = 6$.

1.5.2 Order of Operations

Learning Objective(s)

1 Use the order of operations to simplify expressions, including those with parentheses.

2 Use the order of operations to simplify expressions containing exponents and square roots.

Introduction

People need a common set of rules for performing computation. Many years ago, mathematicians developed a standard **order of operations** that tells you which calculations to make first in an expression with more than one **operation**. Without a standard procedure for making calculations, two people could get two different answers to the same problem. For example, 3 + 5 • 2 has only one correct answer. Is it 13 or 16?

The Order of Addition, Subtraction, Multiplication & Division Operations

Objective 1

First, consider expressions that include one or more of the arithmetic operations: addition, subtraction, multiplication, and division. The order of operations requires that all multiplication and division be performed first, going from left to right in the **expression**. The order in which you compute multiplication and division is determined by which one comes first, reading from left to right.

After multiplication and division has been completed, add or subtract in order from left to right. The order of addition and subtraction is also determined by which one comes first when reading from left to right.

Below, are three examples showing the proper order of operations for expressions with addition, subtraction, multiplication, and/or division.

Example

Problem **Simplify 3 + 5 • 2.**

3 + **5 • 2**	Order of operations tells you to perform multiplication before addition.
3 + 10	Then add.

Answer 3 + 5 • 2 = 13

Example

Problem **Simplify 20 – 16 ÷ 4.**

20 – **16 ÷ 4**	Order of operations tells you to perform division before subtraction.
20 – 4 16	Then subtract.

Answer 20 – 16 ÷ 4 = 16

Example

Problem **Simplify 60 – 30 ÷ 3 • 5 + 7.**

60 – **30 ÷ 3 • 5** + 7	Order of operations tells you to perform multiplication and division first, working from left to right, before doing addition and subtraction.
60 – **10 • 5** + 7 60 – 50 + 7	Continue to perform multiplication and division from left to right.
10 + 7 17	Next, add and subtract from left to right. (Note that addition is not necessarily performed before subtraction.)

Answer 60 – 30 ÷ 3 • 5 + 7 = 17

Grouping Symbols and the Order of Operations

Grouping symbols such as parentheses ( ), brackets [ ], braces $\{\ \}$, and fraction bars can be used to further control the order of the four basic arithmetic operations. The rules of the order of operations require computation within grouping symbols to be completed first, even if you are adding or subtracting within the grouping symbols and you have multiplication outside the grouping symbols. After computing within the grouping symbols, divide or multiply from left to right and then subtract or add from left to right.

Example

Problem **Simplify 900 ÷ (6 + 3 • 8) – 10.**

900 ÷ **(6 + 3 • 8)** – 10	Order of operations tells you to perform what is inside the parentheses first.
900 ÷ (6 + **3 • 8)** – 10 900 ÷ **(6 + 24)** – 10	Simplify the expression in the parentheses. Multiply first.
900 ÷ 30 – 10	Then add 6 + 24.
900 ÷ 30 – 10 30 – 10 20	Now perform division; then subtract.

Answer 900 ÷ (6 + 3 • 8) – 10 = 20

When there are grouping symbols within grouping symbols, compute from the inside to the outside. That is, begin simplifying the innermost grouping symbols first. Two examples are shown.

Example

Problem **Simplify 4 – 3[20 – 3 • 4 – (2 + 4)] ÷ 2.**

4 – 3**[20 – 3 • 4 – (2 + 4)]** ÷ 2	There are brackets and parentheses in this problem. Compute inside the innermost grouping symbols first.
4 – 3[**20 – 3 • 4 – (2 + 4)]** ÷ 2 4 – 3[**20 – 3 • 4 – 6]** ÷ 2	Simplify within parentheses.
4 – 3[**20 – 3 • 4 – 6]** ÷ 2 4 – 3**[20 – 12 – 6]** ÷ 2 4 – 3**[8 – 6]** ÷ 2	Then, simplify within the brackets by multiplying and then subtracting from left to right.
4 – 3(2) ÷ 2	
4 – **3(2) ÷ 2** 4 – **6 ÷ 2** 4 **–** 3	Multiply and divide from left to right.
4 **–** 3 1	Subtract.

Answer 4 – 3[20 – 3 • 4 – (2 + 4)] ÷ 2 = 1

Remember that parentheses can also be used to show multiplication. In the example that follows, the parentheses are not a grouping symbol; they are a multiplication symbol. In this case, since the problem only has multiplication and division, we compute from left to right. Be careful to determine what parentheses mean in any given problem. Are they a grouping symbol or a multiplication sign?

Example

Problem **Simplify 6 ÷ (3)(2).**

$6 \div 3 \cdot 2$	This expression has multiplication and division only. The multiplication operation can be shown with a dot.
$6 \div 3 \cdot 2$ $2 \cdot 2$ 4	Since this expression has only division and multiplication, compute from left to right.

Answer 6 ÷ (3)(2) = 4

Consider what happens if braces are added to the problem above: 6 ÷ $\{(3)(2)\}$. The parentheses still mean multiplication; the additional braces are a grouping symbol. According to the order of operations, compute what is inside the braces first. This problem is now evaluated as 6 ÷ 6 = 1. Notice that the braces caused the answer to change from 1 to 4.

Self Check A

Simplify 40 – (4 + 6) ÷ 2 + 3.

The Order of Operations

1) Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2) Multiply and Divide, from left to right.
3) Add and Subtract, from left to right.

Performing the Order of Operations with Exponents and Square Roots

Objective **2**

So far, our rules allow us to simplify expressions that have multiplication, division, addition, subtraction or grouping symbols in them. What happens if a problem has **exponents** or **square roots** in it? We need to expand our order of operation rules to include exponents and square roots.

If the expression has exponents or square roots, they are to be performed *after* parentheses and other grouping symbols have been simplified and *before* any multiplication, division, subtraction and addition that are outside the parentheses or other grouping symbols.

Note that you compute from more complex operations to more basic operations. Addition and subtraction are the most basic of the operations. You probably learned these first. Multiplication and division, often thought of as repeated addition and subtraction, are more complex and come before addition and subtraction in the order of operations. Exponents and square roots are repeated multiplication and division, and because they're even more complex, they are performed before multiplication and division. Some examples that show the order of operations involving exponents and square roots are shown below.

Example

Problem **Simplify $14 + 28 \div 2^2$.**

$14 + 28 \div 2^2$	This problem has addition, division, and exponents in it. Use the order of operations.
$14 + 28 \div 4$	Simplify 2^2.
$14 + 7$	Perform division before addition.
21	Add.

Answer $14 + 28 \div 2^2 = 21$

Example

Problem **Simplify $3^2 \cdot 2^3$.**

$3^2 \cdot 2^3$	This problem has exponents and multiplication in it.
$9 \cdot 8$	Simplify 3^2 and 2^3.
72	Perform multiplication.

Answer $3^2 \cdot 2^3 = 72$

Example

Problem **Simplify $(3 + 4)^2 + (8)(4)$.**

$\mathbf{(3 + 4)}^2 + (8)(4)$	This problem has parentheses, exponents, and multiplication in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication. Grouping symbols are handled first.
$7^2 + (8)(4)$ $49 + (8)(4)$	Add the numbers inside the parentheses that are serving as grouping symbols. Simplify 7^2.
$49 + 32$	Perform multiplication.
81	Add.

Answer $(3 + 4)^2 + (8)(4) = 81$

Self Check B

Simplify $77 – (1 + 4 – 2)^2$.

The Order of Operations

1) Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2) Evaluate exponents and roots of numbers, such as square roots.
3) Multiply and Divide, from left to right.
4) Add and Subtract, from left to right.

Some people use a saying to help them remember the order of operations. This saying is called PEMDAS or “**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally.” The first letter of each word begins with the same letter of an arithmetic operation.

Please ⇒ **P**arentheses (and other grouping symbols)

Excuse ⇒ **E**xponents

My **D**ear ⇒ **M**ultiplication and **D**ivision (from left to right)

Aunt **S**ally ⇒ **A**ddition and **S**ubtraction (from left to right)

Note: Even though multiplication comes before division in the saying, division could be performed first. Which is performed first, between multiplication and division, is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don’t let the saying confuse you about this!

Summary

The order of operations gives us a consistent sequence to use in computation. Without the order of operations, you could come up with different answers to the same computation problem. (Some of the early calculators, and some inexpensive ones, do NOT use the order of operations. In order to use these calculators, the user has to input the numbers in the correct order.)

1.5.2 Self Check Solutions

Self Check A

Simplify 40 – (4 + 6) ÷ 2 + 3.

38

Compute the addition in parentheses first. 40 – 10 ÷ 2 + 3.
Then, perform division. 40 – 5 + 3. Finally, add and subtract from left to right.

Self Check B

Simplify $77 - (1 + 4 - 2)^2$.

68

Correct. $77 - (1 + 4 - 2)^2 = 77 - (3)^2 = 77 - 9 = 68$

1.6.1 Graphing Data

Learning Objective(s)

1 Read and interpret data from tables and pictographs.
2 Read and interpret data from bar graphs
3 Read and interpret data from line graphs

Introduction

A nurse is collecting blood type data from her patients. When a new patient is checked in, the nurse does a simple finger-prick test to see whether the patient's blood is type A, B, AB, or O. (These are the four possible blood types. Each one also carries a + or – to represent the RH factor, but for our purposes, let's just track the type, not the + or – .) She tracks her results by creating a two-column table with the patient's name and blood type.

Name	Blood Type
Dominique	A
Ilya	O
Raul	AB
Madison	O
Philip	AB
Samuel	B
Josefine	O
Brett	O
Paula	B
Leticia	AB

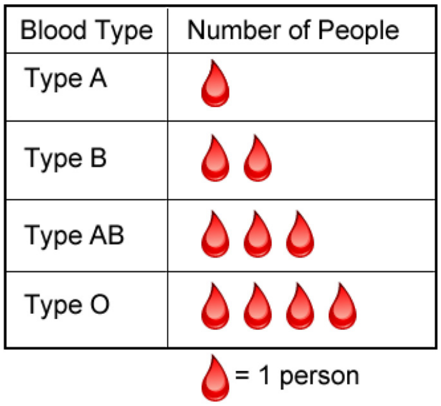
The information in this table is an example of **data**, or information. In this case, the nurse has gathered a fair amount of data about her patients' blood types. By analyzing the data, she can learn more about the range of patients that she serves.

Data helps us make many kinds of decisions. Organizing data into graphs can help us get a clear picture of a situation and can often help us make decisions based on the picture. So how do you take data and make a picture out of it? Let's take a look.

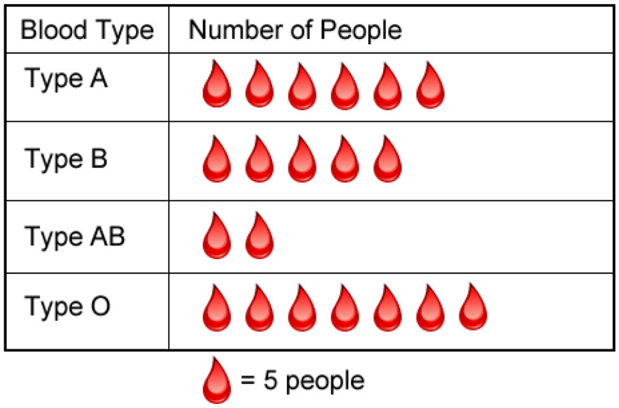
Pictures of Data

Objective 1

Let's return to the data set used previously. If the nurse wanted to represent the data visually, she could use a **pictograph**. Pictographs represent data using images. This visual presentation helps illustrate that for the data in her table, Type O blood is the most common, and Type A blood is the least common.



Interested by the results of this small survey, the nurse continues to document the blood types of her patients until she has surveyed 100 people. She puts all of this data in a table, but she finds that it is hard for her to quickly identify what the data is telling her. She decides to make another pictograph using a different scale.



To read this pictograph, all you need is the scale—the number of people that each blood drop symbol represents. In this graph, each blood drop represents 5 people. There are six drops next to Type A, so 5 • 6 = 30 people had Type A blood. The table below shows the rest of the information.

Blood Type	**Number of People**
Type A	6 drops • 5 people = 30 people
Type B	5 drops • 5 people = 25 people
Type AB	2 drops • 5 people = 10 people
Type O	7 drops • 5 people = 35 people

Example

Problem **The pictograph below shows the number of medals earned at an international competition. How many medals did Japan earn?**

Country	**Medals**
Japan	♦♦♦♦♦
Argentina	♦♦
Germany	♦♦♦♦♦♦
Egypt	♦♦♦♦

♦ = 4 medals

Look at the scale of the pictograph.

Each ♦ represents 4 medals.

5 • 4 = 20

Japan has 5 ♦s, so the total number of medals is 5 • 4 = 20.

Answer Japan earned 20 medals.

Self Check A

Which table accurately represents the data shown in the pictograph below?

Employee	**Hourly wage**
Wayne	$ $ $ $ $
Sarah	$ $ $ $ $ $ $
Leigh	$ $ $ $

$ = $4

A)

Employee	**Hourly wage**
Wayne	$5
Sarah	$7
Leigh	$4

B)

Employee	**Hourly wage**
Wayne	$20
Sarah	$22
Leigh	$19

C)

Employee	**Hourly wage**
Wayne	$10
Sarah	$14
Leigh	$8

D)

Employee	**Hourly wage**
Wayne	$20
Sarah	$28
Leigh	$16

Bar Graphs

Objective **2**

Representing data as pictures doesn't always make sense either. **Bar graphs** are an alternative (and popular) way to represent data sets, especially those with large amounts of data or which do not lend themselves well to individual symbols. In a bar graph, the number of items in a data category is represented by the height or length of bars.

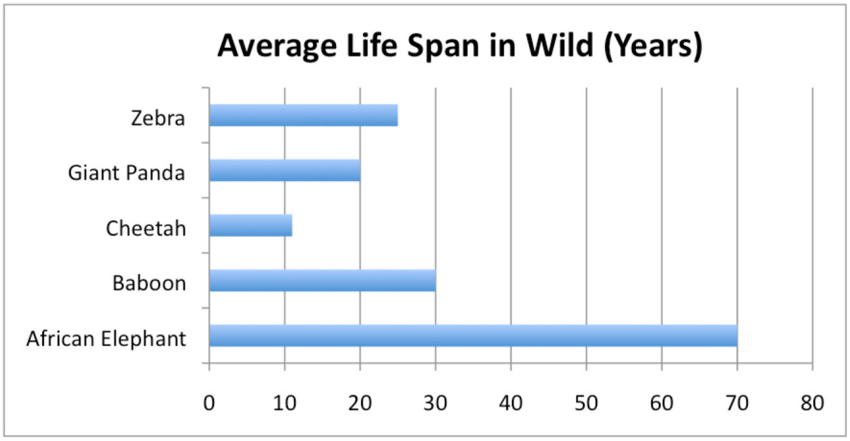
As when reading pictographs, paying attention to the scale is essential—small differences in the height of two bars can sometimes represent thousands of dollars, for example!

Let's look at one example. Here is some information about the average life span of five animals in the wild, presented in a table.

Animal	Average Life Span in the Wild (Years)
Zebra	25
Giant Panda	20
Cheetah	11
Baboon	30
African Elephant	70

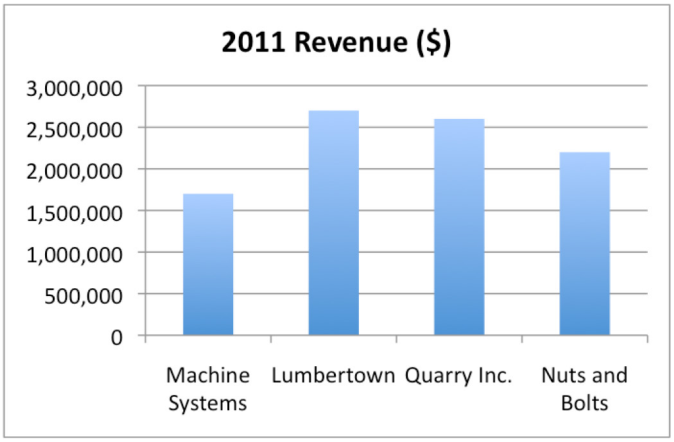
Source: National Geographic, accessed July 2011

This data is fine in a table, but presenting it as a bar graph helps the viewer compare the different life spans more easily. Look at the bar graph below. In this example, the animals are listed on the left side of the graph (also called the **y-axis**), and the life span in years is listed on the bottom (the **x-axis**). The graph shows the information by the length of the bar associated with each animal name.



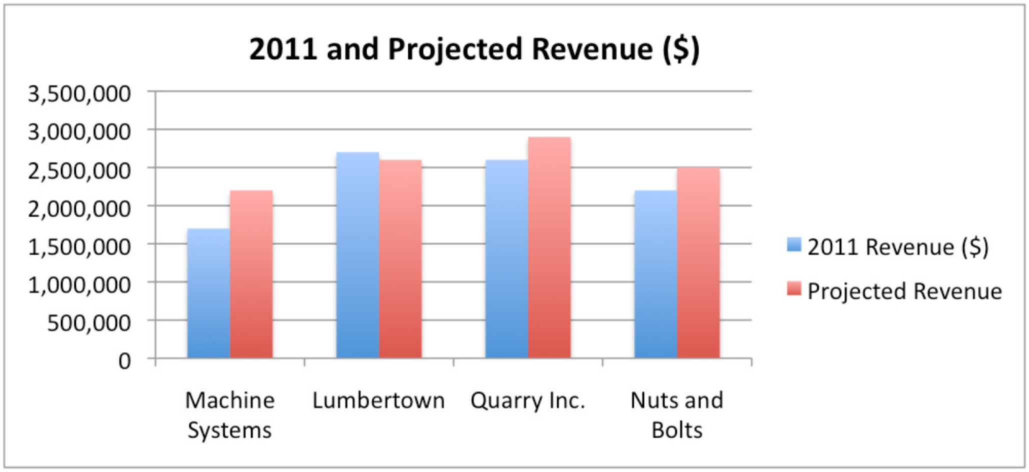
Bar graphs are generally used to compare quantities, not to determine exact quantities, especially when the scale is large, as in the next graph.

The bar graph below shows total revenue for four fictional companies in 2011. Notice that the scale, on the *y*-axis, is very large—each horizontal line represents an increase of $500,000. For this reason, it is difficult to tell exactly how much money each company made in 2011. However, comparing the bars is straightforward—glancing at the data, you can tell that Lumbertown earned the most (a little over $2,500,000), while Machine Systems earned the least (about $1 million less, at just over $1,500,000).



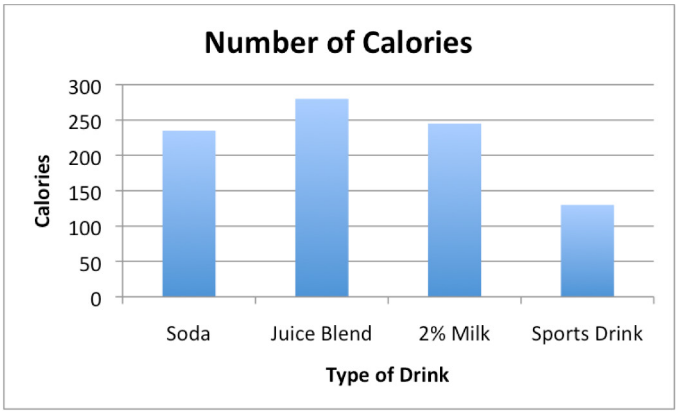
You can also use bar graphs to showcase multiple pieces of information about a specific situation. For example, let's show the next year's projected revenue for each company on the graph that you just looked at. You can leave the existing bars in the graph and just add 4 more.

The blue columns remain, but now they are accompanied by four new red columns that represent the projected revenue for these companies. Again, this data could be expressed in a table—with a bar graph, you gain ease of quick comparison, but lose the detail of the exact values. Looking at this graph tells you that while Lumbertown has the highest revenue for 2011, it is projected to decrease. Conversely, Machine Systems is projected to increase its revenue. Seeing data visually can help you understand the story that the data is telling about a situation.



Example

Problem **Use the graph to list the drinks from the most number of calories to the least number of calories (serving size: 16 oz).**

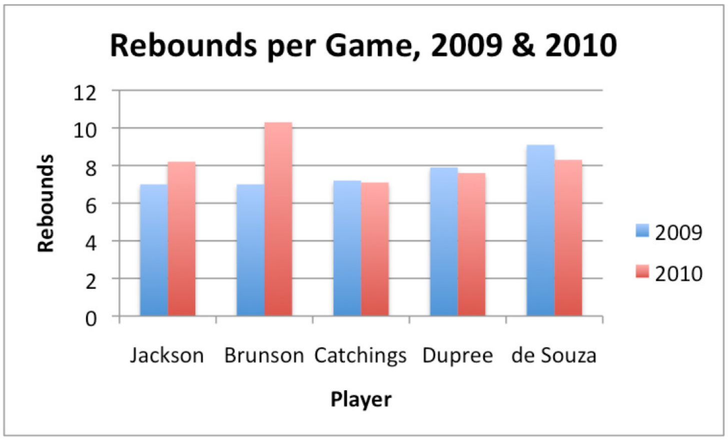


	The y-axis shows total calories, and the x-axis shows the drink. The taller the bar, the more calories the drink has.
Juice Blend (≈275)	The juice blend contains over 250 calories, so it has the most calories per serving.
2% Milk (≈245) Soda (≈230)	Soda and 2% milk are both between 200 and 250 calories, but the bar for 2% milk is taller, so it must contain more calories.
Sports Drink (≈125)	Sports drink has the shortest bar; it contains about 125 calories.

Answer From most to least number of calories per serving:
Juice Blend, 2% Milk, Soda, Sports Drink

Example

Problem **Based on the graph below, which player's rebounding increased the most from 2009 to 2010?**



Source: WNBA.com, accessed July 2011

The *y*-axis shows rebounds per game, and the *x*-axis shows the player's name. A taller bar represents more rebounds per game by the player.

This graph shows two sets of data—one for 2009, in blue, and one for 2010, in red. To compare the data from one year to the next, compare the heights of the two bars for each player.

Jackson and Brunson

Two players had higher rebound averages in 2010 than they did in 2009. This is indicated by the red bar being taller than the blue bar. The other players' red bars are shorter, so their rebounds decreased.

Comparing the sizes of the increases, you can tell that Brunson increased her per game rebounding *more* than Jackson did.

Answer The player whose rebounding increased the most from 2009 to 2010 was Brunson.

Line Graphs

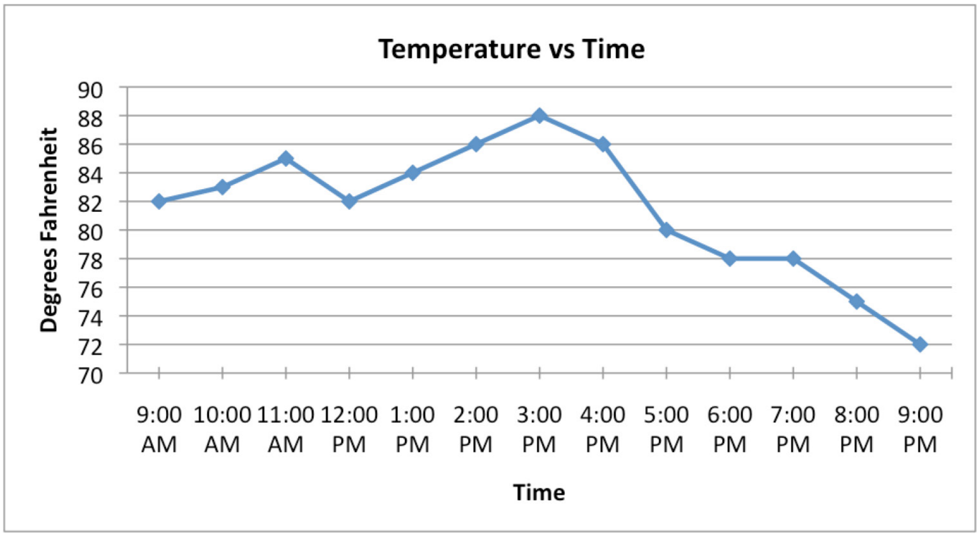
Objective 3

A **line graphs** are often used to relate data over a period of time. In a line graph, the data is shown as individual points on a grid; a trend line connects all data points.

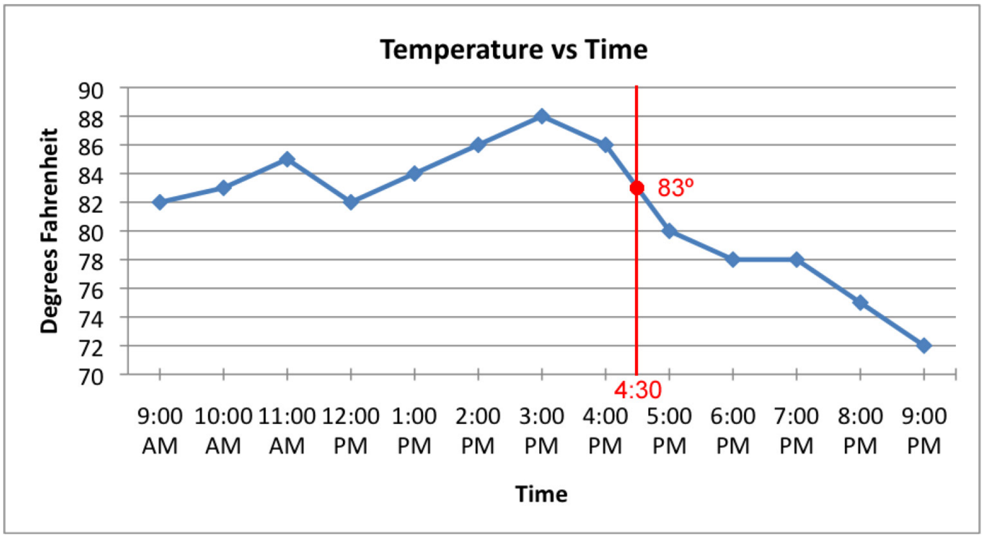
A typical use of a line graph involves the mapping of temperature over time. One example is provided below. Look at how the temperature is mapped on the *y*-axis and the time is mapped on the *x*-axis.

Each point on the grid shows a specific relationship between the temperature and the time. At 9:00 AM, the temperature was 82º. It rose to 83º at 10:00 AM, and then again to 85º at 11:00 AM. It cooled off a bit by noon, as the temperature fell to 82º. What happened the rest of the day?

The data on this graph shows that the temperature peaked at 88º at 3:00 PM. By 9:00 PM that evening, it was down to 72º.



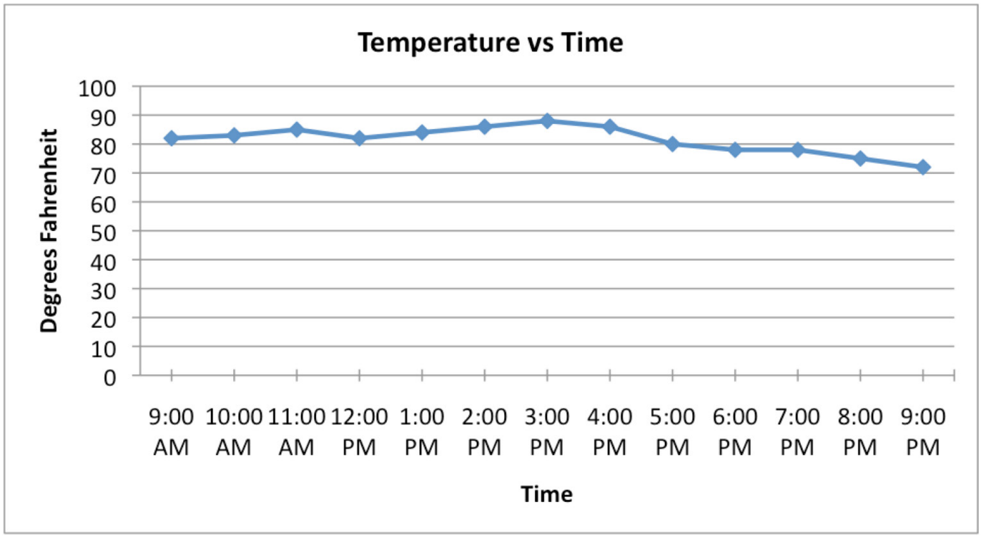
The line segments connecting each data point are important to consider, too. While this graph only provides data points for each hour, you could track the temperature each minute (or second!) if you wanted. The line segments connecting the data points indicate that the temperature vs. time relationship is continuous—it can be read at any point. The line segments also provide an estimate for what the temperature would be if the temperature were measured at any point between two existing readings. For example, if you wanted to estimate the temperature at 4:30, you could find 4:30 on the *x*-axis and draw a vertical line that passes through the trend line; the place where it intersects the graph will be the temperature estimate at that time.



Note that this is just an estimate based on the data—there are many different possible temperature fluctuations between 4:00 PM and 5:00 PM. For example, the temperature could have held steady at 86º for most of the hour, and then dropped sharply to 80º just before 5:00 PM. Alternatively, the temperature could have dropped to 76º due to a sudden storm, and then climbed back up to 80º once the storm passed. In either of these cases, our estimate of 83º would be incorrect! Based on the data, though, 83º seems like a reasonable prediction for 4:30 PM.

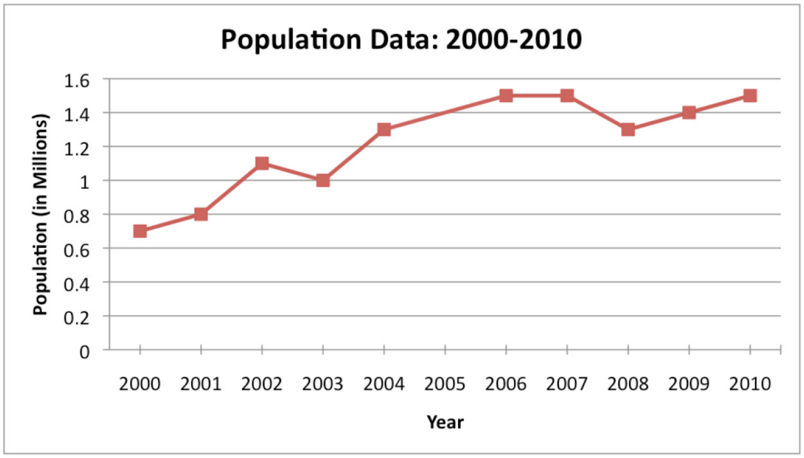
Finally, a quick word about the scale in this graph. Look at the *y*-axis—the vertical line where the Degrees Fahrenheit are listed. Notice that it starts at 70º, and then increases in increments of 2º each time. Since the scale is small and the graph begins at 70º, the temperature data looks pretty volatile—like the temperature went from being warm to hot to very cold! Look at the same data set when plotted on a line graph that begins at 0º and has a scale of 10º. The peaks and valleys are not as apparent!

As you can see, changing the scale of the graph can affect how a viewer perceives the data within the graph.

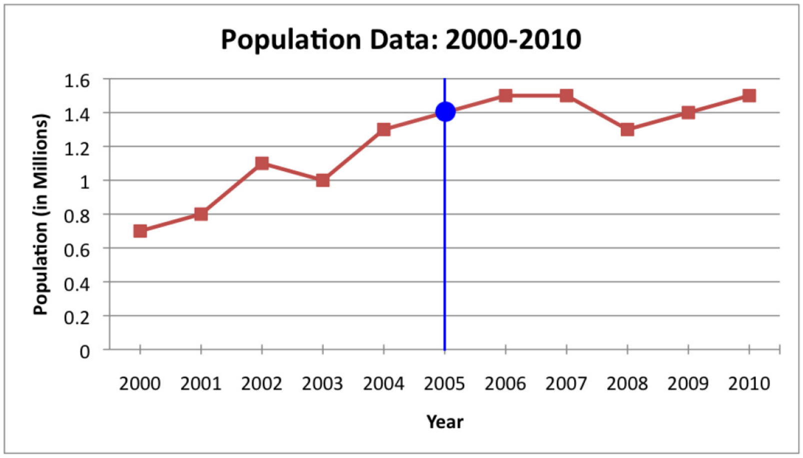


Example

Problem **Population data for a fictional city is given below. Estimate the city's population in 2005.**



Look at the line graph. The population starts at about 0.7 million (or 700,000) in 2000, rises to 0.8 million in 2001, and then again to 1.1 million in 2002. To find the population in 2005, find 2005 on the *x*-axis and draw a vertical line that intersects the trend line.



The lines intersect at 1.4, so 1.4 million (or 1,400,000) would be a good estimate.

Answer The population in 2005 was about 1.4 million.

Summary

Data is mathematical information. Mathematical data is often recorded in tables to organize, or spreadsheets to organize and sort. Graphs can help you see the data visually, which can help you to better understand the data. A pictograph is a graph that uses symbols to represent data. Bar graphs show the frequency of categorical data, using bars instead of symbols. By contrast, line graphs are usually used to relate continuous data over a period of time.

1.6.1 Self Check Solutions

Self Check A

Which table accurately represents the data shown in the pictograph below?

Employee	**Hourly wage**
Wayne	\$ \$ \$ \$ \$
Sarah	\$ \$ \$ \$ \$ \$ \$
Leigh	\$ \$ \$ \$

\$ = \$4

D)

Employee	**Hourly wage**
Wayne	\$20
Sarah	\$28
Leigh	\$16

Each \$ symbol represents \$4, so if you multiply the number of \$s in a row by \$4, you will find that Wayne earns \$20, Sarah earns \$28, and Leigh earns \$16.

1.6.2 Measures of Center

Learning Objective(s)

1 Find the mean, median, and mode of a set of numbers.
2 Find the ramge of a set of numbers.
3 Read and interpret data from box-and-whisker plots.
4 Solve application problems that require the calculation of the mean, median, or range.

Introduction

When trying to describe data, **mean**, **median**, and **mode** are important tools. These measures of center all use data points to approximate and understand a "middle value" or "average" of a given data set. Two more measures of interest are the **range** and **midrange**, which use the greatest and least values of the data set to help describe the spread of the data.

So why would you need to find out the middle of a data set? And why do you need three measures instead of just one? Let's look closely at these measures of center and learn how they can help us understand sets of data.

Mean, Median, and Mode

Objective 1

"Mean" is a mathematical term for "average" which you may already know. Also referred to as the "arithmetic mean," it is found by adding together all the data values in a set and dividing that sum by the number of data items.

You can often find the average of two familiar numbers, such as 10 and 16, in your head without much calculation. What number lies half way between them? 13. A mathematical way to solve this, though, is to add 10 and 16 (which gives you 26) and then divide by 2 (since there are 2 numbers in the data set). 26 ÷ 2 = 13

Knowing the process helps when you need to find the mean of more than two numbers. For example, if you are asked to find the mean of the numbers 2, 5, 3, 4, 5, and 5, first find the sum: 2 + 5 + 3 + 4 + 5 + 5 = 24. Then, divide this sum by the number of numbers in the set, which is 6. So the mean of the data is 24 ÷ 6, or 4.

In the previous data set, notice that the mean was 4 and that the set also contained a value of 4. This does not always occur. Look at the example that follows—the mean is 18, although 18 is not in the data set at all.

Example		
Problem	**Find the mean of the set: 4, 7, 28, 33.**	
	$4 + 7 + 28 + 33 = 72$	Add all the values.
	$\frac{72}{4} = 18$	Divide by 4, the number of values.
Answer	The mean is 18.	

Next, let’s look at the “median.” The median is the middle value when the data is ordered. If there are two middle values, the median is the average of the two middle values.

To calculate the median, you first put your data into numerical order from least to greatest. Then identify the middle value(s).

For example, let’s look at the following values: 4, 5, 1, 3, 2, 7, 6. To find the median of this set, you would put it in order from least to greatest.

1 2 3 4 5 6 7

Then identify the middle value. There are three values to the right of four and three values to the left of four. The middle value is 4, so 4 is the median.

If there is an even number of data items, however, the median will be the mean of the two center data items.

Example		
Problem	**Find the median of the set: 2, 6, 4, 3, 6, 7.**	
	2, 3, 4, 6, 6, 7	Arrange the values from least to greatest.
	2, 3, 4, 6, 6, 7	The set has 2 middle values. So take the mean (average) of the two values.
	$\frac{4+6}{2}=\frac{10}{2}=5$	
Answer	The median is 5.	

Finally, let’s consider the “mode.” The mode is found by looking for the data value that appears most often. If there is a two-way tie for most often, the data is bimodal and you use both data values as the modes. Sometimes there is no mode. This happens when there is no data value that occurs most often. In our example data set (2, 3, 4, 6, 6, 7), the number 5 appears 2 times and all other numbers appear once, so the mode is 6.

Example		
Problem	**Find the mode of the set: 12, 4, 12, 5, 5, 8, 12, 0, 1, 12.**	
	0, 1, 4, 5, 5, 8, 12, 12, 12, 12	Arrange the values from least to greatest

		(although this is not a necessary step, it sometimes helps to find the mode if the numbers are arranged in ascending order).
	0, 1, 4, 5, 5, 8, 12, 12, 12, 12	Find the value that occurs most often.
Answer	The mode is 12.	

Let’s look at an example with some relevant data.

Example		
Problem	**Carlos received the following scores on his mathematics exams: 84, 92, 74, 98, and 82. Find the mean, median, and mode of his scores.**	
	$\frac{84+92+74+98+82}{5}$	To find the mean, add all the tests scores together and divide by the number of tests.
	$\frac{430}{5}=86$	The mean is 86.
	74, 82, 84, 92, 98	To find the median, order the test scores from least to greatest.
	84	There are five scores, so the middle test score is the third in the ordered list. This is the median.
	74, 82, 84, 92, 98	Since each number appears exactly one time, there is no mode.
Answer	The mean is 86. The median is 84. There is no mode.	

What can be learned from the mean, median, and mode of Carlos’ test scores? Notice that these values are not the same.

Both the mean and the median give us a picture of how Carlos is doing. Looking at these measures, you notice that the middle of the data set is in the mid-80s: the mean value is 86, and the median value is 84. That’s all you are really after when using median and mean—finding the center, or middle, of the data. Notice, also, that there is no mode, since Carlos did not score the same on two tests. In the case of test taking, the mode is often meaningless—unless there are a lot of 0s, which could mean that the student didn't do his homework, or really doesn't know what's going on!

Self Check A

During a seven-day period in July, a meteorologist recorded that the median daily high temperature was 91º.

Which of the following are true statements? There may be more than one answer.

i) The high temperature was exactly 91º on each of the seven days.
ii) The high temperature was never lower than 92º.
iii) Half the high temperatures were above 91º and half were below 91º.

Range

Objective **2**

There are other useful measures other than mean, median, and mode to help you analyze a data set. When looking at data, you often want to understand the spread of the data: the gap between the greatest number and the least number. This is the range of the data. To find the range, subtract the least value of the data set from the greatest value. For example, in the data of 2, 5, 3, 4, 5, and 5, the least value is 2 and the greatest value is 5, so the range is 5 – 2, or 3.

Let's look at a couple of examples.

Example		
Problem	**Find the range for the following set of numbers: 2, 4, 7, 10, 14, 35.**	
	range: 35 – 2 = 33	Subtract the least value from the greatest value to find the range.
Answer	The range is 33.	

Example		
Problem	**Find the range and midrange for the following set of numbers: 62, 88, 20, 145, 37, 105, 93, 22.**	
	least number: 20 greatest number: 145	Since the data set isn't ordered from least to greatest, identify the least and greatest numbers.
	range: 145 – 20 = 125	Subtract the least value from the greatest value to find the range.
Answer	The range is 125.	

Box-and Whisker-Plots

Objective 3

Another type of graph that you might see is called a **box-and-whisker plot**. These graphs provide a visual way of understanding both the range and the middle of a data set.

Here is a sample set of 15 numbers to get us started.

12, 5, 18, 20, 11, 9, 3, 5, 7, 18, 12, 15, 6, 10, 11

Creating a box-and-whisker plot from this data requires finding the median of the set. To do this, order the data.

3, 5, 5, 6, 7, 9, 10, 11, 11, 12, 12, 15, 18, 18, 20

This data set has 15 numbers, so the median will be the 8th number in the set: 11.

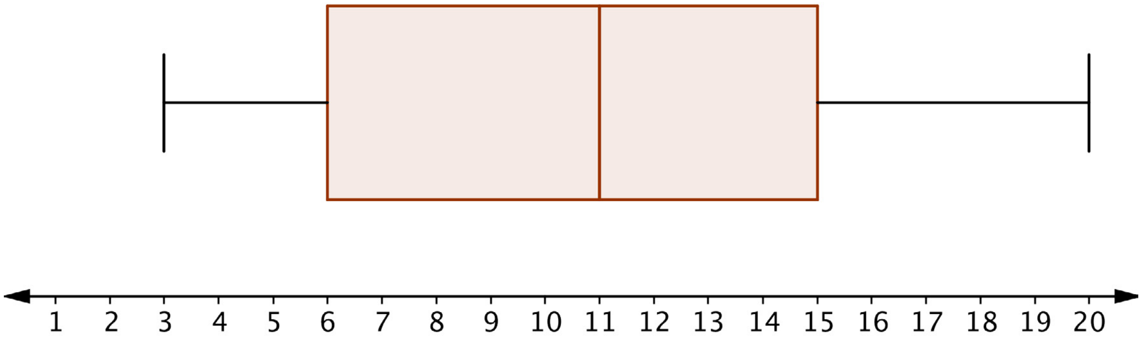
Finding the median of the data set essentially divides it into two—a set of numbers below the median, and a set of numbers above the median. A box-and-whisker plot requires you to find the median of these numbers as well!

Lower set: 3, 5, 5, 6, 7, 9, 10. Median: 6
Upper set: 11, 12, 12, 15, 18, 18, 20. Median: 15

So, the median of the set is 11, the median of the lower half is 6, and the median of the upper half is 15.

3, 5, 5, **6**, 7, 9, 10, **11**, 11, 12, 12, **15**, 18, 18, 20

A box-and-whisker plot for this data set is shown here. Do you see any similarities between the numbers above and the location of the box?



Notice that one “box” (rectangle section) begins at 6 (the median of the lower set) and goes to 11 (the median of the full set), and the other box goes from 11 to 15 (the median of the upper set).

The “whiskers” are the line segments on either end. One stretches from 3 (the least value in the set) to 6, and the other goes from 15 to 20 (the greatest value in the set).

The box-and-whisker plot essentially divides the data set into four sections (or **quartiles**): whisker, box, box, whisker. The size of the quartiles may be different, but *the number of data points* in each quartile is the same.

You can use a box-and-whisker plot to analyze how data in a set are distributed. You can also the box-and-whisker plots to compare two sets of data.

Using Measures of Center to Solve Problems

Objective **4**

Using mean, median, and mode, as well as range and midrange can help you to analyze situations and make decisions about things like which is the best, whether it is more reliable to walk or take the bus to school, or even whether to buy or sell a particular stock on the stock market.

Let’s look at an example of how analyzing data using measures of center can help you to make choices (and even get to school on time!).

FIX THIS

Example

Problem **Below, is a table listing the amount of time it took Marta to get to school by either riding the bus or by walking, on 12 separate days. The times are door to door, meaning the clock starts when she leaves her front door and ends when she enters school.**

Bus	**Walking**
16 min	23 min
14 min	19 min
15 min	21 min
14 min	21 min
28 min	22 min
15 min	20 min

- **Which method of travel is faster?**
- **If she leaves her house 25 minutes before school starts, should she walk or take the bus to be assured of arriving at school on time?**

$$bus: \frac{16+14+15+14+28+15}{6} = 17$$

$$walking: \frac{23+19+21+21+22+20}{6} = 21$$

Determine the mean of each travel method.

bus: 28 –14 = 14
walking: 23 – 19 = 4

Determine the range of each travel method.

bus: 14, 14, 15, 15, 16, 28

$$\frac{15+15}{2} = 15$$

walking: 19, 20, 21, 21, 22, 23

$$\frac{21+21}{2} = 21$$

Determine the median for each travel method.

bus: 14, 15
walking: 21

Determine the mode for each travel method.

	Bus	**Walking**
Mean	17	21
Median	15	21
Mode	14, 15	21
Range	14	4

Answer Looking at the mean, median, and the mode, the faster way to school is riding the bus. The data also shows that the bus is the most variable, with a range of 14, so if Marta wants to be *sure* that she gets to school on time, she should walk.

In the previous example, riding the bus is, *on average*, a faster way to school than walking. This is revealed in the mean of each method, which shows that the bus is 3 minutes faster. The mode and median show an even greater time advantage to riding the bus, and this is due to the one time high value of 28 minutes that isn't really accounted for in these measures. Notice the difference in the mean (17) and the median (15) for riding the bus, which lets you know there is some variance in the data.

As far as getting to school on time is concerned, while not being the fastest method, walking is the most reliable, with consistent values for mean, median, and mode, and a low value for the range, meaning that the spread of the data is very small.

Summary

Measures of center help you to analyze numerical data. The mean (or arithmetic mean) is often called the "average", and is found by dividing the sum of the data items by the number of items. The median is the number that is in the middle when the data is ordered from least to greatest, and the mode is the number that appears most often. The range is the difference between the least number and the greatest number. Box-and-whisker plots use the median and range to help you to interpret the data visually.

8.2.1 Self Check Solutions

Self Check A

During a seven-day period in July, a meteorologist recorded that the median daily high temperature was 91º.

Which of the following are true statements?

i) The high temperature was exactly 91º on each of the seven days.
ii) The high temperature was never lower than 92º.
iii) Half the high temperatures were above 91º and half were below 91º.

iii only
Half the high temperatures were above 91º and half were below 91º since the median will always represent the value where half the data is higher and half the data is lower.

1.7 Areas and Perimeters of Quadrilaterals

Learning Objective(s)

1 Calculate the perimeter of a polygon

2 Calculate the area of trapezoids and parallelograms

Introduction

We started exploring perimeter and area in earlier sections. In this section, we will explore perimeter in general, and look at the area of other quadrilateral (4 sided) figures.

Perimeter

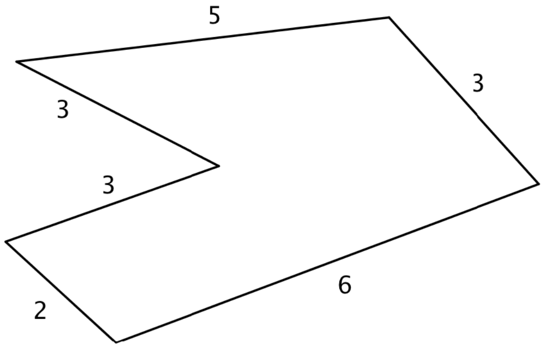
Objective 1

The perimeter of a two-dimensional shape is the distance around the shape. You can think of wrapping a string around a rectangle. The length of this string would be the perimeter of the rectangle. Or walking around the outside of a park, you walk the distance of the park's perimeter. Some people find it useful to think "peRIMeter" because the edge of an object is its rim and peRIMeter has the word "rim" in it.

If the shape is a **polygon**, a shape with many sides, then you can add up all the lengths of the sides to find the perimeter. Be careful to make sure that all the lengths are measured in the same units. You measure perimeter in linear units, which is one dimensional. Examples of units of measure for length are inches, centimeters, or feet.

Example

Problem **Find the perimeter of the given figure. All measurements indicated are inches.**



	$P = 5 + 3 + 6 + 2 + 3 + 3$	Since all the sides are measured in inches, just add the lengths of all six sides to get the perimeter.
Answer	*P* = 22 inches	Remember to include units.

This means that a tightly wrapped string running the entire distance around the polygon would measure 22 inches long.

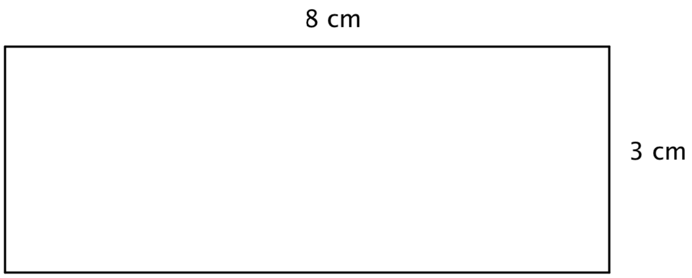
Example

Problem	**Find the perimeter of a triangle with sides measuring 6 cm, 8 cm, and 12 cm.**	
	$P = 6 + 8 + 12$	Since all the sides are measured in centimeters, just add the lengths of all three sides to get the perimeter.
Answer	P = 26 centimeters	

Sometimes, you need to use what you know about a polygon in order to find the perimeter. Let's look at the rectangle in the next example.

Example

Problem **A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the perimeter.**



	$P = 3 + 3 + 8 + 8$	Since this is a rectangle, the opposite sides have the same lengths, 3 cm. and 8 cm. Add up the lengths of all four sides to find the perimeter.
Answer	P = 22 cm	

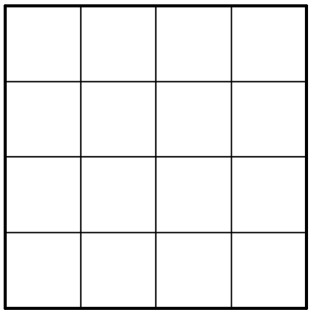
Notice that the perimeter of a rectangle always has two pairs of equal length sides. In the above example you could have also written $P = 2(3) + 2(8) = 6 + 16 = 22$ cm. The formula for the perimeter of a rectangle is often written as $P = 2l + 2w$, where l is the length of the rectangle and w is the width of the rectangle.

Area of Parallelograms

Objective **2**

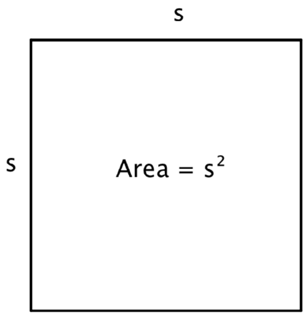
The area of a two-dimensional figure describes the amount of surface the shape covers. You measure area in square units of a fixed size. Examples of square units of measure are square inches, square centimeters, or square miles. When finding the area of a polygon, you count how many squares of a certain size will cover the region inside the polygon.

Let's look at a 4 x 4 square.



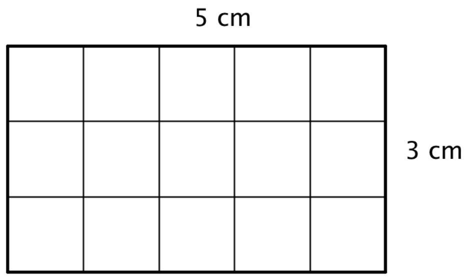
You can count that there are 16 squares, so the area is 16 square units. Counting out 16 squares doesn't take too long, but what about finding the area if this is a larger square or the units are smaller? It could take a long time to count.

Fortunately, you can use multiplication. Since there are 4 rows of 4 squares, you can multiply 4 • 4 to get 16 squares! And this can be generalized to a formula for finding the area of a square with any length, s: *Area* = $s \cdot s = s^2$.

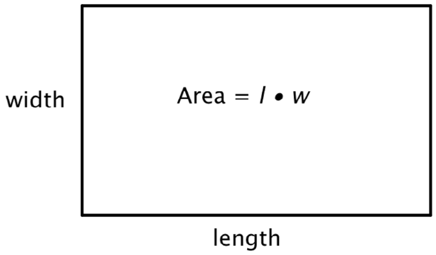


You can write "in^2" for square inches and "ft^2" for square feet.

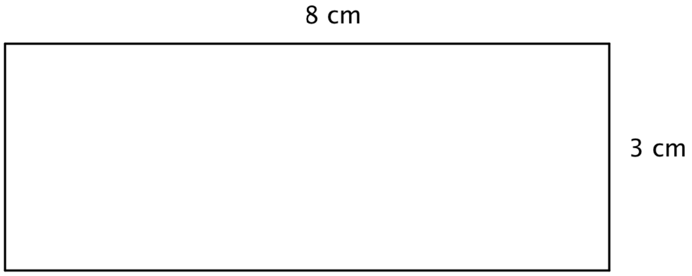
To help you find the area of the many different categories of polygons, mathematicians have developed formulas. These formulas help you find the measurement more quickly than by simply counting. The formulas you are going to look at are all developed from the understanding that you are counting the number of square units *inside* the polygon. Let's look at a rectangle.



You can count the squares individually, but it is much easier to multiply 3 times 5 to find the number more quickly. And, more generally, the area of any rectangle can be found by multiplying *length* times *width.*

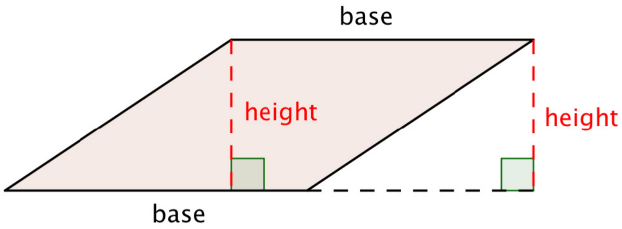


Example

Problem	A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the area. 	
	$A = l \bullet w$	Start with the formula for the area of a rectangle, which multiplies the length times the width.
	$A = 8 \bullet 3$	Substitute 8 for the length and 3 for the width.
Answer	$A = 24\text{ cm}^2$	Be sure to include the units, in this case square cm.

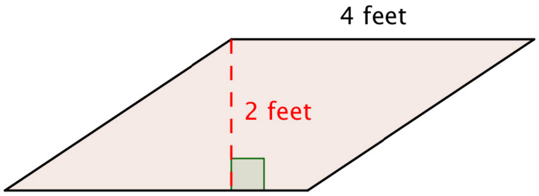
It would take 24 squares, each measuring 1 cm on a side, to cover this rectangle.

The formula for the area of any parallelogram (remember, a rectangle is a type of parallelogram) is the same as that of a rectangle: *Area* = *l* • *w*. Notice in a rectangle, the length and the width are perpendicular. This should also be true for all parallelograms. *Base* (*b*) for the length (of the base), and *height* (*h*) for the width of the line perpendicular to the base is often used. So the formula for a parallelogram is generally written, *A* = *b* • *h.*



Example

Problem **Find the area of the parallelogram.**



$A = b \cdot h$	Start with the formula for the area of a parallelogram: *Area* = *base* • *height*.
$A = 4 \cdot 2$	Substitute the values into the formula.
$A = 8$	Multiply.

Answer The area of the parallelogram is 8 ft^2.

Area of Trapezoids

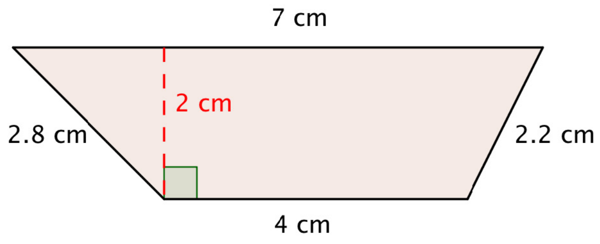
Objective **2**

A trapezoid is a quadrilateral where two sides are parallel, but the other two sides are not. To find the area of a trapezoid, take the average length of the two parallel bases and multiply that length by the height: $A = \frac{(b_1 + b_2)}{2}h$.

An example is provided below. Notice that the height of a trapezoid will always be perpendicular to the bases (just like when you find the height of a parallelogram).

Example

Problem **Find the area of the trapezoid.**



$$A=\frac{(b_1+b_2)}{2}h$$

Start with the formula for the area of a trapezoid.

$$A=\frac{(4+7)}{2}\bullet 2$$

Substitute 4 and 7 for the bases and 2 for the height, and find *A*.

$$A=\frac{11}{2}\bullet 2$$

$$A=11$$

Answer The area of the trapezoid is 11 cm^2.

Area Formulas

Use the following formulas to find the areas of different shapes.

square: $A=s^2$

rectangle: $A=l\bullet w$

parallelogram: $A=b\bullet h$

trapezoid: $A=\frac{(b_1+b_2)}{2}h$

1.8.1 Properties and Laws of Whole Numbers

Learning Objective(s)

1 Simplify by using the addition property of 0.
2 Simplify by using the multiplication property of 1.
3 Identify and use the commutative law of addition.
4 Identify and use the commutative law of multiplication.
5 Identify and use the associative law of addition.
6 Identify and use the associative law of multiplication.

Introduction

Mathematics often involves simplifying numerical expressions. When doing so, you can use laws and properties that apply to particular operations. The multiplication property of 1 states that any number multiplied by 1 equals the same number, and the addition property of zero states that any number added to zero is the same number.

Two important laws are the commutative laws, which state that the order in which you add two numbers or multiply two numbers does not affect the answer. You can remember this because if you *commute* to work you go the same distance driving to work and driving home as you do driving home and driving to work. `You can move numbers around in addition and multiplication expressions because the order in these expressions does not matter.

You will also learn how to simplify addition and multiplication expressions using the associative laws. As with the commutative laws, there are associative laws for addition and multiplication. Just like people may associate with people in different groups, a number may *associate* with other numbers in one group or another. The associative laws allow you to place numbers in different groups using parentheses.

Addition and Multiplication Properties of 0 and 1

Objective 1, 2

The addition property of 0 states that for any number being added to 0, the sum equals that number. Remember that you do not end up with zero as an answer – that only happens when you multiply. Your answer is simply the same as your original number.

Example

Problem **62 + 0 = ?**

62 + 0 = 62	Adding zero to 62 does not add any quantity to the sum, so the number remains 62.

Answer 62 + 0 = 62

Self Check A

112 + 0 = ?

According to the **multiplication property of 1**, the product of 1 and any number results in that number. The answer is simply identical to the original number.

Example

Problem **2,500 • 1 = ?**

2,500 • 1 = 2,500	Multiplying 2,500 by 1 yields the same number.

Answer 2,500 • 1 = 2,500

Self Check B

72,540 x 1 = ?

The Commutative Law of Addition

Objective 3

The **commutative law of addition** states that you can change the position of numbers in an addition expression without changing the sum. For example, 3 + 2 is the same as 2 + 3.

3 + 2 = 5
2 + 3 = 5

You likely encounter daily routines in which the order can be switched. For example, when you get ready for work in the morning, putting on your left glove and right glove is commutative. You could put the right glove on before the left glove, or the left glove on before the right glove. Likewise, brushing your teeth and combing your hair is commutative, because it does not matter which one you do first.

Remember that this law only applies to addition, and not subtraction. For example:

8 – 2 is not the same as 2 – 8.

Below, you will find examples of expressions that have been changed with the commutative law. Note that expressions involving subtraction cannot be changed.

Original Expression	Rewritten Expression
4 + 5	5 + 4
6 + 728	728 + 6
9 + 4 + 1	9 + 1 + 4
9 − 1	cannot be changed
72 − 10	cannot be changed
128 − 100	cannot be changed

You also will likely encounter real life routines that are not commutative. When preparing to go to work, putting on our clothes has to occur before putting on a coat. Likewise, getting in the car has to occur before putting the key in the ignition. In a store, you would need to pick up the items you are buying before proceeding to the cash register for checkout.

Example

Problem **Write the expression 10 + 25 in a different way, using the commutative law of addition, and show that both expressions result in the same answer.**

10 + 25 = 35	Solving the problem yields an answer of 35.
25 + 10	Using the commutative property, you can switch the 10 and the 25 so that they are in different positions.
25 + 10 = 35	Adding 25 to 10 in this new order also yields 35.

Answer 10 + 25 = 35 and 25 + 10 = 35

Self Check C

Rewrite 15 + 12 = 27 in a different way, using the commutative law of addition.

The Commutative Law of Multiplication

Objective **4**

Multiplication also has a commutative law. The **commutative law of multiplication** states that when two or more numbers are being multiplied, their order can be changed without affecting the answer. In the example below, note that 5 multiplied by 4 yields the same result as 4 multiplied by 5. In both cases, the answer is 20.

5 • 4 = 20
4 • 5 = 20

This example shows how numbers can be switched in a multiplication expression.

Example

Problem **Write the expression 30 • 50 in a different way, using the commutative law of multiplication, and show that both expressions result in the same answer.**

30 • 50 = 1,500	Solving the problem yields an answer of 1,500.
50 • 30	Using the commutative law, you can switch the 30 and the 50 so that they are in different positions.
50 • 30 = 1,500	Multiplying 50 and 30 also yields 1,500.

Answer 50 • 30 and 30 • 50 = 1,500

Keep in mind that when you are using the commutative law, only the order is affected. The grouping remains unchanged.

Self Check D

Problem: Rewrite 52 • 46 in a different way, using the commutative law of multiplication.

The Associative Law of Addition

Objective **5**

Below are two ways of simplifying and solving an addition problem. Note that you can add numbers in any order. In the first example, 4 is added to 5 to make 9.

$$4 + 5 + 6 = 9 + 6 = 15$$

Here, the same problem is solved, but this time, 5 is added to 6 to make 11. Note that solving it this way yields the same answer.

$$4 + 5 + 6 = 4 + 11 = 15$$

The **associative law of addition** states that numbers in an addition expression can be regrouped using parentheses. You can remember the meaning of the associative law by remembering that when you *associate* with family members, friends, and co-workers, you end up forming groups with them. In the following expression, parentheses are used to group numbers together so that you know what to add first. Note that when parentheses are present, any numbers within parentheses are numbers you will add first. The expression can be re-written with different groups using the associative law.

$$(4 + 5) + 6 = 9 + 6 = 15$$

$$4 + (5 + 6) = 4 + 11 = 15$$

Here, it is clear that the parentheses do not affect the final answer, the answer is the same regardless of where the parentheses are.

Example

Problem **Rewrite (5 + 8) + 3 using the associative law of addition. Show that the rewritten expression yields the same answer.**

(5 + 8) + 3 = 13 + 3 = 16	The original expression yields an answer of 16.
5 + (8 + 3) = 5 + 11 = 16	Grouping 8 and 3 instead of 5 and 8 results in the same answer of 16.

Answer (5 + 8) + 3 = 16 and 5 + (8 + 3) = 16

When rewriting an expression using the associative law, remember that you are regrouping the numbers and not reversing the order, as in the commutative law.

Self Check E

Rewrite 10 + (5 + 6) using the associative property.

The Associative Law of Multiplication

Objective **6**

Multiplication has an associative law that works exactly the same as the one for addition. The **associative law of multiplication** states that numbers in a multiplication expression can be regrouped using parentheses. The following expression can be rewritten in a different way using the associative law.

$$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4).$$

Here, it is clear that the parentheses do not affect the final answer, the answer is the same regardless of where the parentheses are.

Example

Problem **Rewrite (10 • 200) • 24 using the associative law of multiplication, and show that the rewritten expression yields the same answer.**

(10 • 200) • 24 = 2000 • 24 = 48,000	The original expression yields an answer of 48,000.
10 • (200 • 24) = 10 • 4800 = 48,000	Grouping 200 and 24 instead of 10 and 200 results in the same answer of 48,000.

Answer (10 • 200) • 24 = 48,000 and 10 • (200 • 24) = 48,000

When rewriting an expression using the associative law, remember that you are regrouping the numbers and not changing the order. Changing the order uses the commutative law.

Self Check F
Rewrite 8 • (7 • 6) using the associative property.

Commutative or Associative?

When an expression is being rewritten, you can tell whether it is being rewritten using the commutative or associative laws based on whether the order of the numbers change or the numbers are being regrouped using parentheses.

If an expression is rewritten so that the order of the numbers is changed, the commutative law is being used.

Example

Problem **10 • 2 = 20 is rewritten as 2 • 10 = 20. Was this expression rewritten using the commutative law or the associative law?**

$$\begin{tikzcd} 10 \arrow[drr] & \bullet & 2 \arrow[dll] & = & 20 \\ 2 & \bullet & 10 & = & 20 \end{tikzcd}$$

Rewriting the expression involves switching the order of the numbers. Therefore, the commutative law is being used.

Answer The commutative law is being used to rewrite the expression.

Remember that when you associate with friends and family, typically you are *grouping* yourself with other people. So, if numbers in an expression are regrouped using parentheses and the order of numbers remains the same, then the associative law is being used.

Example

Problem **2 • (4 • 6) = 48 is rewritten as (2 • 4) • 6 = 48. Was this expression rewritten using the commutative law or the associative law?**

$$2 \bullet (\undergroup{4 \bullet 6}) = 48$$

$$\swarrow$$

$$(\overgroup{2 \bullet 4}) \bullet 6 = 48$$

Regrouping using parentheses does not change the order of the numbers. Therefore, the associative law is being used.

Answer The associative law is being used to rewrite the expression.

Self Check G

12 • (6 • 2) = 144 is rewritten as (12 • 6) • 2 = 144. Was this expression rewritten using the commutative law or the associative law?

If there are absolutely no parentheses in a problem that is being rewritten, you can assume the associative law is not being used.

Self Check H

17 • 3 = 51 is rewritten as 3 • 17 = 51. Was this expression rewritten using the commutative law or associative law?

Using the Associative and Commutative Laws

The associative and commutative laws are useful when you have an expression with only addition. Using the commutative law, the numbers can be reordered so that the numbers that are easiest to add are next to each other, and using the associative law, you can group them in any way.

For example, here are some of the ways we can add 6 + 5 + 4 using the associative and commutative laws. Note that the answer is always the same.

(6 + 5) + 4 = 11 + 4 = 15	grouping 6 and 5 to add first
(5 + 6) + 4 = 11 + 4 = 15	reordering 6 and 5
5 + (6 + 4) = 5 + 10 = 15	grouping 6 and 4 to add first

6 + (5 + 4) = 6 + 9 = 15	grouping 5 and 4 to add first
6 + (4 + 5) = 6 + 9 = 15	reordering 4 and 5
(6 + 4) + 5 = 10 + 5 = 15	grouping 6 and 4 to add first

Example

Problem **Write the expression 13 + 28 + 7 a different way to make it easier to simplify. Then simplify.**

13 + 28 + 7	
13 + 7 + 28	Using the commutative property, reorder the numbers 7 and 28 since 13 + 7 is easier to add than 13 + 28.
20 + 28	Using the associative property, group the 13 and 7 together and add them first.
48	Add 20 and 28.

Answer 13 + 28 + 7 = 13 + 7 + 28 = 48

Sometimes the commutative and associative laws can make the problem easy enough to do in your head.

Example

Problem **Jim is buying 8 pears, 7 apples, and 2 oranges. He decided the total number of fruit is 8 + 7 + 2. Use the commutative property to write this expression in a different way. Then find the total.**

8 + 7 + 2	
8 + 2 + 7	Using the commutative property, reorder 2 and 7.
10 + 7	Using the associative property, group the 8 and 2 together and add them first.
17	Add 10 and 7.

Answer 8 + 7 + 2 = 8 + 2 + 7 = 17

This also works when you are multiplying more than two numbers. You can use the commutative and associative laws freely if the expression involves only multiplication.

Example	
Problem	**There are 2 trucks in a garage, and each truck holds 60 boxes. There are 5 laptop computers in each box. Find the number of computers in the garage.**
2 • 60 • 5	In order to find the answer, you need to multiply the number of trucks times the number of boxes in each truck, and, then by the number of computers in each box.
2 • 5 • 60	Using the commutative property, reorder the 5 and the 60. Now you can multiply 2 • 5 first.
10 • 60	Using the associative property, multiply the 2 and the 5, 2 • 5 = 10.
600	Now it's easier to multiply 10 and 60 to get 600.
Answer	There are 600 computers in the garage.

Summary

The addition property of 0 states that for any number being added to zero, the sum is the same number. The multiplication property of 1 states that for any number multiplied by one, that answer is that same number. Zero is called the additive identity, and one is called the multiplicative identity.

When you rewrite an expression by a commutative law, you change the order of the numbers being added or multiplied. When you rewrite an expression using an associative law, you group a different pair of numbers together using parentheses.

You can use the commutative and associative laws to regroup and reorder any number in an expression that involves only addition. You can also use the commutative and associative laws to regroup and reorder any number in an expression that involves only multiplication.

1.8.1 Self Check Solutions

Self Check A

112 + 0 = ?

112

Adding zero to a number does not change a number.

Self Check B

72,540 x 1 = ?

72,540

Multiplying any number by 1 yields the same number, which is in this case 72,540.

Self Check C

Rewrite 15 + 12 = 27 in a different way, using the commutative law of addition.

12 + 15 = 27
The commutative law lets you change the order of the numbers being added.

Self Check D

Problem: Rewrite 52 • 46 in a different way, using the commutative law of multiplication.

46 • 52
The order of numbers is reversed, and the same two numbers are multiplied.

Self Check E

Rewrite 10 + (5 + 6) using the associative property.

(10 + 5) + 6
Here, the numbers are regrouped. Now 10 and 5 are grouped in parentheses instead of 5 and 6.

Self Check F

Rewrite 8 • (7 • 6) using the associative property.

(8 • 7) • 6
Here, the numbers are regrouped. Now 8 and 7 are grouped in parentheses instead of 7 and 6.

Self Check G

12 • (6 • 2) = 144 is rewritten as (12 • 6) • 2 = 144. Was this expression rewritten using the commutative law or the associative law?

associative law
The numbers are being regrouped using parentheses and the order of numbers does not change.

Self Check H

17 • 3 = 51 is rewritten as 3 • 17 = 51. Was this expression rewritten using the commutative law or associative law?

commutative law
The order of numbers is being switched, which shows that the commutative law is being used.

1.8.2 The Distributive Property

Learning Objective(s)

1 Simplify using the distributive property of multiplication over addition.

2 Simplify using the distributive property of multiplication over subtraction.

Introduction

The distributive property of multiplication is a very useful property that lets you simplify expressions in which you are multiplying a number by a sum or difference. The property states that the product of a sum or difference, such as 6(5 – 2), is equal to the sum or difference of the products – in this case, 6(5) – 6(2).

Remember that there are several ways to write multiplication. 3 x 6 = 3(6) = 3 • 6.
3 • (2 + 4) = 3 • 6 = 18.

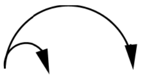
Distributive Property of Multiplication

Objective **1**

The **distributive property of multiplication over addition** can be used when you multiply a number by a sum. For example, suppose you want to multiply 3 by the sum of 10 + 2.

$$3(10 + 2) = ?$$

According to this property, you can add the numbers and then multiply by 3.
3(10 + 2) = 3(12) = 36. Or, you can first multiply each addend by the 3. (This is called **distributing** the 3.) Then, you can add the products.



$$\mathbf{3\,(10 + 2) = 3(10) + 3(2)}$$

The multiplication of 3(10) and 3(2) will each be done before you add.
3(10) + 3(2) = 30 + 6 = 36. Note that the answer is the same as before.

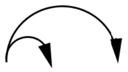
You probably use this property without knowing that you are using it. When a group (let's say 5 of you) order food, and order the same thing (let's say you each order a hamburger for $3 each and a coke for $1 each), you can compute the bill (without tax) in two ways. You can figure out how much each of you needs to pay and multiply the sum times the number of you. So, you each pay (3 + 1) and then multiply times 5. That's 5(3 + 1) = 5(4) = 20. Or, you can figure out how much the 5 hamburgers will cost and the 5 cokes and then find the total. That's 5(3) + 5(1) = 15 + 5 = 20. Either way, the answer is the same, $20.

The two methods are represented by the equations below. On the left side, we add 10 and 2, and then multiply by 3. The expression is rewritten using the distributive property on the right side, where we distribute the 3, then multiply each by 3 and add the results. Notice that the result is the same in each case.

$$\begin{aligned} 3(10+2) &= 3(10)+3(2) \\ 3(12) &= 30+6 \\ 36 &= 36 \end{aligned}$$

The same process works if the 3 is on the other side of the parentheses, as in the example below.

$$\mathbf{(10 + 2)\ 3 = (10)3 + (2)3}$$

Example	
Problem	**Rewrite the expression 5(8 + 4) using the distributive property of multiplication over addition. Then simplify the result.**
 **5 (8 + 4) = 5(8) + 5(4)**	In the original expression, the 8 and the 4 are grouped in parentheses. Using arrows, you can see how the 5 is distributed to each addend. The 8 and 4 are each multiplied by 5.
40 + 20 = 60	The resulting products are added together, resulting in a sum of 60.
Answer	5(8 + 4) = 5(8) + 5(4) = 60

Self Check A
Rewrite the expression 30(2 + 4) using the distributive property of addition.

Distributive Property of Multiplication over Subtraction

Objective **2**

The **distributive property of multiplication over subtraction** is like the distributive property of multiplication over addition. You can subtract the numbers and then multiply, or you can multiply and then subtract as shown below. This is called “distributing the multiplier.”

$$\begin{tikzcd}[column sep=small] 5 \arrow[r, bend left] \arrow[rrr, bend left] & (6 & - & 3) & = 5(6) - 5(3) \end{tikzcd}$$

The same number works if the 5 is on the other side of the parentheses, as in the example below.

$$\begin{tikzcd}[column sep=small] (6 & - & 3) & 5 \arrow[lll, bend right] \arrow[l, bend right] & = (6)5 - (3)5 \end{tikzcd}$$

In both cases, you can then simplify the distributed expression to arrive at your answer. The example below, in which 5 is the outside multiplier, demonstrates that this is true. The expression on the right, which is simplified using the distributive property, is shown to be equal to 15, which is the resulting value on the left as well.

$$\begin{aligned} 5(6-3) &= 5(6)-5(3) \\ 5(3) &= 30-15 \\ 15 &= 15 \end{aligned}$$

Example	
Problem	**Rewrite the expression 20(9 – 2) using the distributive property of multiplication over subtraction. Then simplify.**
$\begin{tikzcd}[column sep=small] 20 \arrow[r, bend left] \arrow[rrr, bend left] & (9 & - & 2) & = 20(9) - 20(2) \end{tikzcd}$	In the original expression, the 9 and the 2 are grouped in parentheses. Using arrows, you can see how the 20 is distributed to each number so that the 9 and 2 are both multiplied by 20 individually.
180 – 40 = 140	Here, the resulting product of 40 is subtracted from the product of 180, resulting in an answer of 140.
Answer	20(9 – 2) = 20(9) – 20(2) = 140

Self Check B

Rewrite the expression 10(15 – 6) using the distributive property of subtraction.

Summary

The distributive properties of addition and subtraction can be used to rewrite expressions for a variety of purposes. When you are multiplying a number by a sum, you can add and then multiply. You can also multiply each addend first and then add the products. This can be done with subtraction as well, multiplying each number in the difference before subtracting. In each case, you are distributing the outside multiplier to each number in the parentheses, so that multiplication occurs with each number before addition or subtraction occurs. The distributive property will be useful in future math courses, so understanding it now will help you build a solid math foundation.

1.8.2 Self Check Solutions

Self Check A

Rewrite the expression 30(2 + 4) using the distributive property of addition.

30(2) + 30(4)

The number 30 is distributed to both the 2 and the 4, so that both 2 and 4 are multiplied by 30.

Self Check B

Rewrite the expression 10(15 – 6) using the distributive property of subtraction.

10(15) – 10(6)

The 10 is correctly distributed so that it is used to multiply the 15 and the 6 separately.

2.1.1 Introduction to Fractions and Mixed Numbers

Learning Objective(s)

1 Identify the numerator and denominator of a fraction.
2 Represent a fraction as part of a whole or part of a set.

Introduction

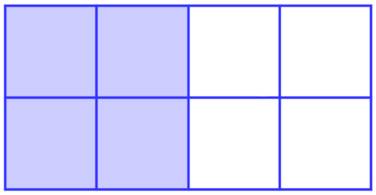
Many problems in mathematics deal with whole numbers, which are used to count whole units of things. For example, you can count students in a classroom and the number of dollar bills. You need other kinds of numbers to describe units that are not whole. For example, an aquarium might be partly full. A group may have a meeting, but only some of the members are present.

Fractions are numbers used to refer to a part of a whole. This includes measurements that cannot be written as whole numbers. For example, the width of a piece of notebook paper is more than 8 inches but less than 9 inches. The part longer than 8 inches is written as a fraction. Here, you will investigate how fractions can be written and used to represent quantities that are parts of the whole.

Identifying Numerators and Denominators

Objective 1

A whole can be divided into parts of equal size. In the example below, a rectangle has been divided into eight equal squares. Four of these eight squares are shaded.



The shaded area can be represented by a fraction. A fraction is written vertically as two numbers with a line between them.
The **denominator** (the bottom number) represents the number of equal parts that make up the whole. The **numerator** (the top number) describes the number of parts that you are describing. So returning to the example above, the rectangle has been divided into 8 equal parts, and 4 of them have been shaded. You can use the fraction $\frac{4}{8}$ to describe the shaded part of the whole.

$$\frac{4}{8} \quad \begin{array}{l} \leftarrow \text{The numerator tells how many parts are shaded.} \\ \leftarrow \text{The denominator tells how many parts are required to make up the whole.} \end{array}$$

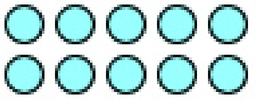
Parts of a Set

Objective **2**

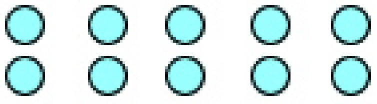
The rectangle model above provides a good, basic introduction to fractions. However, what do you do with situations that cannot be as easily modeled by shading part of a figure? For example, think about the following situation:

Marc works as a Quality Assurance Manager at an automotive plant. Every hour he inspects 10 cars; $\frac{4}{5}$ *of those pass inspection.*

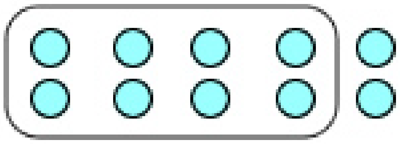
In this case, 10 cars make up the whole group. Each car can be represented as a circle, as shown below.



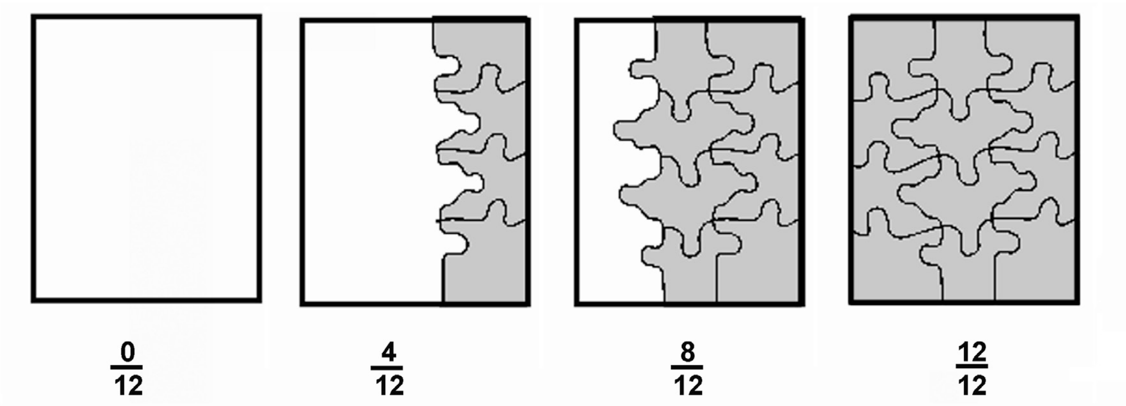
To show $\frac{4}{5}$ of the whole group, you first need to divide the whole group into 5 equal parts. (You know this because the fraction has a denominator of 5.)



To show $\frac{4}{5}$, circle 4 of the equal parts.



Here is another example. Imagine that Aneesh is putting together a puzzle made of 12 pieces. At the beginning, none of the pieces have been put into the puzzle. This means that $\frac{0}{12}$ of the puzzle is complete. Aneesh then puts four pieces together. The puzzle is $\frac{4}{12}$ complete. Soon, he adds four more pieces. Eight out of twelve pieces are now connected. This fraction can be written as $\frac{8}{12}$. Finally, Aneesh adds four more pieces. The puzzle is whole, using all 12 pieces. The fraction can be written as $\frac{12}{12}$.



$\frac{0}{12}$ $\frac{4}{12}$ $\frac{8}{12}$ $\frac{12}{12}$

Note that the number in the denominator cannot be zero. The denominator tells how many parts make up the whole. So if this number is 0, then there are no parts and therefore there can be no whole.

The numerator can be zero, as it tells how many parts you are describing. Notice that in the puzzle example above, you can use the fraction $\frac{0}{12}$ to represent the state of the puzzle when 0 pieces have been placed.

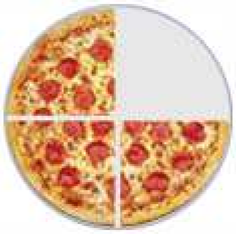
Fractions can also be used to analyze data. In the data table below, 3 out of 5 tosses of a coin came up heads, and 2 out of five tosses came up tails. Out of the total number of coin tosses, the portion that was heads can be written as $\frac{3}{5}$. The portion that was tails can be written as $\frac{2}{5}$.

Coin Toss	Result
1	Heads
2	Tails
3	Heads
4	Heads
5	Tails

Self Check A
Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

Parts of a Whole

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, $\frac{1}{4}$ of the pizza is gone and $\frac{3}{4}$ remains. Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into.

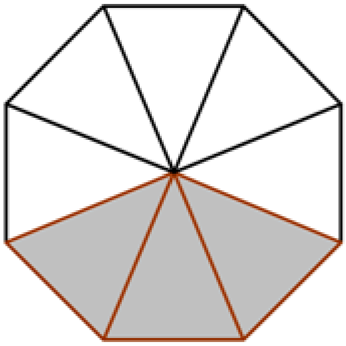


Example

Problem		**Joaquim bakes a blueberry pie for a potluck dinner. The total pie is cut into 6 equal slices. After everybody eats dessert, only one slice of the pie remains. What fraction of the pie remains?**
	$\frac{?}{6}$	The pie was cut into six equal slices, so the denominator of the fraction will be 6.
	$\frac{1}{6}$	Only 1 slice remains, so the numerator of the fraction will be 1.
Answer		$\frac{1}{6}$ of the pie remains.

Example

Problem



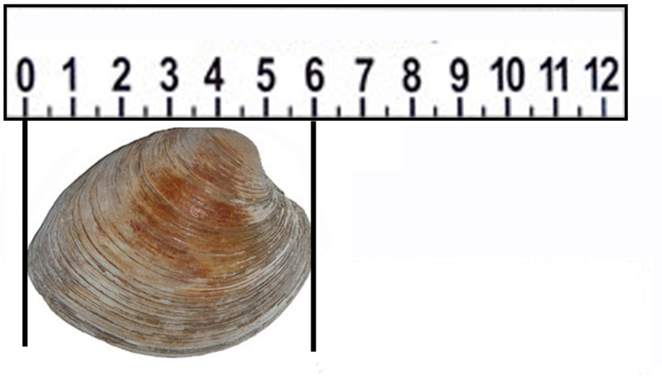
Write a fraction to represent the portion of the octagon that is *not* shaded.

$\frac{?}{8}$	The octagon has eight equal sections, so the denominator of the fraction will be 8.
$\frac{5}{8}$	Five sections are not shaded, so the numerator of the fraction will be 5.
Answer	$\frac{5}{8}$ of the octagon is not shaded.

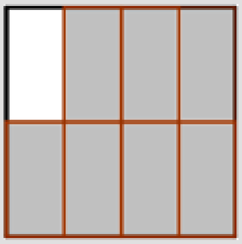
Measurement Contexts

You can use a fraction to represent the quantity in a container. This measuring cup is $\frac{3}{4}$ filled with a liquid. Note that if the cup were $\frac{4}{4}$ full, it would be a whole cup.

You can also use fractions in measuring the length, width, or height of something that is not a full unit. Using a 12-inch ruler, you measure a shell that is 6 inches long. You know that 12 inches equals one foot. So, the length of this shell is $\frac{6}{12}$ of a foot; the 12-inch ruler is the "whole", and the length of the shell is the "part."



Self Check B



Which fraction represents the portion of the shape that is shaded?

Summary

Fractions are used to represent parts of a whole. You can use fractions when describing substances, quantities, or diagrams that are not complete. You also use fractions to describe numbers of people or objects that do not make up a complete group. Fractions are written with a numerator and denominator. The numerator (above the fraction bar) tells the number of parts being described, and the denominator (below the fraction bar) tells the number of parts that make up the whole.

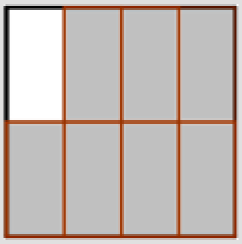
2.1.1 Self Check Solutions

Self Check A

Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

The entire race is 10 miles long, and Daphne will run 5 miles. This means she will run $\frac{5}{10}$ of the race.

Self Check B



Which fraction represents the portion of the shape that is shaded?

$\frac{7}{8}$

The total number of parts that make up the whole, 8, is the denominator (below the fraction bar). The number of parts that are shaded, 7, is the numerator (above the fraction bar).

2.1.2 Proper and Improper Fractions

Learning Objective(s)
1 Identify proper and improper fractions.
2 Change improper fractions to mixed numbers.
3 Change mixed numbers to improper fractions.

Introduction

Mathematicians use three categories to describe fractions: proper, improper, and mixed.

Fractions that are greater than 0 but less than 1 are called **proper fractions**. In proper fractions, the **numerator** is less than the **denominator**. When a fraction has a numerator that is greater than or equal to the denominator, the fraction is an **improper fraction**. An improper fraction is always 1 or greater than 1. And, finally, a **mixed number** is a combination of a whole number and a proper fraction.

Identify Proper and Improper Fractions

Objective 1

In a proper fraction, the numerator is always less than the denominator. Examples of proper fractions include $\frac{1}{2}, \frac{9}{13}$ and $\frac{1}{1001}$.

In an improper fraction, the numerator is always greater than or equal to the denominator. Examples of improper fractions include $\frac{5}{2}, \frac{9}{9}$ and $\frac{25}{20}$.

Self Check A

Identify $\frac{5}{3}$ as a proper or improper fraction.

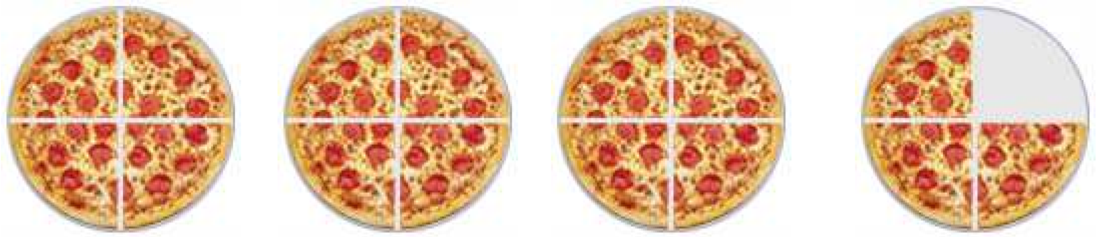
Changing Improper Fractions to Mixed Numbers

Objective 2

An improper fraction can also be written as a **mixed number**. Mixed numbers contain both a whole number and a proper fraction. Examples of mixed numbers include $8\frac{1}{10}, 1\frac{19}{20}$ and $2\frac{1}{2}$.

Let's look at a quick example.

Below are three whole pizzas that are each cut into four pieces. A fourth pizza is there as well, but someone has taken one piece, leaving only three pieces.



You can use fractions to compare the number of pieces you have to the number of pieces that make up *a whole*. In this picture, the denominator is the total number of pieces that make up *one* whole pizza, which is 4. The total number of all pieces of pizza, which is 15, represents the numerator.

You can use the improper fraction $\frac{15}{4}$ to represent the total amount of pizza here. Think: "Each whole pizza is cut into 4 equal pieces, and there are 15 pieces total. So, the total amount of whole pizzas is $\frac{15}{4}$."

As you looked at the image of the pizzas, however, you probably noticed right away that there were 3 full pizzas and one pizza with a piece missing. While you can use the improper fraction $\frac{15}{4}$ to represent the total amount of pizza, it makes more sense here to use a mixed number – a fraction that includes both a whole number and a fractional part. For this pizza scenario, you can use the fraction $3\frac{3}{4}$.

The number of whole pizzas you have is represented by a whole number

$$3\frac{3}{4}$$

The number of pieces in the incomplete pizza is the numerator

The total number of pieces that make up one pizza is the denominator

The mixed number $3\frac{3}{4}$ can be easier to understand than the improper fraction $\frac{15}{4}$. However, both forms are legitimate ways to represent the number of pizzas.

Rewriting an improper fraction as a mixed number can be helpful, because it helps you see more easily about how many whole items you have.

Let's look again at the pizzas above.

The improper fraction $\frac{15}{4}$ means there are 15 total pieces, and 4 pieces makes a whole pizza. If you didn't have the picture, you could change $\frac{15}{4}$ into a mixed fraction by determining:

- How many groups of 4 pieces are there in 15 pieces? Since $15 \div 4 = 3$ with a remainder, there are 3 whole pizzas.
- What is the remainder? The remainder is 3. So, there are 3 pieces of the last pizza left, out of the 4 that would make a whole pizza. So, $\frac{3}{4}$ of a pizza is left.

Now, put the number of whole pizzas with the fraction of a pizza that is left over. The mixed number is $3\frac{3}{4}$.

Writing Improper Fractions as Mixed Numbers

Step 1: Divide the denominator into the numerator.

Step 2: The quotient is the whole number part of the mixed number.

Step 3: The remainder is the numerator of the fractional part of the mixed number.

Step 4: The divisor is the denominator of the fractional part of the mixed number.

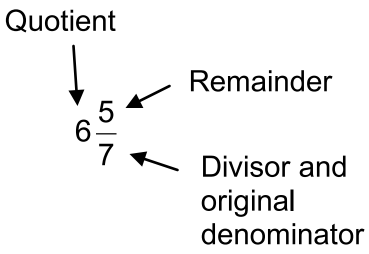
Example

Problem

Write the improper fraction $\frac{47}{7}$ as a mixed number.

$47 \div 7 = 6$, remainder 5

Divide the denominator into the numerator.



The quotient, 6, becomes the whole number.

The remainder, 5, becomes the numerator.

The denominator, which is also used as the divisor, remains as 7.

Answer $\frac{47}{7} = 6\frac{5}{7}$

Self Check B

Change $\frac{12}{5}$ from an improper fraction to a mixed number.

Changing Mixed Numbers to Improper Fractions

Objective **3**

Mixed numbers can also be changed to improper fractions. This is sometimes helpful when doing calculations with mixed numbers, especially multiplication.

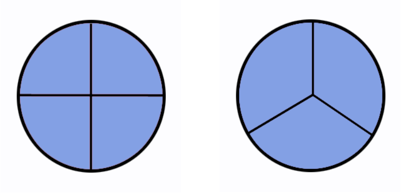
Let's start by considering the idea of one whole as an improper fraction. If you divide a cake into five equal slices, and keep all the slices, the one whole cake is equal to the 5 slices. So, 1 cake is the same as $\frac{5}{5}$ cake.

$\frac{5}{5}$

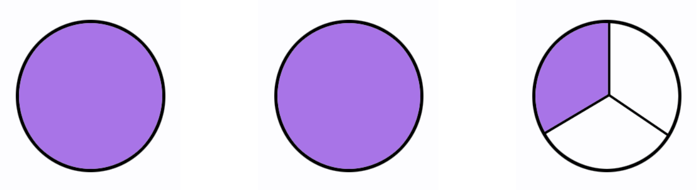
The number of parts present…

… is equal to the number of parts that make up the whole.

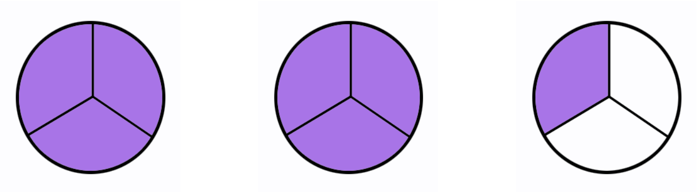
Had you cut the cake into 4 pieces or 3 pieces, as shown below, you could have used the fractions $\frac{4}{4}$ or $\frac{3}{3}$ to represent the whole cake. The fractions may change depending on the number of cuts you make to the cake, but you are still dealing with only one cake.



Let's explore how to write a simple mixed number, $2\frac{1}{3}$, as an improper fraction. The mixed number is represented below. Each full circle represents one whole.



To write an improper fraction, you need to know how many *equal sized pieces* make one whole. You also need to know how many of those pieces you have. Since you have $\frac{1}{3}$, you should divide up all of the circles into 3 pieces.



Each whole circle has 3 pieces. You can multiply the number of whole circles, 2, by 3 to find how many one-third pieces are in the two whole circles. Then you add 1 for the one-third piece in the final, incomplete circle. As you can see from the diagram, there are 7 individual one-third pieces. The improper fraction for $2\frac{1}{3}$ is $\frac{7}{3}$.

Writing Mixed Numbers as Improper Fractions

Step 1. Multiply the denominator of the fraction by the whole number.

Step 2. Add this product to the numerator of the fraction.

Step 3. The sum is the numerator of the improper fraction.

Step 4. The denominator of the improper fraction is the same as the denominator of the fractional part of the mixed number.

Example

Problem **Write $4\frac{3}{4}$ as an improper fraction.**

$4\frac{3}{4}$	
$4 \cdot 4 = 16$	Multiply the denominator of the fraction by the whole number.
$16 + 3 = 19$	Add this result to the numerator of the fraction.
$\frac{19}{4}$	This answer becomes the numerator of the improper fraction.
Answer: $4\frac{3}{4} = \frac{19}{4}$	Notice that the denominator of the improper fraction is the same as the denominator that was in the fractional part of the mixed number

Self Check C

Change $3\frac{5}{6}$ from a mixed number to an improper fraction.

Summary

A fraction can be identified as proper or improper by comparing the numerator and the denominator. Fractions that are less than one are known as proper fractions, and the numerator (the top number) is less than the denominator (the bottom number). A fraction with a numerator that is greater than or equal to the denominator is known as an improper fraction. It represents a number greater than or equal to one. Numbers that are not whole numbers, but are greater than one, can be written as improper fractions or mixed numbers. A mixed number has a whole number part and a fraction part.

2.1.2 Self Check Solutions

Self Check A

Identify $\frac{5}{3}$ as a proper or improper fraction.

The fraction is greater than 1, and the numerator is greater than the denominator, so $\frac{5}{3}$ is an improper fraction.

Self Check B

Change $\frac{12}{5}$ from an improper fraction to a mixed number.

The improper fraction $\frac{12}{5}$ can be thought of as 12 ÷ 5 = 2, with a remainder of 2. So, $2\frac{2}{5}$ is the correct answer.

Self Check C

Change $3\frac{5}{6}$ from a mixed number to an improper fraction.

$3\frac{5}{6} = (6 \cdot 3) + 5 = 23$. The denominator stays the same, so $\frac{23}{6}$ is the improper form.

2.2.1 Factors and Primes

Learning Objective(s)

1 Recognize (by using the divisibility rule) if a number is divisible by 2, 3, 4, 5, 6, 9, or 10.
2 Find the factors of a number.
3 Determine whether a number is prime, composite, or neither.
4 Find the prime factorization of a number.

Introduction

Natural numbers, also called counting numbers (1, 2, 3, and so on), can be expressed as a product of their **factors**. When working with a fraction, you often need to make the fraction as simple as possible. This means that the **numerator** and the **denominator** have no common factors other than 1. It will help to find factors, so that later you can simplify and compare fractions.

Tests of Divisibility

Objective 1

When a natural number is expressed as a product of two other natural numbers, those other numbers are factors of the original number. For example, two factors of 12 are 3 and 4, because 3 • 4 = 12.

When one number can be divided by another number with no remainder, we say the first number is **divisible** by the other number. For example, 20 is divisible by 4 ($20 \div 4 = 5$). If a number is divisible by another number, it is also a **multiple** of that number. For example, 20 is divisible by 4, so 20 is a multiple of 4.

Divisibility tests are rules that let you quickly tell if one number is divisible by another. There are many divisibility tests. Here are some of the most useful and easy to remember:

- A number is divisible by **2** if the last (ones) digit is divisible by 2. That is, the last digit is 0, 2, 4, 6, or 8. (We then say the number is an **even number**.) For example, in the number 236, the last digit is 6. Since 6 is divisible by 2 (6 ÷ 2 = 3), 236 is divisible by 2.

- A number is divisible by **3** if the sum of all the digits is divisible by 3. For example, the sum of the digits of 411 is 4 + 1 + 1 = 6. Since 6 is divisible by 3 ($6 \div 3 = 2$), 411 is divisible by 3.

- A number is divisible by **5** if the last digit is 0 or 5. For example, 275 and 1,340 are divisible by 5 because the last digits are 5 and 0.

- A number is divisible by **10** if the last digit is 0. For example, 520 is divisible by 10 (last digit is 0).

Other useful divisibility tests:

4: A number is divisible by 4 if the last two digits are divisible by 4.

6: A number is divisible by 6 if it is divisible by both 2 and 3.

9: A number is divisible by 9 if the sum of its digits is divisible by 9.

Here is a summary of the most commonly used divisibility rules.

A number is divisible by	Condition	Example
2	The last digit is even (0, 2, 4, 6, 8).	42**6** *yes* 27**3** *no*
3	The sum of the digits is divisible by 3.	642 *yes* (6 + 4 + 2 = 12, 12 is divisible by 3) 721 *no* (7 + 2 + 1 = 10, 10 is not divisible by 3)
4	The last two digits form a number that is divisible by 4.	1**64** *yes* (64 is divisible by 4) 1**35** *no* (35 is not divisible by 4)
5	The last digit is 0 or 5.	68**5** *yes* 43**2** *no*
6	The number is divisible by 2 and 3.	324 *yes* (it is even and 3 + 2 + 4 = 9) 411 *no* (although divisible by 3, it is not even)
9	The sum of the digits is divisible by 9.	279 *yes* (2 + 7 + 9 = 18) 512 *no* (5 + 1 + 2 =8)
10	The last digit is a 0.	62**0** *yes* 23**8** *no*

If you need to check for divisibility of a number without a rule, divide (either using a calculator or by hand). If the result is a number without any fractional part or remainder, then the number is divisible by the **divisor**. If you forget a rule, you can also use this strategy.

Self Check A

Determine whether 522 is divisible by 2, 3, 4, 5, 6, 9, or 10.

Factoring Numbers

Objective **2, 3**

To find all the factors of a number, you need to find all numbers that can divide into the original number without a remainder. The divisibility rules from above will be extremely useful!

Suppose you need to find the factors of 30. Since 30 is a number you are familiar with, and small enough, you should know many of the factors without applying any rules. You can start by listing the factors as they come to mind:

2 • 15
3 • 10
5 • 6

Is that it? Not quite. All natural numbers except 1 also have 1 and the number itself as factors:

1 • 30

The factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30.

When you find one factor of a number, you can easily find another factor—it is the quotient using that first factor as the divisor. For example, once you know 2 is a factor of 30, then 30 ÷ 2 is another factor. A pair of factors whose product is a given number is a **factor pair** of the original number. So, 2 and 15 are a factor pair for 30.

What do you do if you need to factor a greater number and you can't easily see its factors? That's where the divisibility rules will come in quite handy. Here is a general set of steps that you may follow:

1. Begin with 1 and check the numbers sequentially, using divisibility rules or division.
2. When you find a factor, find the other number in the factor pair.
3. Keep checking sequentially, until you reach the second number in the last factor pair you found, *or* until the result of dividing gives a number less than the divisor.

Note that you can stop checking when the result of dividing is less than the number you're checking. This means that you have already found all factor pairs, and continuing the process would find pairs that have been previously found.

Example				
Problem	**Find factors of 165.**	Factors	Explanation	Divisible?
	divisible by 1?	1 • 165 = 165	All numbers are divisible by 1.	Yes
	divisible by 2?		The last digit, 5, is not even, so 165 is not divisible by 2.	No
	divisible by 3?		1 + 6 + 5 = 12, which is divisible by 3, so 165 is divisible by 3.	Yes
	$165 \div 3 = 55$	3 • 55 = 165	Use division to find the other factor.	
	divisible by 4?		Since 165 is not an **even number**, it will not be divisible by any even number. The divisibility test for 4 also applies: 65 is not divisible by 4, so 165 is not divisible by 4.	No
	divisible by 5?		Since the last digit is 5, 165 is divisible by 5.	Yes
	$165 \div 5 = 33$	5 • 33 = 165	Use division to find the other factor.	Yes
	divisible by 6?		Since 165 is not divisible by 2, it is not divisible by 6.	No
	divisible by 7?		There is no divisibility test for 7, so you have to divide. $165 \div 7$ is not a whole number, so it is not divisible by 7.	No

divisible by 8?		Since 165 is not divisible by 2, we know that it cannot be divisible by any other even number. Note also that dividing 165 by 8 would not result in a whole number.	No
divisible by 9?		1 + 6 + 5 = 12, which is not a multiple of 9. 165 is not divisible by 9.	No
divisible by 10?		The ending digit is a 5 not a 0. 165 is not divisible by 10.	No
divisible by 11?		$165 \div 11 = 15$ with no remainder, so 165 is divisible by 11.	Yes
$165 \div 11 = 15$	11 • 15 = 165	We already performed the division to find the other factor that pairs with 11.	Yes
divisible by 12?		165 cannot be divisible by an even number and 12 is even. Also, dividing 165 by 12 would not result in a whole number.	No
divisible by 13?		$165 \div 13$ is not a whole number, so 165 is not divisible by 13.	No
Done checking numbers		Since the result of 165 ÷ 13 is less than 13, you can stop. Any factor greater than 13 would already have been found as the pair of a factor less than 13.	

Answer The factors of 165 are 1, 3, 5, 11, 15, 33, 55, 165.

If a number has exactly two factors, 1 and itself, the number is a **prime number**. A number that has more factors than itself and 1 is called a **composite number**. The number 1 is considered neither prime nor composite, as its only factor is 1. To determine whether a number is prime, composite, or neither, check factors. Here are some examples.

Number	**Composite, Prime, or Neither?**	**Explanation**
1	Neither	1 does not have two different factors, so it is not prime.
2	Prime	2 has only the factors 2 and 1.
3	Prime	3 has only the factors 3 and 1.
4	Composite	4 has more than two factors: 1, 2, and 4, so it is composite.
5, 7,11,13	Prime	Each number has only two factors: 1 and itself.
6, 8, 9,10, 50, 63	Composite	Each number has more than two factors.

Example

Problem Find all the factors of 48.

Answer: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48

Prime Factorization

Objective **4**

A composite number written as a product of only prime numbers is called the **prime factorization** of the number. One way to find the prime factorization of a number is to begin with the prime numbers 2, 3, 5, 7, 11 and so on, and determine whether the number is divisible by the primes.

For example, if you want to find the prime factorization of 20, start by checking if 20 is divisible by 2. Yes, 2 • 10 = 20.

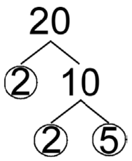
Then factor 10, which is also divisible by 2 (2 • 5 = 10).

Both of those factors are prime, so you can stop. The prime factorization of 20 is 2 • 2 • 5, which you can write using **exponential notation** as 2^2 • 5.

One way to find the prime factorization of a number is to use successive divisions.

$\begin{array}{r} 10 \\ 2\overline{)20} \end{array}$	Divide 20 by 2 to get 10. 2 is being used because it is a prime number and a factor of 20. You could also have started with 5.
$\begin{array}{r} 5 \\ 2\overline{)10} \\ 2\overline{)20} \end{array}$	Then divide 10 by 2 to get 5.
$2 \cdot 2 \cdot 5$	Multiplying these divisors forms the prime factorization of 20.

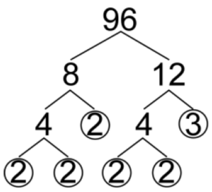
To help you organize the factoring process, you can create a **factor tree**. This is a diagram that shows a factor pair for a composite number. Then, each factor that isn't prime is also shown as a factor pair. You can continue showing factor pairs for composite factors, until you have only prime factors. When a prime number is found as a factor, circle it so you can find it more easily later.

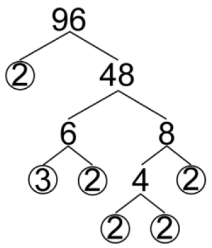


Written using exponential notation, the prime factorization of 20 is again 2^2 • 5.

Notice that you don't *have* to start checking the number using divisibility of prime numbers. You can factor 20 to 4 • 5, and then factor 4 to 2 • 2, giving the same prime factorization: 2 • 2 • 5.

Now look at a more complicated factorization.





Notice that there are two different trees, but they both produce the same result: five 2s and one 3. Every number will only have one, unique prime factorization. You can use any sets of factor pairs you wish, as long as you keep factoring composite numbers.

When you rewrite the prime factorization of 96 (2 • 2 • 2 • 2 • 2 • 3) in exponential notation, the five 2s can be written as 2^5. So, 96 = 2^5 • 3.

Self Check B

When finding the prime factorization of 72, Marie began a tree diagram using the two factors 9 and 8. Which of the following statements are true?

1. Marie started the diagram incorrectly and should have started the tree diagram using the factors 2 and 36.
2. Marie's next set of factor pairs could be 3, 3 and 2, 4.
3. Marie's next set of factor pairs could be 3, 3 and 9, 8.
4. Marie didn't have to use a tree diagram.

Summary

Finding the factors of a natural number means that you find all the possible numbers that will divide into the given number without a remainder. There are many rules of divisibility to help you to find factors more quickly. A prime number is a number that has exactly two factors. A composite number is a number that has more than two factors. The prime factorization of a number is the product of the number's prime factors.

2.2.1Self Check Solutions

Self Check A

Determine whether 522 is divisible by 2, 3, 4, 5, 6, 9, or 10.

A) 2 and 3 only
B) 4 only
C) 2, 3, 6, and 9 only
D) 4, 5, and 10 only

2, 3, 6, and 9
522 is divisible by 2 (the last digit is even) and 3 (5 + 2 + 2 = 9, which is a multiple of 3). Since it is divisible by 2 and 3, it is also divisible by 6. Also, the sum of the digits is divisible by 9, so 522 is divisible by 9. Since the last digit is not 0 or 5, 522 is not divisible by 5 or 10. The number formed by the last two digits, 22, is not divisible by 4, so 522 is not divisibly by 4.

Self Check B

When finding the prime factorization of 72, Marie began a tree diagram using the two factors 9 and 8. Which of the following statements are true?

1. Marie started the diagram incorrectly and should have started the tree diagram using the factors 2 and 36.
2. Marie's next set of factor pairs could be 3, 3 and 2, 4.
3. Marie's next set of factor pairs could be 3, 3 and 9, 8.
4. Marie didn't have to use a tree diagram.

2 and 4 only
Marie's next set of factor pairs could read 3, 3, and 2, 4, as 3 • 3 is a factorization of 9 and 2 • 4 is a factorization of 8. Marie could also find the prime factorization by using successive divisions.

2.2.2 Simplifying Fractions

Learning Objective(s)
1 Find an equivalent fraction with a given denominator.
2 Simplify a fraction to lowest terms.

Introduction

Fractions are used to represent a part of a whole. Fractions that represent the same part of a whole are called **equivalent fractions**. Factoring, multiplication, and division are all helpful tools for working with equivalent fractions.

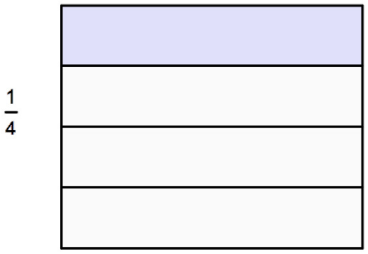
Equivalent Fractions

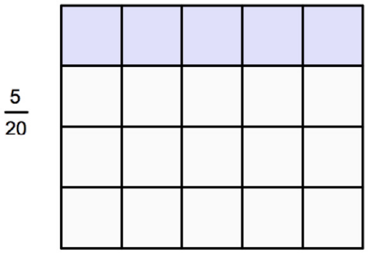
Objective 1

We use equivalent fractions every day. Fifty cents can be 2 quarters, and we have $\frac{2}{4}$ of a dollar, because there are 4 quarters in a dollar. Fifty cents is also 50 pennies out of 100 pennies, or $\frac{50}{100}$ of a dollar. Both of these fractions are the same amount of money, but written with a different numerator and denominator.

Think about a box of crackers that contains 3 packets of crackers. Two of these packets are $\frac{2}{3}$ of the box. Suppose each packet has 30 crackers in it. Two packets are also 60 (30 • 2) crackers out of 90 (30 • 3) crackers. This is $\frac{60}{90}$ of the box. The fractions $\frac{2}{3}$ and $\frac{60}{90}$ both represent two packets of crackers, so they are equivalent fractions.

Equivalent fractions represent the same part of a whole, even if the numerator and denominator are different. For example, $\frac{1}{4}=\frac{5}{20}$. In these diagrams, both fractions represent one of four rows in the rectangle.



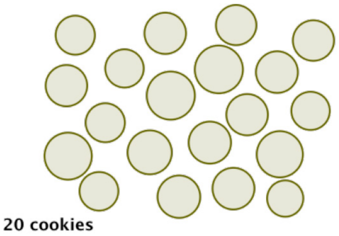


Since $\frac{1}{4}$ and $\frac{5}{20}$ are naming the same part of a whole, they are equivalent. There are many ways to name the same part of a whole using equivalent fractions.

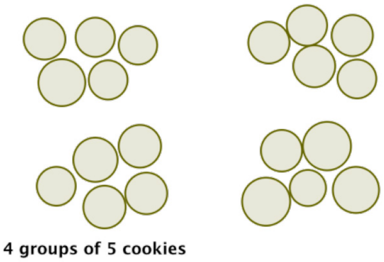
Let's look at an example where you need to find an equivalent fraction.

Example

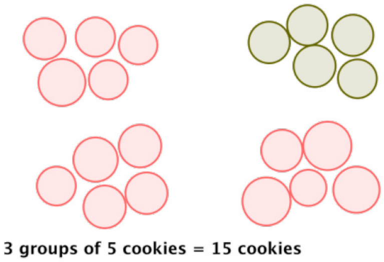
Problem **John is making cookies for a bake sale. He made 20 large cookies, but he wants to give away only 3/4 of them for the bake sale. What fraction of the cookies does he give away, using 20 as the denominator?**



Start with 20 cookies.



Because the denominator of $\frac{3}{4}$ is 4, make 4 groups of cookies, 5 in each group.



$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5}$ because there are 5 cookies in each group.

$$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}$$

Answer He gives away $\frac{15}{20}$ of the cookies.

When you regroup and reconsider the parts and whole, you are multiplying the numerator and denominator by the same number. In the above example, you multiply 4 by 5 to get the needed denominator of 20, so you also need to multiply the numerator 3 by 5, giving the new numerator of 15.

Finding Equivalent Fractions

To find equivalent fractions, multiply or divide *both* the numerator and the denominator by the same number.

Examples:

$$\frac{20}{25}=\frac{20\div 5}{25\div 5}=\frac{4}{5}$$

$$\frac{2}{7}=\frac{2\cdot 6}{7\cdot 6}=\frac{12}{42}$$

Self Check A

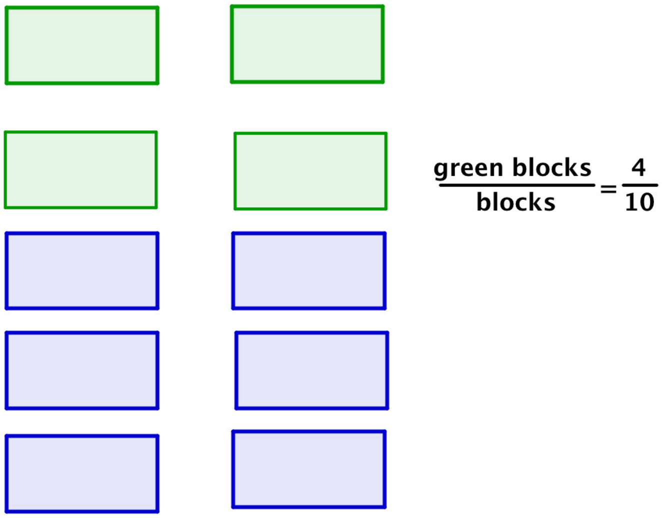
Write an equivalent fraction to $\frac{2}{3}$ that has a denominator of 27.

Simplifying Fractions

Objective **2**

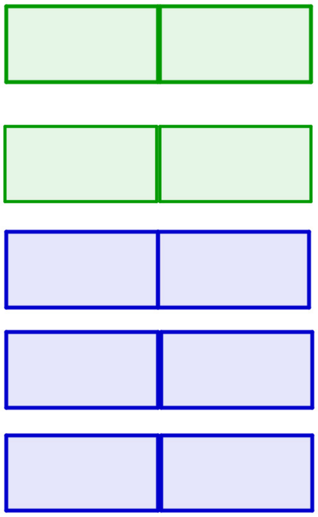
A fraction is in its **simplest form**, or **lowest terms**, when it has the least numerator and the least denominator possible for naming this part of a whole. The numerator and denominator have no common factor other than 1.

Here are 10 blocks, 4 of which are green. So, the fraction that is green is $\frac{4}{10}$. To simplify, you find a common factor and then regroup the blocks by that factor.



Example

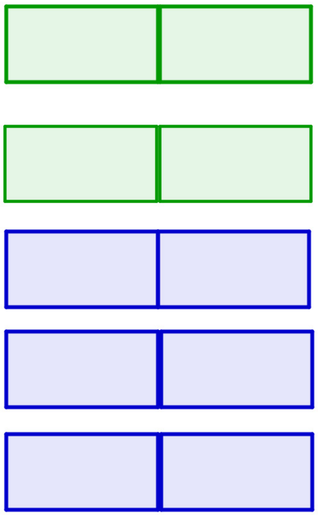
Problem **Simplify $\frac{4}{10}$.**



$$\frac{\text{green blocks}}{\text{blocks}} = \frac{2(2)}{5(2)}$$

We start with 4 green blocks out of 10 total blocks.

Group the blocks in twos, since 2 is a common factor. You have 2 groups of green blocks and a total of 5 groups, each group containing 2 blocks.



$$\frac{\text{green blocks}}{\text{blocks}} = \frac{2(2)}{5(2)} = \frac{2}{5}$$

Now, consider the groups as the part and you have 2 green groups out of 5 total groups.

Answer $\frac{4}{10} = \frac{2}{5}$ The simplified fraction is $\frac{2}{5}$.

Once you have determined a common factor, you can divide the blocks into the groups by dividing both the numerator and denominator to determine the number of groups that you have.

For example, to simplify $\frac{6}{9}$ you find a common factor of 3, which will divide evenly into both 6 and 9. So, you divide 6 and 9 into groups of 3 to determine how many groups of 3 they contain. This gives $\frac{6\div 3}{9\div 3}=\frac{2}{3}$, which means 2 out of 3 groups, and $\frac{2}{3}$ is equivalent to $\frac{6}{9}$.

It may be necessary to group more than one time. Each time, determine a common factor for the numerator and denominator using the tests of divisibility, when possible. If both numbers are even numbers, start with 2. For example:

Example

Problem	**Simplify** $\frac{32}{48}$.	
	$\frac{32}{48}=\frac{32\div 2}{48\div 2}=\frac{16}{24}$	32 and 48 have a common factor of 2. Divide each by 2.
	$\frac{16}{24}=\frac{16\div 2}{24\div 2}=\frac{8}{12}$	16 and 24 have a common factor of 2. Divide each by 2.
	$\frac{8}{12}=\frac{8\div 4}{12\div 4}=\frac{2}{3}$	8 and 12 have a common factor of 4. Divide each by 4.
Answer	$\frac{32}{48}=\frac{2}{3}$	$\frac{2}{3}$ is the simplified fraction equivalent to $\frac{32}{48}$.

In the example above, 16 is a factor of both 32 and 48, so you could have shortened the solution.

$$\frac{32}{48}=$$

$$\frac{2\cdot 16}{3\cdot 16}=$$

$$\frac{2}{3}$$

You can also use **prime factorization** to help regroup the numerator and denominator.

Example

Problem	**Simplify** $\frac{54}{72}$.	
	$\frac{54}{72}=\frac{2\cdot3\cdot3\cdot3}{2\cdot2\cdot2\cdot3\cdot3}$	The prime factorization of 54 is $2\cdot3\cdot3\cdot3$. The prime factorization of 72 is $2\cdot2\cdot2\cdot3\cdot3$.
	$\frac{3\cdot(2\cdot3\cdot3)}{2\cdot2\cdot(2\cdot3\cdot3)}$	Rewrite, finding common factors.
	$\frac{3}{2\cdot2}\cdot1$	$\frac{2\cdot3\cdot3}{2\cdot3\cdot3}=1$
	$\frac{3}{4}$	Multiply: $2\cdot2$.
Answer	$\frac{54}{72}=\frac{3}{4}$	$\frac{3}{4}$ is the simplified fraction equivalent to $\frac{54}{72}$.

Notice that when you *simplify* a fraction, you *divide* the numerator and denominator by the same number, in the same way you *multiply* by the same number to find an *equivalent* fraction with a greater denominator. In the example above, you could have divided the numerator and denominator by 9, a common factor of 54 and 72.

$$\frac{54\div9}{72\div9}=\frac{6}{8}$$

Since the numerator (6) and the denominator (8) still have a common factor, the fraction is not yet in lowest terms. So, again divide by the common factor 2.

$$\frac{6\div2}{8\div2}=\frac{3}{4}$$

Repeat this process of dividing by a common factor until the only common factor is 1.

Simplifying Fractions to Lowest Terms

To simplify a fraction to lowest terms, divide both the numerator and the denominator by their common factors. Repeat as needed until the only common factor is 1.

Self Check B

Simplify $\frac{36}{72}$.

Summary

Multiplication of binomials and polynomials requires use of the distributive property and integer operations. Whether the polynomials are monomials, binomials, or trinomials, carefully multiply each term in one polynomial by each term in the other polynomial. Be careful to watch the addition and subtraction signs and negative coefficients. A product is written in simplified if all of its like terms have been combined.

2.2.2 Self Check Solutions

Self Check A

Write an equivalent fraction to $\frac{2}{3}$ that has a denominator of 27.

$\frac{18}{27}$

The multiplying factor is 9, so the denominator is 3 • 9 = 27 and the numerator is 2 • 9 = 18.

Self Check B

Simplify $\frac{36}{72}$.

$\frac{36}{72}=\frac{36\div 36}{72\div 36}=\frac{1}{2}$. This is in lowest terms since 1 is the only common factor of 1 and 2.

2.2.3 Comparing Fractions

Learning Objective(s)

1 Determine whether two fractions are equivalent.
2 Use > or < to compare fractions.

Introduction

You often need to know when one fraction is greater or less than another fraction. Since a fraction is a part of a whole, to find the greater fraction you need to find the fraction that contains more of the whole. If the two fractions simplify to fractions with a **common denominator**, you can then compare numerators. If the denominators are different, you can find a common denominator first and then compare the numerators.

Determining Equivalent Fractions

Objective 1

Two fractions are **equivalent fractions** when they represent the same part of a whole. Since equivalent fractions do not always have the same numerator and denominator, one way to determine if two fractions are equivalent is to find a common denominator and rewrite each fraction with that denominator. Once the two fractions have the same denominator, you can check to see if the numerators are equal. If they are equal, then the two fractions are equal as well.

One way to find a common denominator is to check to see if one denominator is a factor of the other denominator. If so, the greater denominator can be used as the common denominator.

Example	
Problem	**Are $\frac{2}{6}$ and $\frac{8}{18}$ equivalent fractions?**
Does $\frac{2}{6}=\frac{8}{18}$?	To solve this problem, find a common denominator for the two fractions. This will help you compare the two fractions. Since 6 is a factor of 18, you can write both fractions with 18 as the denominator.
$\frac{2\cdot 3}{6\cdot 3}=\frac{6}{18}$	Start with the fraction $\frac{2}{6}$. Multiply the denominator, 6, by 3 to get a new denominator of 18. Since you multiply the denominator by 3, you must also multiply the numerator by 3.

$\frac{8}{18}$

The fraction $\frac{8}{18}$ already has a denominator of 18, so you can leave it as is.

$\frac{6}{18}$ does not equal $\frac{8}{18}$

Compare the fractions. Now that both fractions have the same denominator, 18, you can compare numerators.

Answer $\frac{2}{6}$ and $\frac{8}{18}$ are not equivalent fractions.

When one denominator is not a factor of the other denominator, you can find a common denominator by multiplying the denominators together.

Example

Problem **Determine whether $\frac{3}{6}$ and $\frac{5}{10}$ are equivalent fractions.**

$6 \cdot 10 = 60$

Use 60 as a common denominator.

$$\frac{3}{6} = \frac{3 \cdot 10}{6 \cdot 10} = \frac{30}{60}$$

Multiply the numerator and denominator of $\frac{3}{6}$ by 10 to get 60 in the denominator.

$$\frac{5}{10} = \frac{5 \cdot 6}{10 \cdot 6} = \frac{30}{60}$$

Multiply numerator and denominator of $\frac{5}{10}$ by 6.

$$\frac{30}{60} = \frac{30}{60}$$

Now that the denominators are the same, compare the numerators.

Answer Yes, $\frac{3}{6}$ and $\frac{5}{10}$ are equivalent fractions.

Since 30 is the value of the numerator for both fractions, the two fractions are equal.

Notice in the above example you can use 30 as the least common denominator since both 6 and 10 are factors of 30. Any common denominator will work.

In some cases you can simplify one or both of the fractions, which can result in a common denominator.

Example		
Problem	**Determine whether $\frac{2}{3}$ and $\frac{40}{60}$ are equivalent fractions.**	
	$\frac{40}{60}=\frac{40\div 10}{60\div 10}=\frac{4}{6}$	Simplify $\frac{40}{60}$. Divide the numerator and denominator by the common factor 10.
	$\frac{4}{6}=\frac{4\div 2}{6\div 2}=\frac{2}{3}$	$\frac{4}{6}$ is still not in lowest terms, so divide the numerator and the denominator again, this time by the common factor 2.
	$\frac{2}{3}=\frac{2}{3}$	Compare the fractions. The numerators and denominators are the same.
Answer	Yes, $\frac{2}{3}$ and $\frac{40}{60}$ are equivalent fractions.	

Note: In the example above you could have used the common factor of 20 to simplify $\frac{40}{60}$ directly to $\frac{2}{3}$.

Determining Equivalent Fractions

To determine whether or not two fractions are equivalent:

Step 1: Rewrite one or both of the fractions so that they have common denominators.
Step 2: Compare the numerators to see if they have the same value. If so, then the fractions are equivalent.

Self Check A

Which of the following fraction pairs are equivalent?

A) $\frac{5}{7}$ *and* $\frac{7}{5}$

B) $\frac{12}{30}$ *and* $\frac{6}{10}$

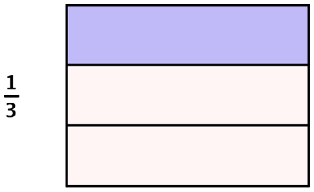
C) $\frac{4}{20}$ *and* $\frac{1}{5}$

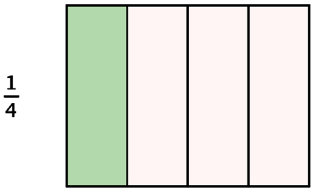
D) $\frac{8}{11}$ *and* $\frac{8}{22}$

Comparing Fractions Using < and >

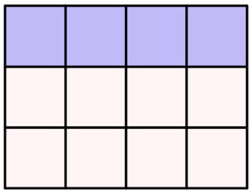
Objective **2**

When given two or more fractions, it is often useful to know which fraction is greater than or less than the other. For example, if the discount in one store is $\frac{1}{3}$ off the original price and the discount in another store is $\frac{1}{4}$ off the original price, which store is offering a better deal? To answer this question, and others like it, you can compare fractions.

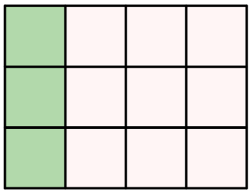




To determine which fraction is greater, you need to find a common denominator. You can then compare the fractions directly. Since 3 and 4 are both factors of 12, you will divide the whole into 12 parts, create equivalent fractions for $\frac{1}{3}$ and $\frac{1}{4}$, and then compare.

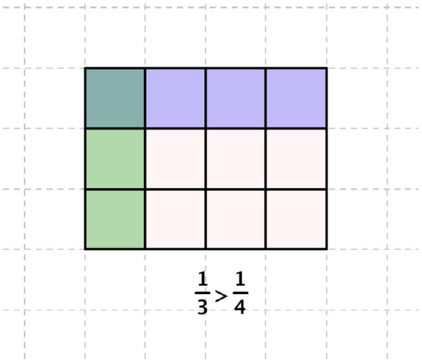


$\frac{1}{3}=\frac{4}{12}$



$\frac{1}{4}=\frac{3}{12}$

Now you see that $\frac{1}{3}$ contains 4 parts of 12, and $\frac{1}{4}$ contains 3 parts of 12. So, $\frac{1}{3}$ is greater than $\frac{1}{4}$.



$\frac{1}{3}>\frac{1}{4}$

As long as the denominators are the same, the fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction as it contains fewer parts of the whole.

Recall that the symbol < means "less than", and the symbol > means "greater than". These symbols are inequality symbols. So, the true statement 3 < 8 is read as "3 is less than 8" and the statement 5 > 3 is read as "5 is greater than 3". One way to help you remember the distinction between the two symbols is to think that the smaller end of the symbol points to the lesser number.

As with comparing whole numbers, the inequality symbols are used to show when one fraction is "greater than" or "less than" another fraction.

Comparing Fractions

To compare two fractions:

Step 1: Compare denominators. If they are different, rewrite one or both fractions with a common denominator.

Step 2: Check the numerators. If the denominators are the same, then the fraction with the greater numerator is the greater fraction. The fraction with the lesser numerator is the lesser fraction. And, as noted above, if the numerators are equal, the fractions are equivalent.

Example

Problem	**Use < or > to compare the two fractions $\frac{4}{5}$ and $\frac{14}{20}$.**	
	Is $\frac{4}{5} > \frac{14}{20}$, or is $\frac{4}{5} < \frac{14}{20}$?	You cannot compare the fractions directly because they have different denominators. You need to find a common denominator for the two fractions.
	$\frac{4}{5} = \frac{?}{20}$	Since 5 is a factor of 20, you can use 20 as the common denominator.
	$\frac{4 \cdot 4}{5 \cdot 4} = \frac{16}{20}$	Multiply the numerator and denominator by 4 to create an equivalent fraction with a denominator of 20.
	$\frac{16}{20} > \frac{14}{20}$	Compare the two fractions. $\frac{16}{20}$ is greater than $\frac{14}{20}$.
Answer	$\frac{4}{5} > \frac{14}{20}$	If $\frac{16}{20} > \frac{14}{20}$, then $\frac{4}{5} > \frac{14}{20}$, since $\frac{4}{5} = \frac{16}{20}$.

Self Check B

Which of the following is a true statement?

A) $\frac{5}{6} < \frac{24}{30}$

B) $\frac{25}{100} > \frac{9}{12}$

C) $\frac{4}{16} > \frac{1}{3}$

D) $\frac{3}{8} < \frac{20}{40}$

Summary

You can compare two fractions with like denominators by comparing their numerators. The fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction as it contains fewer parts of the whole. If two fractions have the same denominator, then equal numerators indicate equivalent fractions.

2.2.3 Self Check Solutions

Self Check A

Which of the following fraction pairs are equivalent?

A) $\frac{5}{7}$ *and* $\frac{7}{5}$ B) $\frac{12}{30}$ *and* $\frac{6}{10}$

C) $\frac{4}{20}$ *and* $\frac{1}{5}$ D) $\frac{8}{11}$ *and* $\frac{8}{22}$

$\frac{4}{20}$ *and* $\frac{1}{5}$

Take the fraction $\frac{1}{5}$ and multiply both the numerator and denominator by 4. You are left with the fraction $\frac{4}{20}$. This means that the two fractions are equivalent.

Self Check B

Which of the following is a true statement?

A) $\frac{5}{6} < \frac{24}{30}$ B) $\frac{25}{100} > \frac{9}{12}$ C) $\frac{4}{16} > \frac{1}{3}$ D) $\frac{3}{8} < \frac{20}{40}$

$\frac{3}{8} < \frac{20}{40}$

Simplifying $\frac{20}{40}$, you get the equivalent fraction $\frac{1}{2}$. Since you still don't have a common denominator, write $\frac{1}{2}$ as an equivalent fraction with a denominator of 8: $\frac{1}{2} = \frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$.

You find that $\frac{3}{8} < \frac{4}{8}$, so $\frac{3}{8} < \frac{20}{40}$ as well.

2.3 Multiplying Fractions and Mixed Numbers

Learning Objective(s)

1 Multiply two or more fractions.
2 Multiply a fraction by a whole number.
3 Multiply two or more mixed numbers.
4 Solve application problems that require multiplication of fractions or mixed numbers.
5 Find the area of triangles.

Introduction

Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions and **mixed numbers**. For example, this recipe will make 4 crumb piecrusts:

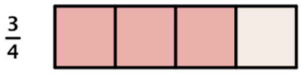
5 cups graham crackers	8 T. sugar
$1\frac{1}{2}$ cups melted butter	$\frac{1}{4}$ tsp. vanilla

Suppose you only want to make 2 crumb piecrusts. You can multiply all the ingredients by $\frac{1}{2}$, since only half of the number of piecrusts are needed. After learning how to multiply a fraction by another fraction, a whole number or a mixed number, you should be able to calculate the ingredients needed for 2 piecrusts.

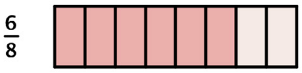
Multiplying Fractions

Objective 1

When you multiply a fraction by a fraction, you are finding a "fraction of a fraction." Suppose you have $\frac{3}{4}$ of a candy bar and you want to find $\frac{1}{2}$ of the $\frac{3}{4}$:



By dividing each fourth in half, you can divide the candy bar into eighths.



Then, choose half of those to get $\frac{3}{8}$.

In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.

Multiplying Two Fractions

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}=\frac{\text{product of the numerators}}{\text{product of the denominators}}$$

Example:

$$\frac{3}{4}\cdot\frac{1}{2}=\frac{3\cdot 1}{4\cdot 2}=\frac{3}{8}$$

Multiplying More Than Two Fractions

$$\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}=\frac{a\cdot c\cdot e}{b\cdot d\cdot f}$$

Example:

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}=\frac{1\cdot 2\cdot 3}{3\cdot 4\cdot 5}=\frac{6}{60}$$

Example		
Problem	$\frac{2}{3}\cdot\frac{4}{5}$	**Multiply.**
	$\frac{2\cdot 4}{3\cdot 5}$	Multiply the numerators and multiply the denominators.
	$\frac{8}{15}$	Simplify, if possible. This fraction is already in lowest terms.
Answer	$\frac{8}{15}$	

If the resulting **product** needs to be simplified to lowest terms, divide the numerator and denominator by common factors.

Example		
Problem	$\frac{2}{3}\cdot\frac{1}{4}$	**Multiply. Simplify the answer.**
	$\frac{2\cdot 1}{3\cdot 4}$	Multiply the numerators and multiply the denominators.
	$\frac{2}{12}$	Simplify, if possible.
	$\frac{2\div 2}{12\div 2}$	Simplify by dividing the numerator and denominator by the common factor 2.
Answer	$\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{6}$	

You can also simplify the problem before multiplying, by dividing common factors.

Example		
Problem	$\frac{2}{3}\cdot\frac{1}{4}$	**Multiply. Simplify the answer.**
	$\frac{2\cdot 1}{3\cdot 4}=\frac{1\cdot 2}{3\cdot 4}$	Reorder the numerators so that you can see a fraction that has a common factor.
	$\frac{1\cdot 1}{3\cdot 2}$	Simplify. $\frac{2}{4}=\frac{2\div 2}{4\div 2}=\frac{1}{2}$
Answer	$\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{6}$	

You do not have to use the “simplify first” shortcut, but it could make your work easier because it keeps the numbers in the numerator and denominator smaller while you are working with them.

Self Check A

$\frac{3}{4}\cdot\frac{1}{3}$ Multiply. Simplify the answer.

Multiplying a Fraction by a Whole Number

Objective **2**

When working with both fractions and whole numbers, it is useful to write the whole number as an **improper fraction** (a fraction where the numerator is greater than or equal to the denominator). All whole numbers can be written with a “1” in the denominator. For example: $2=\frac{2}{1}$, $5=\frac{5}{1}$, and $100=\frac{100}{1}$. Remember that the denominator tells how many parts there are in one whole, and the numerator tells how many parts you have.

Multiplying a Fraction and a Whole Number

$$a\cdot\frac{b}{c}=\frac{a}{1}\cdot\frac{b}{c}$$

Example:

$$4\cdot\frac{2}{3}=\frac{4}{1}\cdot\frac{2}{3}=\frac{8}{3}$$

Often when multiplying a whole number and a fraction the resulting product will be an improper fraction. It is often desirable to write improper fractions as a mixed number for the final answer. You can simplify the fraction before or after rewriting as a mixed number. See the examples below.

Example		
Problem	$7\cdot\frac{3}{5}$	**Multiply. Simplify the answer and write as a mixed number.**
	$\frac{7}{1}\cdot\frac{3}{5}$	Rewrite 7 as the improper fraction $\frac{7}{1}$.
	$\frac{7\cdot 3}{1\cdot 5}=\frac{21}{5}$	Multiply the numerators and multiply the denominators.
	$4\frac{1}{5}$	Rewrite as a mixed number. $21\div 5=4$ with a remainder of 1.
Answer	$7\cdot\frac{3}{5}=4\frac{1}{5}$	

Example		
Problem	$4\cdot\frac{3}{4}$	**Multiply. Simplify the answer and write as a mixed number.**
	$\frac{4}{1}\cdot\frac{3}{4}$	Rewrite 4 as the improper fraction $\frac{4}{1}$.
	$\frac{4\cdot 3}{1\cdot 4}$	Multiply the numerators and multiply the denominators.
	$\frac{12}{4}=3$	Simplify.
Answer	$4\cdot\frac{3}{4}=3$	

Self Check B

$3\cdot\frac{5}{6}$ Multiply. Simplify the answer and write it as a mixed number.

Multiplying Mixed Numbers

Objective 3

If you want to multiply two mixed numbers, or a fraction and a mixed number, you can again rewrite any mixed number as an improper fraction.

So, to multiply two mixed numbers, rewrite each as an improper fraction and then multiply as usual. Multiply numerators and multiply denominators and simplify. And, as before, when simplifying, if the answer comes out as an improper fraction, then convert the answer to a mixed number.

Example		
Problem	$2\frac{1}{5}\cdot 4\frac{1}{2}$	**Multiply. Simplify the answer and write as a mixed number.**
	$2\frac{1}{5}=\frac{11}{5}$	Change $2\frac{1}{5}$ to an improper fraction. 5 • 2 + 1 = 11, and the denominator is 5.
	$4\frac{1}{2}=\frac{9}{2}$	Change $4\frac{1}{2}$ to an improper fraction. 2 • 4 + 1 = 9, and the denominator is 2.

	$\frac{11}{5}\cdot\frac{9}{2}$	Rewrite the multiplication problem, using the improper fractions.
	$\frac{11\cdot9}{5\cdot2}=\frac{99}{10}$	Multiply numerators and multiply denominators.
	$\frac{99}{10}=9\frac{9}{10}$	Write as a mixed number. $99\div10=9$ with a remainder of 9.
Answer	$2\frac{1}{5}\cdot4\frac{1}{2}=9\frac{9}{10}$	

	Example	
Problem	$\frac{1}{2}\cdot3\frac{1}{3}$	**Multiply. Simplify the answer and write as a mixed number.**
	$3\frac{1}{3}=\frac{10}{3}$	Change $3\frac{1}{3}$ to an improper fraction. 3 • 3 + 1 = 10, and the denominator is 3.
	$\frac{1}{2}\cdot\frac{10}{3}$	Rewrite the multiplication problem, using the improper fraction in place of the mixed number.
	$\frac{1\cdot10}{2\cdot3}=\frac{10}{6}$	Multiply numerators and multiply denominators.
	$\frac{10}{6}=1\frac{4}{6}$	Rewrite as a mixed number. $10\div6=1$ with a remainder of 4.
	$1\frac{2}{3}$	Simplify the fractional part to lowest terms by dividing the numerator and denominator by the common factor 2.
Answer	$\frac{1}{2}\cdot3\frac{1}{3}=1\frac{2}{3}$	

As you saw earlier, sometimes it's helpful to look for common factors in the numerator and denominator before you simplify the products.

Example

Problem	$1\frac{3}{5}\cdot 2\frac{1}{4}$	**Multiply. Simplify the answer and write as a mixed number.**
	$1\frac{3}{5}=\frac{8}{5}$	Change $1\frac{3}{5}$ to an improper fraction. 5 • 1 + 3 = 8, and the denominator is 5.
	$2\frac{1}{4}=\frac{9}{4}$	Change $2\frac{1}{4}$ to an improper fraction. 4 • 2 + 1 = 9, and the denominator is 4.
	$\frac{8}{5}\cdot\frac{9}{4}$	Rewrite the multiplication problem using the improper fractions.
	$\frac{8\cdot 9}{5\cdot 4}=\frac{9\cdot 8}{5\cdot 4}$	Reorder the numerators so that you can see a fraction that has a common factor.
	$\frac{9\cdot 8}{5\cdot 4}=\frac{9\cdot 2}{5\cdot 1}$	Simplify. $\frac{8}{4}=\frac{8\div 4}{4\div 4}=\frac{2}{1}$
	$\frac{18}{5}$	Multiply.
	$\frac{18}{5}=3\frac{3}{5}$	Write as a mixed fraction.
Answer	$1\frac{3}{5}\cdot 2\frac{1}{4}=3\frac{3}{5}$	

In the last example, the same answer would be found if you multiplied numerators and multiplied denominators without removing the common factor. However, you would get $\frac{72}{20}$, and then you would need to simplify more to get your final answer.

Self Check C

$1\frac{3}{5}\cdot 3\frac{1}{3}$ Multiply. Simplify the answer and write as a mixed number.

Solving Problems by Multiplying Fractions and Mixed Numbers

Objective **4**

Now that you know how to multiply a fraction by another fraction, by a whole number, or by a mixed number, you can use this knowledge to solve problems that involve multiplication and fractional amounts. For example, you can now calculate the ingredients needed for the 2 crumb piecrusts.

Example

Problem

5 cups graham crackers
$1\frac{1}{2}$ cups melted butter
8 T. sugar
$\frac{1}{4}$ tsp. vanilla

The recipe at left makes 4 piecrusts. Find the ingredients needed to make only 2 piecrusts.

Since the recipe is for 4 piecrusts, you can multiply each of the ingredients by $\frac{1}{2}$ to find the measurements for just 2 piecrusts.

$$5\cdot\frac{1}{2}=\frac{5}{1}\cdot\frac{1}{2}=\frac{5}{2}$$

$2\frac{1}{2}$ cups of graham crackers are needed.

5 cups graham crackers: Since the result is an improper fraction, rewrite $\frac{5}{2}$ as the improper fraction $2\frac{1}{2}$.

$$8\cdot\frac{1}{2}=\frac{8}{1}\cdot\frac{1}{2}=\frac{8}{2}=4$$

4 T. sugar is needed.

8 T. sugar: This is another example of a whole number multiplied by a fraction.

$$\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}$$

$\frac{3}{4}$ cup melted butter is needed.

$1\frac{1}{2}$ cups melted butter: You need to multiply a mixed number by a fraction. So, first rewrite $1\frac{1}{2}$ as the improper fraction $\frac{3}{2}$: 2 • 1 + 1, and the denominator is 2. Then, rewrite the multiplication problem, using the improper fraction in place of the mixed number. Multiply.

$$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$

$\frac{1}{4}$ tsp. vanilla: Here, you multiply a fraction by a fraction.

$\frac{1}{8}$ **tsp. vanilla is needed.**

Answer The ingredients needed for 2 pie crusts are: $2\frac{1}{2}$ cups graham crackers

4 T. sugar

$\frac{3}{4}$ cup melted butter

$\frac{1}{8}$ tsp. vanilla

Often, a problem indicates that multiplication by a fraction is needed by using phrases like "half of," "a third of," or "$\frac{3}{4}$ of."

Example

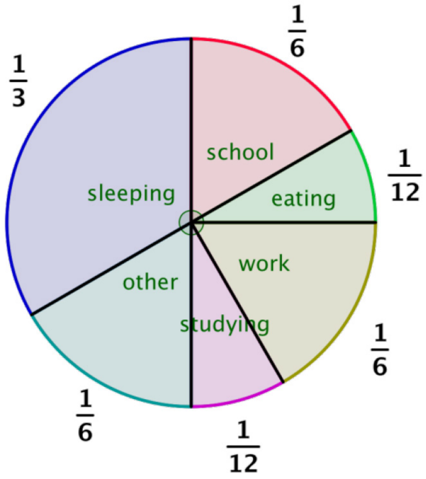
Problem **The cost of a vacation is $4,500 and you are required to pay $\frac{1}{5}$ of that amount when you reserve the trip. How much will you have to pay when you reserve the trip?**

$4{,}500\cdot\frac{1}{5}$	You need to find $\frac{1}{5}$ of 4,500. "Of" tells you to multiply.
$\frac{4{,}500}{1}\cdot\frac{1}{5}$	Change 4,500 to an improper fraction by rewriting it with 1 as the denominator.
$\frac{4{,}500}{5}$	Divide.
900	Simplify.

Answer You will need to pay $900 when you reserve the trip.

Example

Problem



The pie chart at left represents the fractional part of daily activities.
Given a 24-hour day, how many hours are spent sleeping? Attending school? Eating? Use the pie chart to determine your answers.

$\frac{1}{3}\cdot 24 =$ number of hours sleeping	Sleeping is $\frac{1}{3}$ of the pie, so the number of hours spent sleeping is $\frac{1}{3}$ of 24.
$\frac{1}{3}\cdot\frac{24}{1}$	Rewrite 24 as an improper fraction with a denominator of 1.
$\frac{24}{3}=8$ **8 hours sleeping**	Multiply numerators and multiply denominators. Simplify $\frac{24}{3}$ to 8.
$\frac{1}{6}\cdot 24 =$ number of hours spent at school	Attending school is $\frac{1}{6}$ of the pie, so the number of hours spent attending school is $\frac{1}{6}$ of 24.
$\frac{1}{6}\cdot\frac{24}{1}$	Rewrite 24 as an improper fraction with a denominator of 1.
$\frac{24}{6}=4$ **4 hours attending school**	Multiply numerators and multiply denominators. Simplify $\frac{24}{6}$ to 4.

$\frac{1}{12}\cdot 24 =$ number of hours spent eating	Eating is $\frac{1}{12}$ of the pie, so the number of hours spent eating is $\frac{1}{12}$ of 24.
$\frac{1}{12}\cdot\frac{24}{1}$	Rewrite 24 as an improper fraction with a denominator of 1.
$\frac{24}{12}=2$ **2 hours spent eating**	Multiply numerators and multiply denominators. Simplify $\frac{24}{12}$ to 2.

Answer Hours spent:
sleeping: 8 hours
attending school: 4 hours
eating: 2 hours

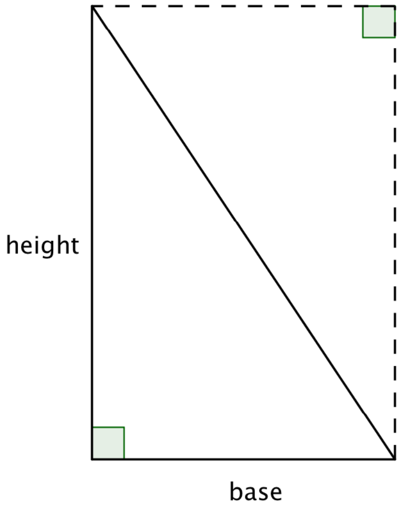
Self Check D

Neil bought a dozen (12) eggs. He used $\frac{1}{3}$ of the eggs for breakfast. How many eggs are left?

Area of Triangles

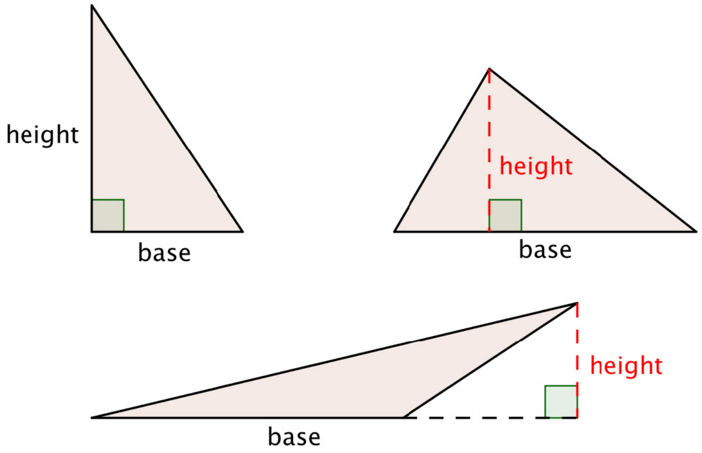
Objective 5

The formula for the area of a triangle can be explained by looking at a right triangle. Look at the image below—a rectangle with the same height and base as the original triangle. The area of the triangle is one half of the rectangle!

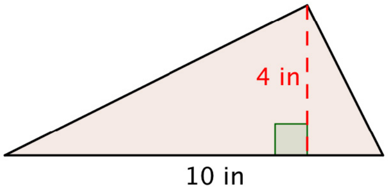


Since the area of two congruent triangles is the same as the area of a rectangle, you can come up with the formula $Area = \frac{1}{2} b \bullet h$ to find the area of a triangle.

When you use the formula for a triangle to find its area, it is important to identify a base and its corresponding height, which is perpendicular to the base.



Example

Problem	**A triangle has a height of 4 inches and a base of 10 inches. Find the area.**  	
	$A = \frac{1}{2} bh$	Start with the formula for the area of a triangle.
	$A = \frac{1}{2} \bullet 10 \bullet 4$	Substitute 10 for the base and 4 for the height.
	$A = \frac{1}{2} \bullet 40$	Multiply.
	$A = 20$	
Answer	$A = 20 \text{ in}^2$	

Summary

You multiply two fractions by multiplying the numerators and multiplying the denominators. Often the resulting product will not be in lowest terms, so you must also simplify. If one or both fractions are whole numbers or mixed numbers, first rewrite each as an improper fraction. Then multiply as usual, and simplify.

2.3 Self Check Solutions

Self Check A

$\frac{3}{4} \cdot \frac{1}{3}$ Multiply. Simplify the answer.

$\frac{3 \cdot 1}{4 \cdot 3} = \frac{3}{12}$, then simplify: $\frac{3 \div 3}{12 \div 3} = \frac{1}{4}$.

Self Check B

$3 \cdot \frac{5}{6}$ Multiply. Simplify the answer and write it as a mixed number.

$2\frac{1}{2}$. Multiplying the two numbers gives $\frac{15}{6}$, and since 15 ÷ 6 = 2R3, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$.

Self Check C

$1\frac{3}{5} \cdot 3\frac{1}{3}$ Multiply. Simplify the answer and write as a mixed number.

$5\frac{1}{3}$. First, rewrite each mixed number as an improper fraction: $1\frac{3}{5} = \frac{8}{5}$ and $3\frac{1}{3} = \frac{10}{3}$. Next, multiply numerators and multiply denominators: $\frac{8}{5} \cdot \frac{10}{3} = \frac{80}{15}$. Then write as a mixed fraction $\frac{80}{15} = 5\frac{5}{15}$. Finally, simplify the fractional part by dividing both numerator and denominator by the common factor 5.

Self Check D

Neil bought a dozen (12) eggs. He used $\frac{1}{3}$ of the eggs for breakfast. How many eggs are left?

$\frac{1}{3}$ of 12 is 4 ($\frac{1}{3} \cdot \frac{12}{1} = \frac{12}{3} = 4$), so he used 4 of the eggs. Because 12 – 4 = 8, there are 8 eggs left.

2.4 Dividing Fractions and Mixed Numbers

Learning Objective(s)

1 Find the reciprocal of a number.
2 Divide two fractions.
3 Divide two mixed numbers.
4 Divide fractions, mixed numbers, and whole numbers.
5 Solve application problems that require division of fractions or mixed numbers.

Introduction

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and there are 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction. Suppose painting a closet with one coat only required $\frac{1}{2}$ quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 6 by the fraction, $\frac{1}{2}$.

Reciprocals

Objective 1

If the **[product]** of two numbers is 1, the two numbers are **[reciprocals]** of each other. Here are some examples:

Original number	Reciprocal	Product
$\frac{3}{4}$	$\frac{4}{3}$	$\frac{3}{4}\cdot\frac{4}{3}=\frac{3\cdot4}{4\cdot3}=\frac{12}{12}=1$
$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{2}\cdot\frac{2}{1}=\frac{1\cdot2}{2\cdot1}=\frac{2}{2}=1$
$3=\frac{3}{1}$	$\frac{1}{3}$	$\frac{3}{1}\cdot\frac{1}{3}=\frac{3\cdot1}{1\cdot3}=\frac{3}{3}=1$
$2\frac{1}{3}=\frac{7}{3}$	$\frac{3}{7}$	$\frac{7}{3}\cdot\frac{3}{7}=\frac{7\cdot3}{3\cdot7}=\frac{21}{21}=1$

In each case, the original number, when multiplied by its reciprocal, equals 1.

To create two numbers that multiply together to give an answer of one, the numerator of one is the denominator of the other. You sometimes say one number is the "flip" of the other number: flip $\frac{2}{5}$ to get the reciprocal $\frac{5}{2}$. In order to find the reciprocal of a mixed number, write it first as an **improper fraction** so that it can be "flipped."

Example		
Problem	**Find the reciprocal of $5\frac{1}{4}$.**	
	$5\frac{1}{4}=\frac{21}{4}$	Rewrite $5\frac{1}{4}$ as an improper fraction. The numerator is 4 • 5 + 1 = 21.
Answer	$\frac{4}{21}$	Find the reciprocal by interchanging ("flipping") the numerator and denominator.

Self Check A

What is the reciprocal of $3\frac{2}{5}$?

Dividing a Fraction or a Mixed Number by a Whole Number

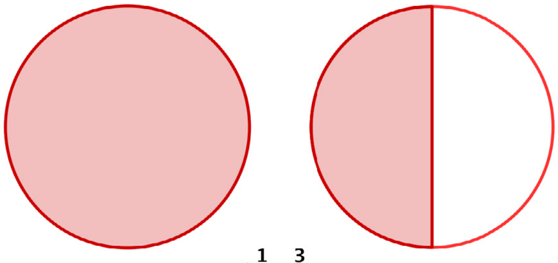
Objective 3, 4

When you divide by a whole number, you multiply by the reciprocal of the divisor. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by 3, 6 ÷3 = 2. You can also multiply 6 by the reciprocal of 3, which is $\frac{1}{3}$, so the multiplication problem becomes $\frac{6}{1}\cdot\frac{1}{3}=\frac{6}{3}=2$.

The same idea will work when the divisor is a fraction. If you have $\frac{3}{4}$ of a candy bar and need to divide it among 5 people, each person gets $\frac{1}{5}$ of the available candy: $\frac{1}{5}$ of $\frac{3}{4}$ is $\frac{1}{5}\cdot\frac{3}{4}=\frac{3}{20}$, so each person gets $\frac{3}{20}$ of a whole candy bar.

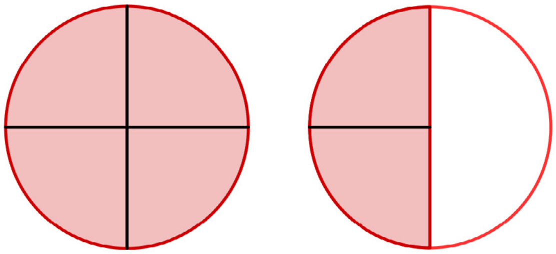
If you have a recipe that needs to be divided in half, you can divide each ingredient by 2, or you can multiply each ingredient by $\frac{1}{2}$ to find the new amount.

Similarly, with a **mixed number**, you can either divide by the whole number or you can multiply by the reciprocal. Suppose you have $1\frac{1}{2}$ pizzas that you want to divide evenly among 6 people.



$1\frac{1}{2}=\frac{3}{2}$

Dividing by 6 is the same as multiplying by the reciprocal of 6, which is $\frac{1}{6}$. Cut the available pizza into six equal-sized pieces.



$\frac{3}{2}$ divided by 6 $= \frac{3}{2} \times \frac{1}{6}$

$\frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$

Each person gets one piece, so each person gets $\frac{1}{4}$ of a pizza.

Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve such division problems.

Example

Problem	**Find $2\frac{2}{3} \div 4$. Write your answer as a mixed number with any fraction part in lowest terms.**	
	$2\frac{2}{3} = \frac{8}{3}$	Rewrite $2\frac{2}{3}$ as an improper fraction. The numerator is $2 \cdot 3 + 2$. The denominator is still 3.
	$\frac{8}{3} \div 4 = \frac{8}{3} \cdot \frac{1}{4}$	Dividing by 4 or $\frac{4}{1}$ is the same as multiplying by the reciprocal of 4, which is $\frac{1}{4}$.
	$\frac{8 \cdot 1}{3 \cdot 4} = \frac{8}{12}$	Multiply numerators and multiply denominators.
	$\frac{2}{3}$	Simplify to lowest terms by dividing numerator and denominator by the common factor 4.
Answer	$2\frac{2}{3} \div 4 = \frac{2}{3}$	

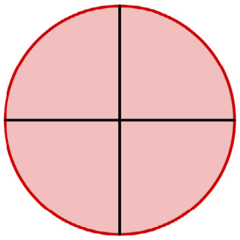
Self Check B

Find $4\frac{3}{5} \div 2$ Simplify the answer and write as a mixed number.

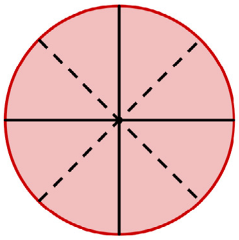
Dividing by a Fraction

Objective **2**

Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many $\frac{1}{2}$ slices are there?



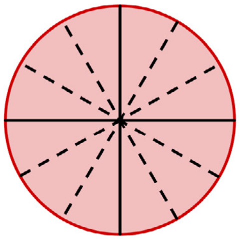
number of slices $= 4$



number of $\frac{1}{2}$ slices $= 8$

There are 8 slices. You can see that dividing 4 by $\frac{1}{2}$ gives the same result as multiplying 4 by 2.

What would happen if you needed to divide each slice into thirds?



number of $\frac{1}{3}$ slices $= 12$

You would have 12 slices, which is the same as multiplying 4 by 3.

Dividing with Fractions

1. Find the reciprocal of the number that follows the division symbol.
2. Multiply the first number (the one **before** the division symbol) by the reciprocal of the second number (the one **after** the division symbol).

   Examples:

$6 \div \frac{2}{3} = 6 \cdot \frac{3}{2}$ and $\frac{2}{5} \div \frac{1}{3} = \frac{2}{5} \cdot \frac{3}{1}$

Any easy way to remember how to divide fractions is the phrase “keep, change, flip”. This means to **KEEP** the first number, **CHANGE** the division sign to multiplication, and then **FLIP** (use the reciprocal) of the second number.

Example		
Problem	$\frac{2}{3} \div \frac{1}{6}$	**Divide.**
	$\frac{2}{3} \cdot \frac{6}{1}$	Multiply by the reciprocal: Keep $\frac{2}{3}$, change $\div$ to •, and flip $\frac{1}{6}$.
	$\frac{2 \cdot 6}{3 \cdot 1} = \frac{12}{3}$	Multiply numerators and multiply denominators.
	$\frac{12}{3} = 4$	Simplify.
Answer	$\frac{2}{3} \div \frac{1}{6} = 4$	

Example		
Problem	$\frac{3}{5} \div \frac{2}{3}$	**Divide.**
	$\frac{3}{5} \cdot \frac{3}{2}$	Multiply by the reciprocal: Keep $\frac{3}{5}$, change $\div$ to •, and flip $\frac{2}{3}$.
	$\frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10}$	Multiply numerators and multiply denominators.
Answer	$\frac{3}{5} \div \frac{2}{3} = \frac{9}{10}$	

When solving a division problem by multiplying by the reciprocal, remember to write all whole numbers and mixed numbers as improper fractions. The final answer should be simplified and written as a mixed number.

Example

Problem	$2\frac{1}{4} \div \frac{3}{4}$	**Divide.**
	$\frac{9}{4} \div \frac{3}{4}$	Write $2\frac{1}{4}$ as an improper fraction.
	$\frac{9}{4} \cdot \frac{4}{3}$	Multiply by the reciprocal: Keep $\frac{9}{4}$, change ÷ to •, and flip $\frac{3}{4}$.
	$\frac{9 \cdot 4}{4 \cdot 3} = \frac{36}{12}$	Multiply numerators and multiply denominators.
	$\frac{36}{12} = 3$	Simplify.
Answer	$2\frac{1}{4} \div \frac{3}{4} = 3$	

Example

Problem	$3\frac{1}{5} \div 2\frac{1}{10}$	**Divide. Simplify the answer and write as a mixed number.**
	$\frac{16}{5} \div \frac{21}{10}$	Write $3\frac{1}{5}$ and $2\frac{1}{10}$ as improper fractions.
	$\frac{16}{5} \cdot \frac{10}{21}$	Multiply by the reciprocal of $\frac{21}{10}$.
	$\frac{16 \cdot 10}{21 \cdot 5}$	Multiply numerators, multiply denominators, and regroup.
	$\frac{16 \cdot 2}{21 \cdot 1}$	Simplify: $\frac{10}{5} = \frac{2}{1}$.
	$\frac{16 \cdot 2}{21 \cdot 1} = \frac{32}{21}$	Multiply.
	$\frac{32}{21} = 1\frac{11}{21}$	Rewrite as a mixed number.
Answer	$3\frac{1}{5} \div 2\frac{1}{10} = 1\frac{11}{21}$	

Self Check C

Find $5\frac{1}{3} \div \frac{2}{3}$. Simplify the answer and write as a mixed number.

Dividing Fractions or Mixed Numbers to Solve Problems

Objective **5**

Using multiplication by the reciprocal instead of division can be very useful to solve problems that require division and fractions.

Example

Problem **A cook has $18\frac{3}{4}$ pounds of ground beef. How many quarter-pound burgers can he make?**

$18\frac{3}{4} \div \frac{1}{4}$	You need to find how many quarter pounds there are in $18\frac{3}{4}$, so use division.
$\frac{75}{4} \div \frac{1}{4}$	Write $18\frac{3}{4}$ as an improper fraction.
$\frac{75}{4} \cdot \frac{4}{1}$	Multiply by the reciprocal.
$\frac{75 \cdot 4}{4 \cdot 1}$	Multiply numerators and multiply denominators.
$\frac{4}{4} \cdot \frac{75}{1}$	Regroup and simplify $\frac{4}{4}$, which is 1.

Answer 75 burgers

Example

Problem **A child needs to take $2\frac{1}{2}$ tablespoons of medicine per day in 4 equal doses. How much medicine is in each dose?**

$2\frac{1}{2} \div 4$	You need to make 4 equal doses, so you can use division.
$\frac{5}{2} \div 4$	Write $2\frac{1}{2}$ as an improper fraction.
$\frac{5}{2} \cdot \frac{1}{4}$	Multiply by the reciprocal.
$\frac{5 \cdot 1}{2 \cdot 4} = \frac{5}{8}$	Multiply numerators and multiply denominators. Simplify, if possible.

Answer $\frac{5}{8}$ tablespoon in each dose.

Self Check D

How many $\frac{2}{5}$-cup salt shakers can be filled from 12 cups of salt?

Summary

Division is the same as multiplying by the reciprocal. When working with fractions, this is the easiest way to divide. Whether you divide by a number or multiply by the reciprocal of the number, the result will be the same. You can use these techniques to help you solve problems that involve division, fractions, and/or mixed numbers.

2.4 Self Check Solutions

Self Check A

What is the reciprocal of $3\frac{2}{5}$?

$\frac{5}{17}$. First, write $3\frac{2}{5}$ as an improper fraction , $\frac{17}{5}$. The reciprocal of $\frac{17}{5}$ is found by interchanging (“flipping”) the numerator and denominator.

Self Check B

Find $4\frac{3}{5} \div 2$ Simplify the answer and write as a mixed number.

Write $4\frac{3}{5}$ as the improper fraction $\frac{23}{5}$. Then multiply by $\frac{1}{2}$, the reciprocal of 2. This gives the improper fraction $\frac{23}{10}$, and the mixed number is $23 \div 10 = 2R3$, $2\frac{3}{10}$.

Self Check C

Find $5\frac{1}{3} \div \frac{2}{3}$. Simplify the answer and write as a mixed number.

Write $5\frac{1}{3}$ as an improper fraction, $\frac{16}{3}$. Then multiply by the reciprocal of $\frac{2}{3}$, which is $\frac{3}{2}$, giving you $\frac{16}{3} \cdot \frac{3}{2} = \frac{3}{3} \cdot \frac{16}{2} = \frac{16}{2} = 8$.

Self Check D

How many $\frac{2}{5}$-cup salt shakers can be filled from 12 cups of salt?

$12 \div \frac{2}{5}$ will show how many salt shakers can be filled. Write 12 as $\frac{12}{1}$ and multiply by the reciprocal ("flip") of $\frac{2}{5}$, giving you $\frac{12}{1} \cdot \frac{5}{2} = \frac{60}{2} = 30$.

2.5 Adding and Subtracting Fractions and Mixed Numbers with Like Denominators

Learning Objective(s)

1 Add fractions with like denominators.
2 Subtract fractions with like denominators.
3 Add mixed numbers with like denominators.
4 Subtract mixed numbers with like denominators.
5 Solve application problems that require the addition of fractions or mixed numbers.

Introduction

Fractions are used in many areas of everyday life: recipes, woodworking, rainfall, timecards, and measurements, to name just a few. Sometimes you have parts of wholes that you need to combine. Just as you can add whole numbers, you can add fractions and mixed numbers. Consider, for example, how to determine the monthly rainfall if you know the daily rainfall in inches. You have to add fractions. Also, consider several painters who are working to paint a house together with multiple cans of paint. They might add the fractions of what remains in each can to determine if there is enough paint to finish the job or if they need to buy more.

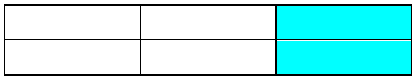
Adding Fractions with Like Denominators

Objective 1

When the pieces are the same size, they can easily be added. Consider the pictures below showing the fractions $\frac{3}{6}$ and $\frac{2}{6}$.

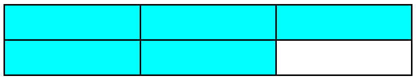
This picture represents $\frac{3}{6}$ shaded because 3 out of 6 blocks are shaded.

This picture represents $\frac{2}{6}$ shaded because 2 out of 6 blocks are shaded.



If you add these shaded blocks together, you are adding $\frac{3}{6} + \frac{2}{6}$.

You can create a new picture showing 5 shaded blocks in a rectangle containing 6 blocks.



So, $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.

Without drawing rectangles and shading boxes, you can get this answer simply by adding the numerators, 3 + 2, and keeping the denominator, 6, the same. This procedure works for adding any fractions that have the same denominator, called **like denominators**.

Example		
Problem	$\frac{3}{5}+\frac{1}{5}$	**Add.**
	$\frac{3+1}{5}$ $\frac{4}{5}$	Since the denominator of each fraction is 5, these fractions have like denominators. So, add the numerators and write the sum over the denominator, 5.
Answer	$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$	

Example		
Problem	$\frac{3}{8}+\frac{5}{8}$	**Add. Simplify the answer.**
	$\frac{3}{8}+\frac{5}{8}=\frac{3+5}{8}$ $=\frac{8}{8}$	The denominators are alike, so add the numerators.
	$\frac{8}{8}=1$	Simplify the fraction.
Answer	$\frac{3}{8}+\frac{5}{8}=1$	

Example

Problem	$\frac{11}{12}+\frac{5}{12}$	**Add. Simplify the answer and write as a mixed number.**
	$\frac{11}{12}+\frac{5}{12}=\frac{11+5}{12}$ $=\frac{16}{12}$	The denominators are alike, so add the numerators.
	$\frac{16}{12}=\frac{16\div 4}{12\div 4}=\frac{4}{3}$	Simplify the fraction. 16 and 12 have a common factor of 4.
	$\frac{4}{3}=1\frac{1}{3}$	Write the improper fraction as a mixed number, by dividing: 4 ÷ 3 = 1 with a remainder of 1.
Answer	$\frac{11}{12}+\frac{5}{12}=1\frac{1}{3}$	

In the previous example, the fraction was simplified and then converted to a mixed number. You could just as easily have first converted the improper fraction to a mixed number and then simplified the fraction in the mixed number. Notice that the same answer is reached with both methods.

$$\frac{16}{12}=1\frac{4}{12}$$

The fraction $\frac{4}{12}$ can be simplified. $\frac{4}{12}=\frac{4\div 4}{12\div 4}=\frac{1}{3}$

But, don't forget about the 1 that is part of the mixed number! The final answer is $1\frac{1}{3}$.

Adding Fractions with Like Denominators

1. Add the numerators (the number in the top of each fraction).
2. Keep the denominator (the bottom number) the same.
3. Simplify to lowest terms.

Self Check A

$\frac{7}{10}+\frac{8}{10}$ Add. Simplify the answer and write as a mixed number.

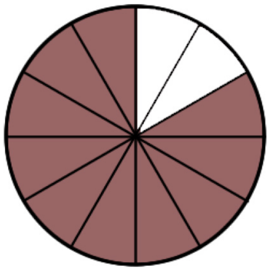
Sometimes subtraction, rather than addition, is required to solve problems that involve fractions. Suppose you are making pancakes and need $4\frac{1}{2}$ cups of flour but you only have $2\frac{3}{4}$ cups. How many additional cups will you have to get to make the pancakes? You can solve this problem by subtracting the mixed numbers.

Subtracting Fractions with Like Denominators

Objective **2**

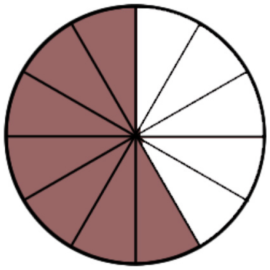
The most simple fraction subtraction problems are those that have two proper fractions with a **common denominator**. That is, each denominator is the same. The process is just as it is for addition of fractions with **like denominators**, except you subtract! You subtract the second numerator from the first and keep the denominator the same.

Imagine that you have a cake with equal-sized pieces. Some of the cake has already been eaten, so you have a fraction of the cake remaining. You could represent the cake pieces with the picture below.



The cake is cut into 12 equal pieces to start. Two are eaten, so the remaining cake can be represented with the fraction $\frac{10}{12}$. If three more pieces of cake are eaten, what fraction of the cake is left? You can represent that problem with the expression $\frac{10}{12}-\frac{3}{12}$.

If you subtract 3 pieces, you can see below that $\frac{7}{12}$ of the cake remains.



You can solve this problem without the picture by subtracting the numerators and keeping the denominator the same:

$$\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$$

Subtracting Fractions with Like Denominators

If the denominators (bottoms) of the fractions are the same, subtract the numerators (tops) and keep the denominator the same. *Remember to simplify the resulting fraction, if possible.*

Example

Problem	$\frac{6}{7}-\frac{1}{7}$	**Subtract.**
	$\frac{6-1}{7}=\frac{5}{7}$	Both fractions have a denominator of 7, so subtract the numerators and keep the same denominator.
Answer	$\frac{6}{7}-\frac{1}{7}=\frac{5}{7}$	

Example

Problem	$\frac{5}{9}-\frac{2}{9}$	**Subtract. Simplify the answer.**
	$\frac{5}{9}-\frac{2}{9}=\frac{3}{9}$	The fractions have a **like denominator**, also known as a common denominator, so subtract the numerators.
	$\frac{3\div 3}{9\div 3}=\frac{1}{3}$	Simplify the fraction.
Answer	$\frac{5}{9}-\frac{2}{9}=\frac{1}{3}$	

Self Check B

$\frac{11}{16}-\frac{7}{16}$ Subtract and simplify the answer.

Adding Mixed Numbers

Objective **3**

Just as you can add whole numbers and proper fractions, you can also add mixed numbers. To add mixed numbers, add the whole numbers together and the fraction parts of the mixed numbers together and then recombine to express the value as a mixed number. The steps for adding two mixed numbers are shown in the examples below.

You can keep the whole numbers and the fractions together using a vertical method for adding mixed numbers as shown below.

Example		
Problem	$2\frac{1}{8}+3\frac{3}{8}$	**Add. Simplify the answer and write as a mixed number.**
	$\begin{array}{r} 2\frac{1}{8} \\ +\ 3\frac{3}{8} \\ \hline \end{array}$	Arrange the mixed numbers vertically so the whole numbers align and the fractions align.
	$\begin{array}{r} 2\frac{1}{8} \\ +\ 3\frac{3}{8} \\ \hline 5\frac{4}{8} \end{array}$	Add whole numbers. Add fractions.
	$5\frac{4}{8}=5\frac{1}{2}$	Simplify the fraction.
Answer	$2\frac{1}{8}+3\frac{3}{8}=5\frac{1}{2}$	

When adding mixed numbers you may need to regroup if the fractional parts add to more than one whole.

Example		
Problem	$6\frac{5}{7}+8\frac{4}{7}$	*Add. Simplify the answer and* **write as a mixed number.**
	$\begin{array}{r} 6\frac{5}{7} \\ +\ 8\frac{4}{7} \\ \hline \end{array}$	Arrange the mixed numbers vertically so the whole numbers align and the fractions align.
	$\begin{array}{r} 6\frac{5}{7} \\ +\ 8\frac{4}{7} \\ \hline 14\frac{9}{7} \end{array}$	Add whole numbers. Add fractions.

	$\frac{9}{7}=1\frac{2}{7}$	Write the improper fraction as a mixed number.
	$14+1\frac{2}{7}=15\frac{2}{7}$	Combine whole numbers and fraction to write a mixed number.
Answer	$6\frac{5}{7}+8\frac{4}{7}=15\frac{2}{7}$	

Self Check C

$3\frac{7}{9}+1\frac{4}{9}$ Add. Simplify the answer and write as a mixed number.

Subtracting Mixed Numbers

Objective **4**

Subtracting mixed numbers works much the same way as adding mixed numbers. To subtract mixed numbers, subtract the whole number parts of the mixed numbers and then subtract the fraction parts in the mixed numbers. Finally, combine the whole number answer and the fraction answer to express the answer as a mixed number.

Example		
Problem	$6\frac{4}{5}-3\frac{1}{5}$	**Subtract. Simplify the answer and write as a mixed number.**
	$6-3=3$ $\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$	Subtract the whole numbers and subtract the fractions.
	$3\frac{3}{5}$	Combine the fraction and the whole number. Make sure the fraction in the mixed number is simplified.
Answer	$6\frac{4}{5}-3\frac{1}{5}=3\frac{3}{5}$	

Sometimes it might be easier to express the mixed number as an improper fraction first and then solve. Consider the example below.

Example		
Problem	$8\frac{1}{3}-4\frac{2}{3}$	**Subtract. Simplify the answer and write as a mixed number.**
	$8\frac{1}{3}=\frac{8\cdot3+1}{3}=\frac{24+1}{3}=\frac{25}{3}$ $4\frac{2}{3}=\frac{4\cdot3+2}{3}=\frac{12+2}{3}=\frac{14}{3}$	Write each mixed number as an improper fraction.
	$\frac{25}{3}-\frac{14}{3}=\frac{11}{3}$	Since the fractions have a like denominator, subtract the numerators.
	$\frac{11}{3}=3\frac{2}{3}$	Write the answer as a mixed number. Divide 11 by 3 to get 3 with a remainder of 2.
Answer	$8\frac{1}{3}-4\frac{2}{3}=3\frac{2}{3}$	

Since addition is the inverse operation of subtraction, you can check your answer to a subtraction problem with addition. In the example above, if you add $4\frac{2}{3}$ to your answer of $3\frac{2}{3}$, you should get $8\frac{1}{3}$.

$$4\frac{2}{3}+3\frac{2}{3}$$

$$4+3+\frac{2}{3}+\frac{2}{3}$$

$$7+\frac{4}{3}$$

$$7+1\frac{1}{3}$$

$$8\frac{1}{3}$$

Subtracting Mixed Numbers with Regrouping

Objective 4

Sometimes when subtracting mixed numbers, the fraction part of the second mixed number is larger than the fraction part of the first number. Consider the problem:

$7\frac{1}{6}-3\frac{5}{6}$. The standard procedure would be to subtract the fractions, but $\frac{1}{6}-\frac{5}{6}$ would result in a negative number. You don't want that!
You can regroup one of the whole numbers from the first number, writing the first mixed number in a different way:

$$7\frac{1}{6}=7+\frac{1}{6}=6+1+\frac{1}{6}$$

$$6+\frac{6}{6}+\frac{1}{6}=6+\frac{7}{6}=6\frac{7}{6}$$

Now, you can write an equivalent problem to the original:

$$6\frac{7}{6}-3\frac{5}{6}$$

Then, you just subtract like you normally subtract mixed numbers:

$$6 - 3 = 3$$

$$\frac{7}{6}-\frac{5}{6}=\frac{2}{6}=\frac{1}{3}$$

So, the answer is $3\frac{1}{3}$.

Example		
Problem	$7\frac{1}{4}-3\frac{3}{4}$	**Subtract. Simplify the answer and write as a mixed number.**
	$7\frac{1}{4}=6+1+\frac{1}{4}$ $6+\frac{4}{4}+\frac{1}{4}$ $6+\frac{5}{4}$ $6\frac{5}{4}$	Since the second fraction part, $\frac{3}{4}$, is larger than the first fraction part, $\frac{1}{4}$, regroup one of the whole numbers and write it as $\frac{4}{4}$.

	$7\frac{1}{4}-3\frac{3}{4}$ $6\frac{5}{4}-3\frac{3}{4}$	Rewrite the subtraction expression using the equivalent fractions.
	$6-3=3$ $\frac{5}{4}-\frac{3}{4}=\frac{2}{4}$	Subtract the whole numbers, subtract the fractions.
	$\frac{2}{4}=\frac{1}{2}$	Simplify the fraction
	$3\frac{1}{2}$	Combine the whole number and the fraction.
Answer	$7\frac{1}{4}-3\frac{3}{4}=3\frac{1}{2}$	

Sometimes a mixed number is subtracted from a whole number. In this case, you can also rewrite the whole number as a mixed number in order to perform the subtraction. You use an equivalent mixed number that has the same denominator as the fraction in the other mixed number.

	Example	
Problem	$8-4\frac{2}{5}$	**Subtract. Simplify the answer and write as a mixed number.**
	$8=7+1$ $7+\frac{5}{5}$ or $7\frac{5}{5}$	Regroup one from the whole number and write it as $\frac{5}{5}$.
	$7\frac{5}{5}-4\frac{2}{5}$	Rewrite the subtraction expression using the equivalent fractions.
	$7-4=3$ $\frac{5}{5}-\frac{2}{5}=\frac{3}{5}$	Subtract the whole numbers, subtract the fractions.
	$3\frac{3}{5}$	Combine the whole number and the fraction.
Answer	$8-4\frac{2}{5}=3\frac{3}{5}$	

Subtracting Mixed Numbers

If the fractional part of the mixed number being subtracted is larger than the fractional part of the mixed number from which it is being subtracted, or if a mixed number is being subtracted from a whole number, follow these steps:

1. Subtract 1 from the whole number part of the mixed number being subtracted.
2. Add that 1 to the fraction part to make an improper fraction. For example, $7\frac{2}{3}=6+\frac{3}{3}+\frac{2}{3}=6\frac{5}{3}$.
3. Then, subtract as with any other mixed numbers.

Alternatively, you can change both numbers to improper fractions and then subtract.

Self Check D

$15-13\frac{1}{4}$ Subtract. Simplify the answer and write as a mixed number.

Adding and Subtracting Fractions to Solve Problems

Objective **5**

Knowing how to add fractions is useful in a variety of situations. When reading problems, look for phrases that help you know you want to add the fractions.

Example

Problem **A stack of pamphlets is placed on top of a book. If the stack of pamphlets is $3\frac{1}{4}$ inches thick and the book is $5\frac{3}{4}$ inches thick, how high is the pile?**

$3\frac{1}{4}+5\frac{3}{4}$	Find the total height of the pile by adding the thicknesses of the stack of pamphlets and the book.
$3+5+\frac{1}{4}+\frac{3}{4}$	Group the whole numbers and fractions to make adding easier.
$8+\frac{1}{4}+\frac{3}{4}$	Add whole numbers.
$\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1$	Add fractions.
$8+1=9$	Combine whole number and fraction.

Answer The pile is 9 inches high.

Knowing how to subtract fractions and mixed numbers is useful in a variety of situations. When reading problems, look for key words that indicate that the problem can be solved using subtraction.

Example

Problem **Sherry loves to quilt, and she frequently buys fabric she likes when she sees it. She purchased 5 yards of blue print fabric and decided to use $2\frac{3}{8}$ yards of it in a quilt. How much of the blue print fabric will she have left over after making the quilt?**

$5-2\frac{3}{8}$	Write an expression using subtraction to describe the situation.
$4\frac{8}{8}-2\frac{3}{8}$	Rewrite the whole number as a mixed number.
$4\frac{8}{8}-2\frac{3}{8}=2\frac{5}{8}$	Subtract. Check that the mixed number is simplified.

Answer Sherry has $2\frac{5}{8}$ yards of blue print fabric left over.

Summary

Adding and subtracting fractions with like denominators involves adding or subtracting the numerators and keeping the denominator the same. Always simplify the answer. Adding mixed numbers involves adding the fractional parts, adding the whole numbers, and then recombining them as a mixed number.

When subtracting mixed numbers, if the fraction in the second mixed number is larger than the fraction in the first mixed number, rewrite the first mixed number by regrouping one whole as a fraction. Alternatively, rewrite all fractions as improper fractions and then subtract. This process is also used when subtracting a mixed number from a whole number.

2.5 Self Check Solutions

Self Check A

$\frac{7}{10}+\frac{8}{10}$ Add. Simplify the answer and write as a mixed number.

$\frac{7+8}{10}=\frac{15}{10}=1\frac{5}{10}=1\frac{1}{2}$.

Self Check B

$\frac{11}{16}-\frac{7}{16}$ Subtract and simplify the answer.

$\frac{11}{16}-\frac{7}{16}=\frac{11-7}{16}=\frac{4}{16}=\frac{1}{4}$

Self Check C

$3\frac{7}{9}+1\frac{4}{9}$ Add. Simplify the answer and write as a mixed number.

Adding the fractions: $\frac{7}{9}+\frac{4}{9}=\frac{11}{9}=1\frac{2}{9}$. Adding the whole numbers, 3+1 = 4. Combining these, $4+1\frac{2}{9}=5\frac{2}{9}$.

Self Check D

$15-13\frac{1}{4}$ Subtract. Simplify the answer and write as a mixed number.

Regrouping, $15=14+1=14+\frac{4}{4}=14\frac{4}{4}$

$15-13\frac{1}{4}=14\frac{4}{4}-13\frac{1}{4}$

Subtracting the whole numbers, 14-13 = 1. Subtracting fractions, $\frac{4}{4}-\frac{1}{4}=\frac{3}{4}$

$15-13\frac{1}{4}=1\frac{3}{4}$

2.6 Adding and Subtracting Fractions and Mixed Numbers with Unlike Denominators

Learning Objective(s)

1 Find the least common multiple (LCM) of two or more numbers.
2 Find the Least Common Denominator
3 Add fractions with unlike denominators.
4 Add mixed numbers
5 Subtract fractions with unlike denominators..
6 Subtract mixed numbers without regrouping.
7 Subtract mixed numbers with regrouping.
8 Solve application problems that require the subtraction of fractions or mixed numbers.

Finding Least Common Multiples

Objective **1**

Sometimes fractions do not have the same denominator. They have **unlike denominators**. Think about the example of the house painters. If one painter has $\frac{2}{3}$ can of paint and his painting partner has $\frac{1}{2}$ can of paint, how much do they have in total? How can you add these fractions when they do not have like denominators?

The answer is that you can rewrite one or both of the fractions so that they have the same denominator. This is called finding a **common denominator**. While any common denominator will do, it is helpful to find the **least common multiple** of the two numbers in the denominator because this will save having to simplify at the end. The least common multiple is the least number that is a multiple of two or more numbers. Least common multiple is sometimes abbreviated LCM.

There are several ways to find common multiples, some of which you used when comparing fractions. To find the least common multiple (LCM), you can list the multiples of each number and determine which multiples they have in common. The least of these numbers will be the least common multiple. Consider the numbers 4 and 6. Some of their multiples are shown below. You can see that they have several common multiples, and the *least* of these is 12.

4	8	**12**	16	20	**24**	28	32	**36**	40	44	**48**	52	56	**60**	64
6	**12**	18	**24**	30	**36**	42	**48**	54	**60**	66	68				

Example		
Problem	**Find the least common multiple of 30 and 50.**	
	30, 60, 90, 120, **150**, 180, 210, 240	List some multiples of 30.
	50, 100, **150**, 200, 250	List some multiples of 50.
	150 is found on both lists of multiples.	Look for the least number found on both lists.
Answer	The least common multiple of 30 and 50 is 150.	

The other method for finding the least common multiple is to use **prime factorization**. This is the method you need for working with rational expressions. The following shows how the factor method works with the numeric example, 4 and 6.

Start by finding the prime factorization of each denominator:

$$4 = 2 \bullet 2$$
$$6 = 3 \bullet 2$$

Identify the greatest number of times any factor appears in either factorization and multiply those factors to get the least common multiple. For 4 and 6, it would be:

$$3 \bullet 2 \bullet 2 = 12$$

Notice that 2 is included twice, because it appears twice in the prime factorization of 4. 12 is the least common multiple of 4 and 6.

The next example also shows how to use prime factorization.

Example		
Problem	**Find the least common multiple of 28 and 40.**	
	$28 = 2 \bullet 2 \bullet 7$	Write the prime factorization of 28.
	$40= 2 \bullet 2 \bullet 2 \bullet 5$	Write the prime factorization of 40.
	$2 \bullet 2 \bullet 2 \bullet 5 \bullet 7= 280$	Write the factors the greatest number of times they appear in either factorization and multiply.
Answer	The least common multiple of 28 and 40 is 280.	

Self Check A

Find the least common multiple of 12 and 80.

Finding Least Common Denominators

Objective **2**

You can use the least common multiple of two denominators as the **least common denominator** for those fractions. Then you rewrite each fraction using the same denominator.

The example below shows how to use the least common multiple as the least common denominator.

Example		
Problem	**Rewrite the fractions $\frac{2}{3}$ and $\frac{1}{2}$ as fractions with a least common denominator.**	
	Multiples of 3: 3, **6**, 9, 12 Multiples of 2: 2, 4, **6** 6 is the least common denominator.	Find the least common multiple of the denominators. This is the least common denominator.
	$\frac{2}{3} \cdot \frac{2}{2} = \frac{4}{6}$	Rewrite $\frac{2}{3}$ with a denominator of 6.
	$\frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$	Rewrite $\frac{1}{2}$ with a denominator of 6.
Answer	The fraction $\frac{2}{3}$ can be rewritten as $\frac{4}{6}$. The fraction $\frac{1}{2}$ can be rewritten as $\frac{3}{6}$.	

Self Check B

Find the least common denominator of $\frac{3}{4}$ and $\frac{1}{6}$. Then express each fraction using the least common denominator.

Adding Fractions with Unlike Denominators

Objective **3**

To add fractions with unlike denominators, first rewrite them with like denominators. Then, you know what to do! The steps are shown below.

Adding Fractions with Unlike Denominators

1. Find a common denominator.
2. Rewrite each fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify to lowest terms, expressing improper fractions as mixed numbers.

You can always find a common denominator by multiplying the two denominators together. See the example below.

Example		
Problem	$\frac{2}{3}+\frac{1}{5}$	**Add. Simplify the answer.**
	$3 \cdot 5 = 15$	Since the denominators are not alike, find a common denominator by multiplying the denominators.
	$\frac{2}{3}\cdot\frac{5}{5}=\frac{10}{15}$	Rewrite each fraction with a denominator of 15.
	$\frac{1}{5}\cdot\frac{3}{3}=\frac{3}{15}$	
	$\frac{10}{15}+\frac{3}{15}=\frac{13}{15}$	Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.
Answer	$\frac{2}{3}+\frac{1}{5}=\frac{13}{15}$	

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.

Example		
Problem	$\frac{3}{7}+\frac{2}{21}$	**Add. Simplify the answer.**
	Multiples of 7: 7, 14, **21** Multiples of 21: **21**	Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 7 and 21.
	$\frac{3}{7}\cdot\frac{3}{3}=\frac{9}{21}$ $\frac{2}{21}$	Rewrite each fraction with a denominator of 21.
	$\frac{9}{21}+\frac{2}{21}=\frac{11}{21}$	Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.
Answer	$\frac{3}{7}+\frac{2}{21}=\frac{11}{21}$	

You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

Example		
Problem	$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}$	**Add. Simplify the answer and write as a mixed number.**
	4 = 2 • 2 6 = 3 • 2 8 = 2 • 2 • 2 LCM: 2 • 2 • 2 • 3 = 24	Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 4, 6 and 8.
	$\frac{3}{4}\cdot\frac{6}{6}=\frac{18}{24}$ $\frac{1}{6}\cdot\frac{4}{4}=\frac{4}{24}$ $\frac{5}{8}\cdot\frac{3}{3}=\frac{15}{24}$	Rewrite each fraction with a denominator of 24.

	$\frac{18}{24}+\frac{4}{24}+\frac{15}{24}=\frac{37}{24}$	Add the fractions by adding the numerators and keeping the denominator the same.
	$\frac{37}{24}=1\frac{13}{24}$	Write the improper fraction as a mixed number and simplify the fraction.
Answer	$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}=1\frac{13}{24}$	

Self Check C

$\frac{2}{3}+\frac{4}{5}+\frac{1}{12}$ Add. Simplify the answer and write as a mixed number.

Adding Mixed Numbers

Objective **4**

When adding mixed numbers you may also need to find a common denominator first. Consider the example below.

Example		
Problem	$8\frac{5}{6}+7\frac{4}{9}$	*Add. Simplify the answer and* **write as a mixed number.**
	Multiples of 6: 6, 12, **18** Multiple of 9: 9, **18**	Find a least common denominator for the fractions.
	$\frac{5}{6}\cdot\frac{3}{3}=\frac{15}{18}$ $\frac{4}{9}\cdot\frac{2}{2}=\frac{8}{18}$	Express each fraction with a denominator of 18.
	$\begin{array}{r} 8\frac{15}{18} \\ +\ 7\frac{8}{18} \\ \hline \end{array}$	Arrange the mixed numbers vertically so the whole numbers align and the fractions align.

$$\begin{array}{r} 8\frac{15}{18} \\ +\ 7\frac{8}{18} \\ \hline 15\frac{23}{18} \end{array}$$

Add whole numbers. Add fractions.

$$\frac{23}{18}=1\frac{5}{18}$$

Write the improper fraction as a mixed number.

$$15+1+\frac{5}{18}$$

$$16\frac{5}{18}$$

Combine whole numbers and fraction to write a mixed number.

Answer

$$8\frac{5}{6}+7\frac{4}{9}=16\frac{5}{18}$$

Self Check D

$3\frac{3}{5}+1\frac{4}{9}$ Add. Simplify the answer and write as a mixed number.

Subtracting Fractions with Unlike Denominators

Objective **5**

If the denominators are not the same (they have **unlike denominators**), you must first rewrite the fractions with a common denominator. The **least common denominator**, which is the least common multiple of the denominators, is the most efficient choice, but any common denominator will do. Be sure to check your answer to be sure that it is in simplest form. You can use prime factorization to find the **least common multiple** (LCM), which will be the least common denominator (LCD). See the example below.

Example		
Problem	$\frac{1}{5}-\frac{1}{6}$	**Subtract. Simplify the answer.**
	$5 \bullet 6 = 30$	The fractions have unlike denominators, so you need to find a common denominator. Recall that a common denominator can be found by multiplying the two denominators together.
	$\frac{1}{5}\cdot\frac{6}{6}=\frac{6}{30}$	Rewrite each fraction as an equivalent fraction with a denominator of 30.
	$\frac{1}{6}\cdot\frac{5}{5}=\frac{5}{30}$	
	$\frac{6}{30}-\frac{5}{30}=\frac{1}{30}$	Subtract the numerators. Simplify the answer if needed.
Answer	$\frac{1}{5}-\frac{1}{6}=\frac{1}{30}$	

The example below shows using multiples to find the least common multiple, which will be the least common denominator.

Example		
Problem	$\frac{5}{6}-\frac{1}{4}$	**Subtract. Simplify the answer.**
	Multiples of 6: 6, **12,** 18, 24 Multiples of 4: 4, 8 **12,** 16, 20 12 is the least common multiple of 6 and 4.	Find the least common multiple of the denominators – this is the least common denominator.
	$\frac{5}{6}\cdot\frac{2}{2}=\frac{10}{12}$	Rewrite each fraction with a denominator of 12.
	$\frac{1}{4}\cdot\frac{3}{3}=\frac{3}{12}$	
	$\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$	Subtract the fractions. Simplify the answer if needed.
Answer	$\frac{5}{6}-\frac{1}{4}=\frac{7}{12}$	

Self Check E

$\frac{2}{3} - \frac{1}{6}$ Subtract and simplify the answer.

Subtracting Mixed Numbers

Objective 6

Sometimes you have to find a common denominator in order to solve a mixed number subtraction problem.

Example		
Problem	$7\frac{1}{2} - 2\frac{1}{3}$	**Subtract. Simplify the answer and write as a mixed number.**
	$2 \cdot 3 = 6$	Recall that a common denominator can easily be found by multiplying the denominators together.
	$\frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$ $\frac{1}{3} \cdot \frac{2}{2} = \frac{2}{6}$	Rewrite each fraction using the common denominator 6.
	$\frac{3}{6} - \frac{2}{6} = \frac{1}{6}$	Subtract the fractions.
	$7 - 2 = 5$	Subtract the whole numbers.
	$5\frac{1}{6}$	Combine the whole number and the fraction.
Answer	$7\frac{1}{2} - 2\frac{1}{3} = 5\frac{1}{6}$	

Self Check F

$9\frac{4}{5} - 4\frac{2}{3}$ Subtract. Simplify the answer and write it as a mixed number.

Subtracting Mixed Numbers with Regrouping

Objective **7**

The regrouping approach shown in the last section will also work with unlike denominators.

Example

Problem	$7\frac{1}{5}-3\frac{1}{4}$	**Subtract. Simplify the answer and write as a mixed number.**
	Multiples of 5: 5, 10, 15, **20,** 25 Multiples of 4: 4, 8, 12, 16, **20,** 24	Find a least common denominator. 20 is the least common multiple, so use it for the least common denominator.
	$\frac{1}{5}\cdot\frac{4}{4}=\frac{4}{20}$ and $\frac{1}{4}\cdot\frac{5}{5}=\frac{5}{20}$	Rewrite each fraction using the common denominator.
	$7\frac{4}{20}-3\frac{5}{20}$	Write the expression using the mixed numbers with the like denominator.
	$7\frac{4}{20}=6+1+\frac{4}{20}$ $6+\frac{20}{20}+\frac{4}{20}$ $6+\frac{24}{20}$ $6\frac{24}{20}$	Since the second fraction part, $\frac{5}{20}$, is larger than the first fraction part, $\frac{4}{20}$, regroup one of the whole numbers and write it as $\frac{20}{20}$.
	$7\frac{4}{20}-3\frac{5}{20}$ $6\frac{24}{20}-3\frac{5}{20}$	Rewrite the subtraction expression using the equivalent fractions.
	$6-3=3$ $\frac{24}{20}-\frac{5}{20}=\frac{19}{20}$	Subtract the whole numbers, subtract the fractions.
	$3\frac{19}{20}$	Combine the whole number and the fraction.
Answer	$7\frac{1}{5}-3\frac{1}{4}=3\frac{19}{20}$	

Self Check G

$6\frac{1}{2}-2\frac{5}{6}$ Subtract. Simplify the answer and write as a mixed number.

Adding and Subtracting Fractions to Solve Problems

Objective **8**

Example

Problem

A cake recipe requires $2\frac{1}{4}$ cups of milk and $1\frac{1}{2}$ cups of melted butter. If these are the only liquids, how much liquid is in the recipe?

$2\frac{1}{4}+1\frac{1}{2}$	Find the total amount of liquid by adding the quantities.
$2+1+\frac{1}{4}+\frac{1}{2}$	Group the whole numbers and fractions to make adding easier.
$3+\frac{1}{4}+\frac{1}{2}$	Add whole numbers.
$3+\frac{1}{4}+\frac{2}{4}=3+\frac{3}{4}$	Add fractions. Recall that $\frac{1}{2}=\frac{2}{4}$.
$3\frac{3}{4}$	Combine whole number and fraction.

Answer

There are $3\frac{3}{4}$ cups of liquid in the recipe.

Self Check H

What is the total rainfall in a three-day period if it rains $3\frac{1}{4}$ inches the first day, $\frac{3}{8}$ inch the second day, and $2\frac{1}{2}$ inches on the third day?

Example

Problem **Pilar and Farouk are training for a marathon. On a recent Sunday, they both completed a run. Farouk ran $12\frac{7}{8}$ miles and Pilar ran $14\frac{3}{4}$ miles. How many more miles did Pilar run than Farouk?**

$14\frac{3}{4}-12\frac{7}{8}$	Write an expression using subtraction to describe the situation.
$14\frac{6}{8}-12\frac{7}{8}$	Rewrite the mixed numbers using the least common denominator.
$14\frac{6}{8}$ $13+1+\frac{6}{8}$ $13+\frac{8}{8}+\frac{6}{8}$ $13\frac{14}{8}$	Since the fraction part of the second mixed number is larger than the fraction part of the first mixed number, regroup one as a fraction and rewrite the first mixed number.
$13\frac{14}{8}-12\frac{7}{8}$	Write the subtraction expression in its new form.
$1\frac{7}{8}$	Subtract.

Answer Pilar ran $1\frac{7}{8}$ miles more than Farouk.

Example

Problem

Mike and Jose are painting a room. Jose used $\frac{2}{3}$ of a can of paint and Mike used $\frac{1}{2}$ of a can of paint. How much more paint did Jose use? Write the answer as a fraction of a can.

$$\frac{2}{3}-\frac{1}{2}$$

Write an expression using subtraction to describe the situation.

$$\frac{2\bullet 2}{3\bullet 2}=\frac{4}{6}$$

$$\frac{1\bullet 3}{2\bullet 3}=\frac{3}{6}$$

Rewrite the fractions using a common denominator.

$$\frac{4}{6}-\frac{3}{6}=\frac{1}{6}$$

Subtract. Check that the fraction is simplified.

Answer

Jose used $\frac{1}{6}$ of a can more paint than Mike.

Self Check I

Mariah's sunflower plant grew $18\frac{2}{3}$ inches in one week. Her tulip plant grew $3\frac{3}{4}$ inches in one week. How many more inches did the sunflower grow in a week than the tulip?

Summary

To adding or subtracting fractions with unlike denominators, first find a common denominator. The least common denominator is easiest to use. The least common multiple can be used as the least common denominator.

2.6 Self Check Solutions

Self Check A

Find the least common multiple of 12 and 80.

240
2 • 2 • 2 • 2 • 3 • 5 = 240.

Self Check B

Find the least common denominator of $\frac{3}{4}$ and $\frac{1}{6}$. Then express each fraction using the least common denominator.

LCD: 12

$$\frac{3}{4}\cdot\frac{3}{3}=\frac{9}{12}, \quad \frac{1}{6}\cdot\frac{2}{2}=\frac{2}{12}$$

Self Check C

$\frac{2}{3}+\frac{4}{5}+\frac{1}{12}$ Add. Simplify the answer and write as a mixed number.

$$\frac{40}{60}+\frac{48}{60}+\frac{5}{60}=\frac{93}{60}=1\frac{33}{60}=1\frac{11}{20}.$$

Self Check D

$3\frac{3}{5}+1\frac{4}{9}$ Add. Simplify the answer and write as a mixed number.

$$3+1+\frac{3}{5}+\frac{4}{9} \quad 4+\frac{27}{45}+\frac{20}{45}=4+\frac{47}{45}= 4+1\frac{2}{45}= 5\frac{2}{45}.$$

Self Check E

$\frac{2}{3}-\frac{1}{6}$ Subtract and simplify the answer.

$$\frac{4}{6}-\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$$

Self Check F

$9\frac{4}{5}-4\frac{2}{3}$ Subtract. Simplify the answer and write it as a mixed number.

9 – 4 = 5; $\frac{4}{5}-\frac{2}{3}=\frac{12}{15}-\frac{10}{15}=\frac{2}{15}$. Combining them gives $5\frac{2}{15}$.

Self Check G

$6\frac{1}{2}-2\frac{5}{6}$ Subtract. Simplify the answer and write it as a mixed number.

$6\frac{1}{2}-2\frac{5}{6}=6\frac{3}{6}-2\frac{5}{6}$. Regrouping, $5\frac{9}{6}-2\frac{5}{6}=3\frac{4}{6}=3\frac{2}{3}$

Self Check H

What is the total rainfall in a three-day period if it rains $3\frac{1}{4}$ inches the first day, $\frac{3}{8}$ inch the second day, and $2\frac{1}{2}$ inches on the third day?

$6\frac{1}{8}$ inches

$3\frac{2}{8}+\frac{3}{8}+2\frac{4}{8}=5\frac{9}{8}=6\frac{1}{8}$

Self Check I

Mariah's sunflower plant grew $18\frac{2}{3}$ inches in one week. Her tulip plant grew $3\frac{3}{4}$ inches in one week. How many more inches did the sunflower grow in a week than the tulip?

$18\frac{2}{3}-3\frac{3}{4}=18\frac{8}{12}-3\frac{9}{12}=17\frac{20}{12}-3\frac{9}{12}=14\frac{11}{12}$ inches.

3.1.1 Decimals and Fractions

Learning Objective(s)
1 Read and write numbers in decimal notation.
2 Write decimals as fractions.
3 Write fractions as decimals.

Introduction

In addition to fraction notation, decimal notation is another way to write numbers between 0 and 1. Decimals can also be used to write numbers between any two whole numbers. For example, you may have to write a check for $2,003.38. Or, in measuring the length of a room, you may find that the length is between two whole numbers, such as 35.24 feet. In this topic you will focus on reading and writing decimal numbers, and rewriting them in fraction notation.

To read or write numbers written in decimal notation, you need to know the **place value** of each digit, that is, the value of a digit based on its position within a number. With decimal numbers, the position of a numeral in relation to the decimal point determines its place value. For example, the place value of the 4 in 45.6 is in the *tens* place, while the place value of 6 in 45.6 is in the *tenths* place.

Decimal Notation

Objective 1

Decimal numbers are numbers whose place values are based on 10s. Whole numbers are actually decimal numbers that are greater than or equal to zero. The place-value chart can be extended to include numbers less than one, which are sometimes called **decimal fractions**. A decimal point is used to separate the whole number part of the number and the fraction part of the number.

Let's say you are measuring the length of a driveway and find that it is 745 feet. You would say this number as *seven hundred forty-five.* Then, a more accurate measurement shows that it is 745.36 feet. Let's place this number in a place-value chart.

What you want to examine now are the place values of the decimal part, which are underlined in the chart below.

Decimal Numbers	Hundreds	Tens	Ones	Decimal Point	Tenths	Hundredths
745.36	7	4	5	.	3	6

Notice how the place-value names start from the decimal point. To the left of the decimal point are the ones, tens, and hundreds places, where you put digits that represent whole numbers that are greater than or equal to zero. To the right of the decimal point are the tenths and hundredths, where you put digits that represent numbers that are fractional parts of one, numbers that are more than zero and less than one.

Again, the place value of a number depends on how far away it is from the decimal point. This is evident in the chart below, where each number has the digit "4" occupying a different place value.

Decimal Numbers	Thousands	Hundreds	Tens	Ones	Decimal Point	Tenths	Hundredths	Thousandths
0.004				0	.	0	0	4
0.04				0	.	0	4	
0.4				0	.	4		
4				4	.			
40			4	0	.			
400		4	0	0	.			
4000	4	0	0	0	.			

Imagine that as a large balloon deflates, the volume of air inside it goes from 1,000 liters, to 100 liters, to 10 liters, to 1 liter. Notice that you're dividing a place value by ten as you go to the right. You divide 100 by 10 to get to the tens place. This is because there are 10 tens in 100. Then, you divide 10 by 10 to get to the ones place, because there are 10 ones in 10.

Now, suppose the balloon continues to lose volume, going from 1 liter, to 0.1 liters, to 0.01 liters, and then to 0.001 liters. Notice that you continue to divide by 10 when moving to decimals. You divide 1 by 10 ($\frac{1}{10}$) to get to the tenths place, which is basically breaking one into 10 pieces. And to get to the hundreds place, you break the tenth into ten more pieces, which results in the fraction $\frac{1}{100}$. The relationship between decimal places and fractions is captured in the table below.

Word Form	Decimal Notation	Fraction Notation
one thousand	1,000	$\frac{1,000}{1}$
one hundred	100	$\frac{100}{1}$
ten	10	$\frac{10}{1}$
one	1	$\frac{1}{1}$
one tenth	0.1	$\frac{1}{10}$
one hundredth	0.01	$\frac{1}{100}$
one thousandth	0.001	$\frac{1}{1,000}$

Consider a number with more digits. Suppose a fisherman has a net full of fish that weighs 1,357.924 kilograms. To write this number, you need to use the thousands place, which is made up of 10 hundreds. You also use the thousandths place, which is 1/10 of a hundredth. In other words, there are ten thousandths in one hundredth.

Decimal Numbers	no *th* side					*th* side		
	Thousands	Hundreds	Tens	Ones	Decimal point	Tenths	Hundredths	Thousandths
1,357.924	1	3	5	7	**.**	**9**	**2**	**4**

As you can see, moving from the decimal point to the left is ones, tens, hundreds, thousands, etc. This is the "no *th* side," which are the numbers greater than or equal to one. Moving from the decimal point to the right is tenths, hundredths, thousandths. This is the the "*th* side," which are the numbers less than 1.

1, 3 5 7 .9 2 4

no *th* side *th* side

The pattern going to the right or the left from the decimal point is the same – but there are two big differences:

1. The place values to the right of the decimal point all end in "th".
2. There is no such thing as "oneths." From your work with fractions, you know that 5 and $\frac{5}{1}$ are the same.

Example

Problem **What is the place value of 8 in 4,279.386?**

Decimal Numbers	no *th* side					*th* side		
	Thousands	Hundreds	Tens	Ones	Decimal point	Tenths	**Hundredths**	Thousandths
4,279.386	4	2	7	9	.	**3**	**8**	**6**

Write the number in a place-value chart. Read the value of the 8 from the chart.

Answer In the number 4,279.386, the 8 is in the hundredths place.

Self Check A
What is the place value of the 7 in 324.2671?

Reading Decimals

The easiest way to read a decimal number is to read the decimal fraction part as a fraction. (Don't simplify the fraction though.) Suppose you have 0.4 grams of yogurt in a cup. You would say, "4 tenths of a gram of yogurt," as the 4 is in the tenths place.

Note that the denominator of the fraction written in fraction form is always a power of ten, and the number of zeros in the denominator is the same as the number of decimal places to the right of the decimal point. See the examples in the table below for further guidance.

Decimal Notation	Fraction Notation	Word Form
0.5	$\frac{5}{10}$	five tenths
0.34	$\frac{34}{100}$	thirty-four hundredths
0.896	$\frac{896}{1{,}000}$	eight hundred ninety-six thousandths

Notice that 0.5 has *one* decimal place. Its equivalent fraction, $\frac{5}{10}$, has a denominator of 10—which is 1 followed by *one* zero. In general, when you are converting decimals to fractions, the denominator is always 1, followed by the number of zeros that correspond to the number of decimal places in the original number.

Another way to determine which number to place in the denominator is to use the place value of the last digit without the "ths" part. For example, if the number is 1.458, the 8 is in the thousandths place. Take away the "ths" and you have a thousand, so the number is written as $1\frac{458}{1{,}000}$.

Example

Problem **Write 0.68 in word form.**

0.68 = $\frac{68}{100}$ = sixty-eight hundredths

Note that the number is read as a fraction.

Also note that the denominator has 2 zeros, the same as the number of decimal places in the original number.

Answer The number 0.68 in word form is sixty-eight hundredths.

Recall that a **mixed number** is a combination of a whole number and a fraction. In the case of a decimal, a mixed number is also a combination of a whole number and a fraction, where the fraction is written as a decimal fraction.

To read mixed numbers, say the whole number part, the word "and" (representing the decimal point), and the number to the right of the decimal point, followed by the name and the place value of the last digit. You can see this demonstrated in the diagram below, in which the last digit is in the ten thousandths place.

One thousand four hundred seventy-five

1,475.3014

and three thousand fourteen *ten thousandths*

Another way to think about this is with money. Suppose you pay $15,264.25 for a car. You would read this as *fifteen thousand, two hundred sixty-four dollars and twenty-five cents*. In this case, the "cents" means "hundredths of a dollar," so this is the same as saying *fifteen thousand, two hundred sixty-four and twenty-five hundredths.* A few more examples are shown in the table below.

Decimal Notation	Fraction Notation	Word Form
9.4	$9\frac{4}{10}$	Nine and four tenths
87.49	$87\frac{49}{100}$	Eighty-seven and forty-nine hundredths
594.236	$594\frac{236}{1{,}000}$	Five hundred ninety-four and two hundred thirty-six thousandths

Example

Problem **Write 4.379 in word form.**

$4.379 = 4\frac{379}{1{,}000} =$ four and three hundred seventy-nine thousandths

The decimal fraction is read as a fraction.

Note that the denominator has 3 zeros, the same as the number of decimal places in the original number.

Answer The number 4.379 in word form is four and three hundred seventy-nine thousandths.

Self Check B
Write 2.364 in word form.

Objective **2**

Writing Decimals as Simplified Fractions

As you have seen above, every decimal can be written as a fraction. To convert a decimal to a fraction, place the number after the decimal point in the numerator of the fraction and place the number 10, 100, or 1,000, or another power of 10 in the denominator. For example, 0.5 would be written as $\frac{5}{10}$. You'll notice that this fraction can be further simplified, as $\frac{5}{10}$ reduces to $\frac{1}{2}$, which is the final answer.

Let's get more familiar with this relationship between decimal places and zeros in the denominator by looking at several examples. Notice that in each example, the number of decimal places is different.

Example		
Problem	**Write 0.6 as a simplified fraction.**	
	$0.6 = \frac{6}{10}$	The last decimal place is tenths, so use 10 for your denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal.
	$\frac{6 \div 2}{10 \div 2} = \frac{3}{5}$	Simplify the fraction.
Answer	$0.6 = \frac{3}{5}$	

Let's look at an example in which a number with two decimal places is written as a fraction.

Example		
Problem	**Write 0.64 as a simplified fraction.**	
	$0.64 = \frac{64}{100}$	The last decimal place is hundredths, so use 100 for your denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal.
	$\frac{64 \div 4}{100 \div 4} = \frac{16}{25}$	Simplify the fraction.
Answer	$0.64 = \frac{16}{25}$	

Now, examine how this is done in the example below using a decimal with digits in three decimal places.

Example

Problem	**Write 0.645 as a simplified fraction.**	
	$0.645 = \frac{645}{1000}$	Note that there are 3 zeros in the denominator, which is the same as the number of decimal places in the original decimal.
	$\frac{645 \div 5}{1000 \div 5} = \frac{129}{200}$	Simplify the fraction.
Answer	$0.645 = \frac{129}{200}$	

You can write a fraction as a decimal even when there are zeros to the right of the decimal point. Here is an example in which the only digit greater than zero is in the thousandths place.

Example

Problem	**Write 0.007 as a simplified fraction.**	
	$0.007 = \frac{7}{1000}$	Note that 7 is in the thousandths place, so you write 1,000 in the denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal. The fraction cannot be simplified further.
Answer	$0.007 = \frac{7}{1000}$	

When writing decimals greater than 1, you only need to change the decimal part to a fraction and keep the whole number part. For example, 6.35 can be written as $6\frac{35}{100}$.

Example

Problem **Write 8.65 as a simplified mixed fraction.**

$8.65 = 8\,\frac{65}{100} = 8\,\frac{13}{20}$

Rewrite 0.65 as $\frac{65}{100}$

Note that the number of zeros in the denominator is two, which is the same as the number of decimal places in the original decimal.

Then simplify $\frac{65}{100}$ by dividing numerator and denominator by 5.

Answer $8.65 = 8\frac{13}{20}$

Self Check C
Write 0.25 as a fraction.

Writing Fractions as Decimals

Objective 3

Just as you can write a decimal as a fraction, every fraction can be written as a decimal. To write a fraction as a decimal, divide the numerator (top) of the fraction by the denominator (bottom) of the fraction. Use long division, if necessary, and note where to place the decimal point in your answer. For example, to write $\frac{3}{5}$ as a decimal, divide 3 by 5, which will result in 0.6.

Example

Problem **Write $\frac{1}{2}$ as a decimal.**

$$\begin{array}{r} 0.5 \\ 2\overline{)1.0} \\ -1\,0 \\ \hline 0 \end{array}$$

Using long division, you can see that dividing 1 by 2 results in 0.5.

Answer $\frac{1}{2} = 0.5$

Note that you could also have thought about the problem like this: $\frac{1}{2} = \frac{?}{10}$, and then solved for ?. One way to think about this problem is that 10 is five times greater than 2, so ? will have to be five times greater than 1. What number is five times greater than 1? Five is, so the solution is $\frac{1}{2} = \frac{5}{10}$.

Now look at a more complex example, where the final digit of the answer is in the thousandths place.

Example		
Problem	**Write $\frac{3}{8}$ as a decimal.**	
	$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ -\underline{2\ 4} \\ 60 \\ -\ \underline{56} \\ 40 \\ -\ \underline{40} \\ 0 \end{array}$	Using long division, you can see that dividing 3 by 8 results in 0.375.
Answer	$\frac{3}{8}$ = 0.375	

Converting from fractions to decimals sometimes results in answers with decimal numbers that begin to repeat. For example, $\frac{2}{3}$ converts to 0.666, a repeating decimal, in which the 6 repeats infinitely. You would write this as $0.\overline{6}$, with a bar over the first decimal digit to indicate that the 6 repeats. Look at this example of a problem in which two consecutive digits in the answer repeat.

Example

Problem **Convert $\frac{4}{11}$ to a decimal.**

$$\begin{array}{r} 0.3636 \\ 11\overline{)4.0000} \\ -\underline{3\,3} \\ 70 \\ -\underline{66} \\ 40 \\ -\underline{33} \\ 70 \\ -\underline{66} \\ 4 \end{array}$$

$0.\overline{36}$

Using long division, you can see that dividing 4 by 11 results in 0.36 repeating. As a result, this is written with a line over it as $0.\overline{36}$.

Answer $\frac{4}{11} = 0.\overline{36}$

With numbers greater than 1, keep the whole number part of the mixed number as the whole number in the decimal. Then use long division to convert the fraction part to a decimal. For example, $2\frac{3}{20}$ can be written as 2.15.

Example

Problem **Convert $2\frac{1}{4}$ to a decimal.**

$$\begin{array}{r} 0.25 \\ 4\overline{)1.00} \\ \underline{-\ 8} \\ 20 \\ -\underline{20} \\ 0 \end{array}$$

2 + 0.25 = 2.25

Knowing that the whole number 2 will remain the same during the conversion, focus only on the decimal part. Using long division, you can see that dividing 1 by 4 results in 0.25.

Now bring back the whole number 2, and the resulting fraction is 2.25.

Answer $2\frac{1}{4} = 2.25$

Tips on Converting Fractions to Decimals

To write a fraction as a decimal, divide the numerator (top) of the fraction by the denominator (bottom) of the fraction.

In the case of repeating decimals, write the repeating digit or digits with a line over it. For example, 0.333 repeating would be written as $0.\overline{3}$.

Summary

Decimal notation is another way to write numbers that are less than 1 or that combine whole numbers with decimal fractions, sometimes called mixed numbers. When you write numbers in decimal notation, you can use an extended place-value chart that includes positions for numbers less than one. You can write numbers written in fraction notation (fractions) in decimal notation (decimals), and you can write decimals as fractions. You can always convert between fractional notation and decimal notation.

3.1.1 Self Check Solutions

Self Check A

What is the place value of the 7 in 324.2671?

The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.

Self Check B

Write 2.364 in word form.

two and three hundred sixty-four thousandths.

2.364 is the same as $2\frac{364}{1{,}000}$, so in addition to the whole number 2, you have three hundred sixty-four thousandths.

Self Check C

Write 0.25 as a fraction.

The number 0.25 can be written as $\frac{25}{100}$, which reduces to $\frac{1}{4}$.

3.2.1 Ordering and Rounding Decimals

Learning Objective(s)

1 Use a number line to assist with comparing decimals.
2 Compare decimals, beginning with their digits from left to right.
3 Use < or > to compare decimals.
4 Round a given decimal to a specified place.

Introduction

Decimal numbers are a combination of whole numbers and numbers between whole numbers. It is sometimes important to be able to compare decimals to know which is greater. For example, if someone ran the 100-meter dash in 10.57 seconds, and someone else ran the same race in 10.67 seconds, you can compare the decimals to determine which time is faster. Knowing how to compare decimals requires an understanding of decimal place value, and is similar to comparing whole numbers.

When working with decimals, there are times when a precise number isn't needed. When that's true, rounding decimal numbers is helpful. For example, if the pump at the gas station shows that you filled a friend's car with 16.478 gallons of gasoline, you may want to round the number and just tell her that you filled it with 16.5 gallons.

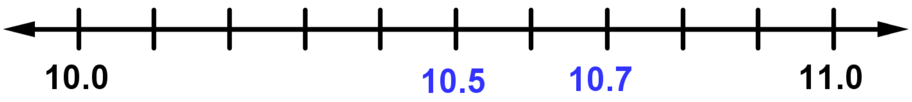
Comparing Decimals

Objective **1, 2, 3**

You can use a number line to compare decimals. The number that is further to the right is greater. Examine this method in the example below.

Example

Problem **Use < or > for [ ] to write a true sentence: 10.5 [ ] 10.7.**



Here, the digits in the *tenths* places differ. The numbers are both plotted on a number line ranging from 10.0 to 11.0. Because 10.5 is to the left of 10.7 on the number line, 10.5 is less than (<) 10.7.

Answer 10.5 < 10.7

Another approach to comparing decimals is to compare the digits in each number, beginning with the greatest place value, which is on the left. When one digit in a decimal number is greater than the corresponding digit in the other number, then that decimal number is greater.

For example, first compare the tenths digits. If they are equal, move to the hundredths place. If these digits are not equal, the decimal with the greater digit is the greater decimal number. Observe how this is done in the examples below.

Example

Problem **Use < or > for [ ] to write a true sentence: 35.689 [ ] 35.679.**

35.689 35.679

↑ ↑

In the hundredths place,
8 is greater than 7

Here, the numbers in the tens, ones, and tenths places, 35.6, are the same. However, the digits in the hundredths places differ. Because 8 is greater than 7, 35.689 is greater than 35.679.

Answer 35.689 > 35.679

If more than two digits in the two numbers differ, focus on the digit in the greatest place value. Look at this example in which two sums of money are compared.

Example

Problem **Use < or > for [ ] to write a true sentence: $45.67 [ ] $45.76.**

$45.67 $45.76

↑ ↑

In the tenths place,
6 is less than 7

Here, the place value that determines which amount is greater is not the hundredths place, but the tenths place. Because 6 tenths is less than 7 tenths, $45.67 is less than $45.76.

Answer $45.67 < $45.76

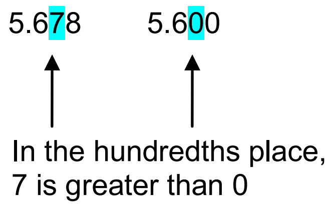
If one number has more decimal places than another, you may use 0's as placeholders in the number with fewer decimal places to help you compare. For example, if you are comparing 4.75 and 4.7, you may find it helpful to write 4.7 as 4.70 so that each number has three digits. Note that adding this extra 0 does not change the value of the decimal; you are adding 0 hundredths to the number. You can add placeholder 0's as long as you remember to add the zeros at the end of the number, to the right of the decimal point.

This is demonstrated in the example below.

Example

Problem	**Use < or > for [ ] to write a true sentence: 5.678 [ ] 5.6.**

Here, you can insert two zeros in the hundredths and thousandths places of 5.6 so that you have the same number of digits for each. Then, you can compare digits. Look for the greatest place value that has different values in each number. Here, it is the hundredths place.



Because 7 is greater than 0, 5.678 is greater than 5.600.

Answer	5.678 > 5.6

Strategies for Comparing Decimals

- Use a number line to assist with comparing decimals, as you did with whole numbers.
- Compare decimals beginning with their digits from left to right. When two digits are not equal, the one with the greater digit is the larger number.

Self Check A

Use < or > for [ ] to write a true sentence: 45.675 [ ] 45.649.

Rounding Decimals

Objective 4

Rounding with decimals is like rounding with whole numbers. As with whole numbers, you round a number to a given place value. Everything to the right of the given place value becomes a zero, and the digit in the given place value either stays the same or increases by one.

With decimals, you can “drop off” the zeros at the end of a number without changing its value. For example, 0.20 = 0.2, as $\frac{20}{100}$ simplifies to $\frac{2}{10}$. Of course, you cannot drop zeros before the decimal point: 200 ≠ 20.

The zeros that occur at the end of a decimal number are called **trailing zeros**.

Dropping Zeros with Whole Numbers and Decimals	
Dropping zeros at the end of a whole number changes the value of a number.	7200 ≠ 72 200,000 ≠ 2
Dropping zeros at the end of a decimal does not change the number’s value.	36.00 = 36 1.00000 = 1

One way to think of it is to consider the number “thirty-six dollars.” This amount can be written equally well one of two ways:

$36 = $36.00

Any zero at the very end of a decimal number can be dropped:

18.25000 = 18.2500 = 18.250 = 18.25

Example

Problem	**A sprinter ran a race in 7.354 seconds. What was the sprinter’s time, rounded to the nearest tenth of a second?**
$7.\underline{3}54$	Look at the first digit to the right of the tenths digit.
$7.3\underline{5}4 \rightarrow 7.400$	Since 5 = 5, round the 3 up to 4.
7.400	Change all digits to the right of the given place value into zeros. This is an intermediate step that you don’t actually write down.
7.4	Since 0.400 = 0.4, the zeros are not needed and should be dropped.
Answer	7.354 rounded to the nearest tenth is 7.4.

In the above example, the digit next to the selected place value is 5, so you round up. Let’s look at a case in which the digit next to the selected place value is less than 5.

Example

Problem **Round 7.354 to the nearest hundredth.**

$7.3\underline{5}4$	Look at the first digit to the right of the hundredths digit.
$7.354 \rightarrow 7.350$	Since 4 < 5, leave 5 as is.
7.35	The zeros to the right of the given place value are not needed and should be dropped.

Answer 7.354 rounded to the nearest hundredth is 7.35.

Sometimes you're asked to round a decimal number to a place value that is in the whole number part. Remember that you may not drop zeros to the left of the decimal point.

Example

Problem **Round 1,294.6374 to the nearest hundred.**

$1{,}\underline{2}94.6374$	Look at the first digit to the right of the hundreds place.
$1{,}294.6374 \rightarrow 1{,}300.000$	9 is greater than 5, so round the 2 up to 3.
1,300	Zeros to the left of the decimal must be included. Zeros to the right of the decimal can be dropped and should be dropped.

Answer 1,294.6374 rounded to the nearest hundred is 1,300.

Tip on Rounding Decimals

When you round, everything to the right of the given place value becomes a zero, and the digit in the given place value either stays the same or rounds up one. Trailing zeros after the decimal point should be dropped.

Self Check B

Round 10.473 to the nearest tenth.

Summary

Ordering, comparing, and rounding decimals are important skills. You can figure out the relative sizes of two decimal numbers by using number lines and by comparing the digits in the same place value of the two numbers. To round a decimal to a specific place value, change all digits to the right of the given place value to zero, and then round the digit in the given place value either up or down. Trailing zeros after the decimal point should be dropped.

3.1.2 Self Check Solutions

Self Check A

Use < or > for [ ] to write a true sentence: 45.675 [ ] 45.649.

45.675 > 45.645, because the 7 in 45.6**7**5 is greater than the corresponding 4 in 45.6**4**9.

Self Check B

Round 10.473 to the nearest tenth.

10.5
We round to the nearest tenth, and round up, because the 7 in 10.473 is greater than 5.

3.2 Adding and Subtracting Decimals

Learning Objective(s)

1 Add two or more decimals.
2 Subtract two or more decimals, with and without regrouping.
3 Solve application problems that require decimal addition or subtraction.

Introduction

Since dollars and cents are typically written as decimals, you often need to work with decimals. Knowing how to add and subtract **decimal numbers** is essential when you deposit money to (and withdraw money from) your bank account; perform an incorrect calculation, and you may be costing yourself some cash!

When adding or subtracting decimals, it is essential that you pay attention to the **place value** of the digits in the numbers you are adding or subtracting. This will be the key idea in the discussion that follows. Let's begin with an everyday example that illustrates this idea before moving into more general techniques.

Adding Decimals

Objective 1

Suppose Celia needs \$0.80 to ride the bus from home to her office. She reaches into her purse and pulls out the following coins: 3 quarters, 1 dime and 2 pennies. Does she have enough money to ride the bus?

Take a moment to think about this problem. Does she have enough money? Some people may solve it like this: "I know each quarter is 25¢, so three quarters is 75¢. Adding a dime brings me to 85¢, and then another two pennies is 87¢. So, Celia does have enough money to ride the bus."

This problem provides a good starting point for our conversation because you can use your knowledge about pocket change to understand the basics about how to add decimals. The coins you use every day can all be represented as whole cent values, as shown above. But they can also be represented as decimal numbers, too, because quarters, dimes, nickels and pennies are each worth less than one whole dollar.

Currency	**Value (cents)**	**Value (dollars)**
Dollar bill	100¢	\$1
Quarter	25¢	\$0.25
Dime	10¢	\$0.10
Nickel	5¢	\$0.05
Penny	1¢	\$0.01

Celia has 87¢. You can also write this amount in terms of the number of dollars she has: \$0.87. The table below shows a step-by-step approach to adding the coins in terms of cents and also as dollars. As you review the table, pay attention to the place values.

Coin combination	Value (cents)	Value (dollars)
Quarter	25¢	$0.25
Quarter	25¢	$0.25
Quarter	25¢	$0.25
Dime	10¢	$0.10
Penny	1¢	$0.01
+ Penny	+ 1¢	+ $0.01
Eighty-seven cents	87¢	$0.87

When you add whole numbers, as shown in the **Value (cents)** column above, you line up the numbers so that the digits in the ones place-value column are aligned.

In order to keep the numbers in the proper place-value column when adding decimals, align the decimal points. This will keep the numbers aligned; ones to ones, tenths to tenths, hundredths to hundredths, and so on. Look at the column titled **Value (dollars)**. You will see that place value is maintained, and that the decimal points align from top to bottom.

Adding Decimals

To add decimals:

- Align the decimal points, which will allow all the digits to be aligned according to their place values.
- Add just as you would add whole numbers, beginning on the right and progressing to the left.
- Write the decimal point in the sum, aligned with the decimal points in the numbers being added.

Example

Problem **Add. 0.23 + 4.5 + 20.32**

0.23 4.5 + 20.32	Write the numbers so that the decimal points are aligned.
0.23 4.50 + 20.32	*Optional:* Write an extra “0” at the end of 4.5 to keep the numbers in the correct position. (Adding this zero does not change the value of the decimal or the sum of the three numbers.)
0.23 4.50 + 20.32 25.05	Add. Begin at the right and move left. Align the decimal point in the sum with the decimal points in the numbers being added.

Answer 0.23 + 4.5 + 20.32 = 25.05

Example

Problem	**Add. 4.041 + 8 + 510.042**	
	$\begin{array}{r@{}l} 4&.041\\ 8&\\ \underline{+\ 510}&\underline{.042} \end{array}$	Write the numbers so that the decimal points are aligned.
	$\begin{array}{r@{}l} 4&.041\\ 8&.000\\ \underline{+\ 510}&\underline{.042} \end{array}$	As presented in the problem, the number 8 does not have a decimal point. You can rewrite this number as 8.0, 8.00, or 8.000 without changing the value of the number. Using 8.000 will allow you to align it with the other two numbers.
	$\begin{array}{r@{}l} 4&.041\\ 8&.000\\ \underline{+\ 510}&\underline{.042}\\ 522&.083 \end{array}$	Add. Begin at the right and move left. Align the decimal point in the sum with the decimal points in the numbers being added.
Answer	4.041 + 8 + 510.042 = 522.083	

Self Check A
Add: 0.08 + 0.156

Subtracting Decimals

Objective **2**

Subtracting decimals uses the same setup as adding decimals: line up the decimal points, and then subtract.

In cases where you are subtracting two decimals that extend to different place values, it often makes sense to add extra zeros to make the two numbers line up—this makes the subtraction a bit easier to follow.

Subtracting Decimals

To subtract decimals:

- Align the decimal points, which will allow all of the digits to be aligned according to their place values.
- Subtract just as you would subtract whole numbers, beginning on the right and progressing to the left.
- Align the decimal point in the difference directly below the decimal points in the numbers that were subtracted.

Example

Problem **Subtract. 39.672 – 5.431**

$$\begin{array}{r} 39.672 \\ \underline{-\ \ 5.431} \end{array}$$

Write the numbers so that the decimal points are aligned.

$$\begin{array}{r} 39.672 \\ \underline{-\ \ 5.431} \\ 34.241 \end{array}$$

Subtract. Begin at the right and move left.

Align the decimal point in the difference with the decimal points in the numbers being subtracted.

Answer 39.672 – 5.431 = 34.241

Example

Problem **Subtract. 0.9 – 0.027**

$$\begin{array}{r} 0.9 \\ \underline{-0.027} \end{array}$$

Write the numbers so that the decimal points are aligned.

$$\begin{array}{r} 0.900 \\ \underline{-0.027} \end{array}$$

Optional: Write two extra “0”s after 9. This will help you line up the numbers and perform the subtraction.

$$\begin{array}{r} 0.900 \\ \underline{-0.027} \end{array}$$

Regroup as needed.

$$\begin{array}{r} 0.8{}^{1}00 \\ \underline{-0.0\ 27} \end{array}$$

$$\begin{array}{r} 0.89{}^{1}0 \\ \underline{-0.02\ 7} \\ 0.87\ 3 \end{array}$$

Subtract.

Answer 0.9 – 0.027 = 0.873

Self Check B

Subtract. 43.21 – 8.1

Solving Problems

Objective **3**

In adding and subtracting decimals, you may have noticed that as long as you line up the decimal points in the numbers you are adding or subtracting, you can operate upon them as you would whole numbers.

Determining whether you need to add or subtract in a given situation is also straightforward. If two quantities are being combined, then add them. If one is being withdrawn from the other, then subtract them.

Example

Problem **Javier has a balance of $1,800.50 in his personal checking account. He pays two bills out of this account: a $50.23 electric bill, and a $70.80 cell phone bill.**

How much money is left in Javier's checking account after he pays these bills?

$\begin{array}{r} 1800.50 \\ -\underline{\ \ 50.23} \end{array}$	Since Javier is paying out money, you will subtract, starting with the electric bill.
$\begin{array}{r} 1800.50 \\ -\underline{\ \ 50.23} \\ 1750.27 \end{array}$	Align the decimals and subtract, regrouping as needed.
$\begin{array}{r} 1750.27 \\ -\underline{\ \ 70.80} \end{array}$	Javier has $1,750.27 remaining after he pays his electric bill. Next, subtract his cell phone bill, $70.80, from this new amount.
$\begin{array}{r} 1750.27 \\ -\underline{\ \ 70.80} \\ 1679.47 \end{array}$	Align the decimals and subtract, regrouping as needed.

Answer Javier has $1,679.47 left in his checking account after paying his bills.

Self Check C

Helene ran the 100-meter dash twice on Saturday. The difference between her two times was 0.3 seconds. Which pair of numbers below could have been her individual race times?

A) 14.22 and 14.25 seconds B) 14.22 and 17.22 seconds

C) 14.22 and 14.58 seconds D) 14.22 and 13.92 seconds

Summary

When adding or subtracting decimals, you must always align the decimal points, which will allow the place-value positions to fall in place. Then add or subtract as you do with whole numbers, regrouping as necessary. You can use these operations to solve real-world problems involving decimals, especially those with money.

3.2 Self Check Solutions

Self Check A

Add: 0.08 + 0.156

Line up the decimal points and then add. The correct answer is 0.236.

Self Check B

Subtract. 43.21 – 8.1

Line up the two numbers so that the decimal points are aligned, and then subtract. The difference is 35.11.

Self Check C

Helene ran the 100-meter dash twice on Saturday. The difference between her two times was 0.3 seconds. Which pair of numbers below could have been her individual race times?

14.22 and 13.92 seconds
14.22 – 13.92 = 0.3; the difference between Helene’s two race times is 0.3 seconds.

3.3.1 Multiplying Decimals

Learning Objective(s)

1 Multiply two or more decimals.
2 Multiply a decimal by a power of 10.
3 Circumference and area of a circle
4 Solve application problems that require decimal multiplication.

Introduction

As with whole numbers, sometimes you run into situations where you need to multiply or divide decimals. And just as there is a correct way to multiply and divide whole numbers, so, too, there is a correct way to multiply and divide decimals.

Imagine that a couple eats dinner at a Japanese steakhouse. The bill for the meal is $58.32—which includes a tax of $4.64. To calculate the tip, they can double the tax. So if they know how to multiply $4.64 by 2, the couple can figure out how much they should leave for the tip.

Here’s another problem. Andy just sold his van that averaged 20 miles per gallon of gasoline. He bought a new pickup truck and took it on a trip of 614.25 miles. He used 31.5 gallons of gas to make it that far. Did Andy get better gas mileage with the new truck?

Both of these problems can be solved by multiplying or dividing decimals. Here’s how to do it.

Multiplying Decimals

Objective **1**

Multiplying decimals is the same as multiplying whole numbers except for the placement of the decimal point in the answer. When you multiply decimals, the decimal point is placed in the **product** so that the number of decimal places in the product is the sum of the decimal places in the **factors**.

Let’s compare two multiplication problems that look similar: 214 • 36, and 21.4 • 3.6.

$$\begin{array}{r} 214 \\ \underline{\text{x}\ \ 36} \\ 1284 \\ \underline{6420} \\ \mathbf{7{,}704} \end{array} \qquad\qquad \begin{array}{r} 21.4 \\ \underline{\text{x}\ \ 3.6} \\ 1284 \\ \underline{6420} \\ \mathbf{77.04} \end{array}$$

Notice how the digits in the two solutions are exactly the same – the multiplication does not change at all. The difference lies in the placement of the decimal point in the final answers: 214 • 36 = 7,704, and 21.4 • 3.6 = 77.04.

To find out where to put the decimal point in a decimal multiplication problem, count the total number of decimal places in each of the factors.

21.4	the first factor has one decimal place
3.6	the second factor has one decimal place
77.04	the product will have 1 + 1 = 2 decimal places

Note that the decimal points do not have to be aligned as for addition and subtraction.

Example

Problem **3.04 • 6.1 = ?**

3.04		Set up the problem.
$\underline{\text{x 6.1}}$		
304		Multiply.
$\underline{18240}$		
18544		Add.
		Count the total number of decimal places in the factors and insert the decimal point in the product.
3.04	←	2 decimal places.
$\underline{\text{x 6.1}}$	←	1 decimal place.
304		
$\underline{18240}$		
18.544	←	3 decimal places.

Answer 3.04 • 6.1 = 18.544

Sometimes you may need to insert zeros in front of the product so that you have the right number of decimal places. See the final answer in the example below:

Example

Problem **0.037 • 0.08 = ?**

0.037	Set up the problem.
$\underline{\text{x 0.08}}$	
296	Multiply.
	Count the total number of decimal places in the factors and insert the decimal point in the product.

	0.037	←	3 decimal places.
	x 0.08	←	2 decimal places.
	0.00296	←	5 decimal places.
Answer	0.037 • 0.08 = 0.00296		Note that you needed to add zeros before 296 to get the 5 decimal places.

If one or more zeros occur on the right in the product, they are not dropped until after the decimal point is inserted.

Example			
Problem	**2.04 • 1.95 = ?**		
	2.04		Set up the problem.
	x 1.95		
	1020		Multiply.
	18360		
	20400		
	39780		Add.
	2.04	←	2 decimal places.
	x 1.95	←	2 decimal places.
	1020		
	18360		
	20400		
	3.9780	←	4 decimal places.
Answer	2.04 • 1.95 = 3.978		Answer can omit the final trailing 0.

Multiplying Decimals

To multiply decimals:

- Set up and multiply the numbers as you do with whole numbers.
- Count the total number of decimal places in both of the factors.
- Place the decimal point in the product so that the number of decimal places in the product is the sum of the decimal places in the factors.
- Keep all zeros in the product when you place the decimal point. You can drop the zeros on the right once the decimal point has been placed in the product. If the number of decimal places is greater than the number of digits in the product, you can insert zeros in front of the product.

Self Check A

Multiply. 51.2 • 3.08

Multiplying by Tens

Objective **2**

Take a moment to multiply 4.469 by 10. Now do 4.469 • 100. Finally, do 4.469 • 1,000. Notice any patterns in your products?

4.469	4.469	4.469
x 10	x 100	x 1000
44.690	446.900	4469.000

Notice that the products keep getting greater by one place value as the multiplier (10, 100, and 1,000) increases. In fact, the decimal point moves to the right by the same number of zeros in the power of ten multiplier.

4.469 • 10 = 44.69 ^ 4.469 • 100 = 446.9 ^ 4.469 • 1,000 = 4469**.** ^

You can use this observation to help you quickly multiply any decimal by a power of ten (10, 100, 1,000, etc).

Example		
Problem	**0.03 • 100 = ?**	
	0.03 • **100** = ?	100 has two zeros.
	0.03 • 100 = 3	Move the decimal point two places to the right to find the product.
Answer	0.03 • 100 = 3	

Multiplying a Decimal by a Power of Ten

To multiply a decimal number by a power of ten (such as 10, 100, 1,000, etc.), count the number of zeros in the power of ten. Then move the decimal point that number of places to the right.

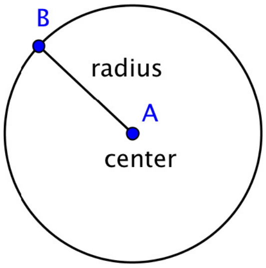
For example, 0.054 • 100 = 5.4. The multiplier 100 has two zeros, so you move the decimal point in 0.054 two places to the right—for a product of 5.4.

Circumference and Area of Circles

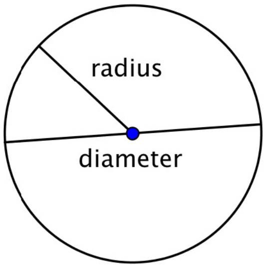
Circles are a common shape. You see them all over—wheels on a car, Frisbees passing through the air, compact discs delivering data. These are all circles.

A circle is a two-dimensional figure just like polygons and quadrilaterals. However, circles are measured differently than these other shapes—you even have to use some different terms to describe them. Let's take a look at this interesting shape.

A circle represents a set of points, all of which are the same distance away from a fixed, middle point. This fixed point is called the center. The distance from the center of the circle to any point on the circle is called the **radius**.



When two radii (the plural of radius) are put together to form a line segment across the circle, you have a **diameter**. The diameter of a circle passes through the center of the circle and has its endpoints on the circle itself.



The diameter of any circle is two times the length of that circle's radius. It can be represented by the expression $2r$, or "two times the radius." So if you know a circle's radius, you can multiply it by 2 to find the diameter; this also means that if you know a circle's diameter, you can divide by 2 to find the radius.

The distance around a circle is called the **circumference**. (Recall, the distance around a polygon is the perimeter.)

One interesting property about circles is that the ratio of a circle's circumference and its diameter is the same for all circles. No matter the size of the circle, the ratio of the circumference and diameter will be the same.

Some actual measurements of different items are provided below. The measurements are accurate to the nearest millimeter or quarter inch (depending on the unit of measurement used). Look at the ratio of the circumference to the diameter for each one—although the items are different, the ratio for each is approximately the same.

Item	**Circumference (*C*) (rounded to nearest hundredth)**	**Diameter (*d*)**	**Ratio $\frac{C}{d}$**
Cup	253 mm	79 mm	$\frac{253}{79} = 3.2025...$
Quarter	84 mm	27 mm	$\frac{84}{27} = 3.1111...$
Bowl	37.25 in	11.75 in	$\frac{37.25}{11.75} = 3.1702...$

The circumference and the diameter are approximate measurements, since there is no precise way to measure these dimensions exactly. If you were able to measure them more precisely, however, you would find that the ratio $\frac{C}{d}$ would move towards 3.14 for each of the items given. The mathematical name for the ratio $\frac{C}{d}$ is **pi**, and is represented by the Greek letter π.

π is a non-terminating, non-repeating decimal, so it is impossible to write it out completely. The first 10 digits of π are 3.141592653; it is often rounded to 3.14 or estimated as the fraction $\frac{22}{7}$. Note that both 3.14 and $\frac{22}{7}$ are *approximations* of π, and are used in calculations where it is not important to be precise.

Since you know that the ratio of circumference to diameter (or π) is consistent for all circles, you can use this number to find the circumference of a circle if you know its diameter.

$$\frac{C}{d} = \pi \text{, so } C = \pi d$$

Also, since $d = 2r$, then $C = \pi d = \pi(2r) = 2\pi r$.

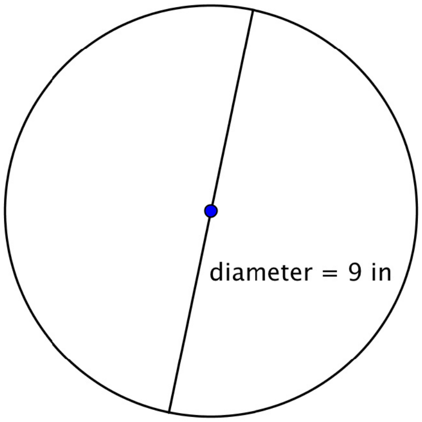
Circumference of a Circle

To find the circumference (C) of a circle, use one of the following formulas:

If you know the diameter (d) of a circle: $C = \pi d$
If you know the radius (r) of a circle: $C = 2\pi r$

Example

Problem **Find the circumference of the circle.**



$C = \pi d$ $C = \pi \bullet 9$ $C \approx 3.14 \bullet 9$ $C \approx 28.26$	To calculate the circumference given a diameter of 9 inches, use the formula $C = \pi d$. Use 3.14 as an approximation for π . Since you are using an approximation for π , you cannot give an exact measurement of the circumference. Instead, you use the symbol $\approx$ to indicate "approximately equal to."

Answer The circumference is 9π or approximately 28.26 inches.

Example

Problem **Find the circumference of a circle with a radius of 2.5 yards.**

$C = 2\pi r$ $C = 2\pi \bullet 2.5$ $C = \pi \bullet 5$ $C \approx 3.14 \bullet 5$ $C \approx 15.7$	To calculate the circumference of a circle given a radius of 2.5 yards, use the formula $C = 2\pi r$. Use 3.14 as an approximation for π .

Answer The circumference is 5π or approximately 15.7 yards.

Self Check D

A circle has a radius of 8 inches. What is its circumference, rounded to the nearest inch?

Area

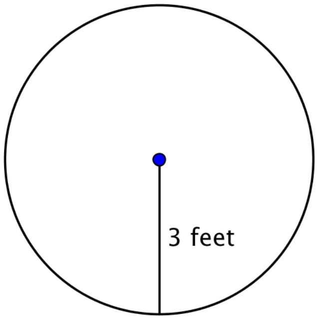
π is an important number in geometry. You have already used it to calculate the circumference of a circle. You use π when you are figuring out the *area* of a circle, too.

Area of a Circle

To find the area (*A*) of a circle, use the formula: $A = \pi r^2$

Example

Problem **Find the area of the circle.**



$$A = \pi r^2$$
$$A = \pi \bullet 3^2$$
$$A = \pi \bullet 9$$
$$A \approx 3.14 \bullet 9$$
$$A \approx 28.26$$

To find the area of this circle, use the formula $A = \pi r^2$.

Remember to write the answer in terms of square units, since you are finding the area.

Answer The area is 9π or approximately 28.26 feet^2.

Self Check E
A button has a diameter of 20 millimeters. What is the area of the button? Use 3.14 as an approximation of π.

Solving Problems by Multiplying Decimals

Objective 4

Now let's return to the first of the problems from the beginning of this section. You know how to multiply decimals now. Let's put that knowledge to the test.

Example

Problem **A couple eats dinner at a Japanese steakhouse. The bill for the meal totals $58.32—which includes a tax of $4.64. To calculate the tip, they can double the tax. How much tip should the couple leave?**

$\begin{array}{r} 4.64 \\ \underline{x\ \ \ 2} \end{array}$	Set up a multiplication problem.
$\begin{array}{r} 4.64 \\ \underline{x\ \ \ 2} \\ 928 \end{array}$	Multiply.
$\begin{array}{r} 4.64 \\ \underline{x\ \ \ 2} \\ 9.28 \end{array}$	Count the number of decimal places in the two factors, and place the decimal point accordingly.

Answer The couple should leave a tip of $9.28.

3.3.1 Self Check Solutions

Self Check A

Multiply. 51.2 • 3.08

To find the product, multiply 512 • 308 = 157696. Count the total number of decimal places in the factors, 3, and then place a decimal point in the product so that the product has three decimal places as well. The answer is 157.696.

Self Check B

A circle has a radius of 8 inches. What is its circumference, rounded to the inch?

$2\pi \bullet 8 = 50.24$. Rounded to the nearest inch is 50 inches.

Self Check C

A button has a diameter of 20 millimeters. What is the area of the button? Use 3.14 as an approximation of π.

The radius will be 10 millimeters. The area will be
$\pi(10)^2 = 3.14 \bullet 100 = 314$ square millimeters

3.3.2 Dividing Decimals

Learning Objective(s)

1 Divide by a decimal.
2 Divide a decimal by a power of 10.
3 Solve application problems that require decimal division.

Dividing Decimals

Objective 1

To divide decimals, you will once again apply the methods you use for dividing whole numbers. Look at the two problems below. How are the methods similar?

```
        8 6 7              8.6 7
3 ) 2 6 0 1        3 ) 2 6.0 1
   -2 4               -2 4
      2 0                2 0
     -1 8               -1 8
        2 1                2 1
      - 2 1              - 2 1
          0                  0
```

Notice that the division occurs in the same way—the only difference is the placement of the decimal point in the **quotient**.

Example

Problem **18.32 ÷ 8 = ?**

```
8 ) 1 8.3 2
```

Set up the problem.

```
         2.2 9
8 ) 1 8.3 2
   -1 6
      2 3
     -1 6
        7 2
      - 7 2
          0
```

Divide.

```
         2.2 9
8 ) 1 8.3 2
```

Place decimal point in the quotient. It should be placed directly above the decimal point in the dividend.

Answer 18.32 ÷ 8 = 2.29

But what about a case where you are dividing by a decimal, as in the problem below?

```
     __________
0.3 ) 2 6 0.1
```

In cases like this, you can use powers of 10 to help create an easier problem to solve. In this case, you can multiply the **divisor**, 0.3, by 10 to move the decimal point 1 place to the right. If you multiply the divisor by 10, then you also have to multiply the **dividend** by 10 to keep the quotient the same. The new problem, with its solution, is shown below.

Example

Problem **260.1 ÷ 0.3 = ?**

```
      ________
0.3 ) 2 6 0.1
```

Set up the problem.

```
      __________
 3. ) 2 6 0 1.
```

Multiply divisor and dividend by 10 to create a whole number divisor.

```
        8 6 7
    _________
3 ) 2 6 0 1
   -2 4
    ‾‾‾
      2 0
     -1 8
      ‾‾‾
        2 1
      - 2 1
      ‾‾‾‾‾
          0
```

Divide.

Answer 260.1 ÷ 0.3 = 867

Often, the dividend will still be a decimal after multiplying by a power of 10. In this case, the placement of the decimal point must align with the decimal point in the dividend.

Example

Problem **15.275 ÷ 3.25 = ?**

```
3.2 5 ) 1 5.2 7 5
```

Set up the problem.

```
3 2 5.) 1 5 2 7.5
```

Multiply divisor and dividend by 100 to create a whole number divisor.

```
              4.7
3 2 5.) 1 5 2 7.5
       -1 3 0 0
          2 2 7 5
         -2 2 7 5
                0
```

Divide. 325 goes into 1527 four times, so the number 4 is placed above the digit 7.

The decimal point in the quotient is placed directly above the decimal point in the dividend.

Answer 15.275 ÷ 3.25 = 4.7

Dividing Decimals

Dividing Decimals by Whole Numbers
Divide as you would with whole numbers. Then place the decimal point in the quotient directly above the decimal point in the dividend.

Dividing by Decimals
To divide by a decimal, multiply the divisor by a power of ten to make the divisor a whole number. Then multiply the dividend by the same power of ten. You can think of this as moving the decimal point in the dividend the same number of places to the right as you move the decimal point in the divisor.

Then place the decimal point in the quotient directly over the decimal point in the dividend. Finally, divide as you would with whole numbers.

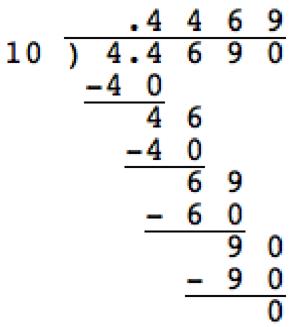
Self Check B
Divide: 25.095 ÷ 0.5.

Dividing by Tens

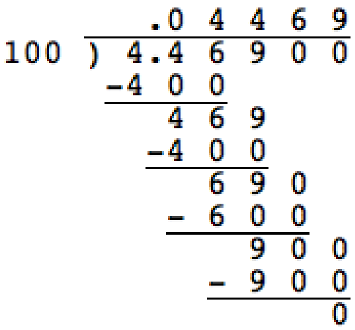
Objective **2**

Recall that when you multiply a decimal by a power of ten (10, 100, 1,000, etc), the placement of the decimal point in the product will move to the right according to the number of zeros in the power of ten. For instance, 4.12 • 10 = 41.2.

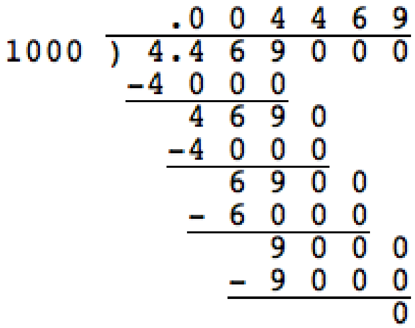
Multiplication and division are inverse operations, so you can expect that if you divide a decimal by a power of ten, the decimal point in the quotient will also correspond to the number of zeros in the power of ten. The difference is that the decimal point moves to the right when you multiply; it moves to the left when you divide.



```
          .4 4 6 9
      ___________
10 ) 4.4 6 9 0
    -4 0
    ----
       4 6
      -4 0
      ----
         6 9
       - 6 0
       -----
           9 0
         - 9 0
         -----
             0
```



```
            .0 4 4 6 9
       _______________
100 ) 4.4 6 9 0 0
     -4 0 0
     ------
        4 6 9
       -4 0 0
       ------
          6 9 0
        - 6 0 0
        -------
            9 0 0
          - 9 0 0
          -------
                0
```



```
              .0 0 4 4 6 9
        ___________________
1000 ) 4.4 6 9 0 0 0
      -4 0 0 0
      --------
         4 6 9 0
        -4 0 0 0
        --------
           6 9 0 0
         - 6 0 0 0
         ---------
             9 0 0 0
           - 9 0 0 0
           ---------
                   0
```

In the examples above, notice that each quotient still contains the digits 4469—but as another 0 is added to the end of each power of ten in the divisor, the decimal point moves an additional place to the left in the quotient.

Dividing by Powers of Ten

To divide a decimal by a power of ten (10, 100, 1,000, etc.), count the number of zeros in the divisor. Then move the decimal point in the dividend that number of decimal places to the left; this will be your quotient.

Example

Problem **31.05 ÷ 10 = ?**

31.05 ÷ **10** = ?	10 has one zero.
31.05 ÷ 10 = 3.105	Move the decimal point one place to the left in the dividend; this is the quotient.

Answer 31.05 ÷ 10 = 3.105

Self Check C

Divide. 0.045 ÷ 100

Solving Problems by Dividing Decimals

Objective **6**

Example

Problem **Andy just sold his van that averaged 20 miles per gallon of gasoline. He bought a new pickup truck and took it on a trip of 614.25 miles. He used 31.5 gallons of gas for the trip. Did Andy get better gas mileage with the new truck?**

```
          ___________
3 1.5 )   6 1 4.2 5
```

Set up a division problem.

```
          ___________
3 1 5.)   6 1 4 2.5
```

Make the divisor a whole number by multiplying by 10; do the same to the dividend.

```
              1 9.5
          ___________
3 1 5.)   6 1 4 2.5
         -3 1 5
          2 9 9 2
         -2 8 3 5
            1 5 7 5
           -1 5 7 5
                  0
```

Divide. Insert a decimal point in the quotient so that it is directly above the decimal point in the dividend.

Answer Andy gets 19.5 miles per gallon now. He used to get 20 miles per gallon. He does not get better gas mileage with the new truck.

Example

Problem **Zoe is training for a race. Her last 5 times were 15.2, 17.5, 16.3, 18.1, and 17.8 seconds. Find her mean time.**

```
   16.3
   15.2
   17.5
   16.3
   18.1
 + 17.8
   83.1
```

Recall that to find the mean, we add the values and divide by the number of values.

Start by adding the 5 times.

```
       1 6.6 4
5.)    8 3.1 0
      -5
       3 3
       3 0
         3 2
        -3 0
           2 0
          -2 0
             0
```

Now divide by 5 to find the mean. Insert a decimal point in the quotient so that it is directly above the decimal point in the dividend.

Answer Her mean time is 16.64 seconds

Summary

Learning to multiply and divide with decimals is an important skill. In both cases, you work with the decimals as you have worked with whole numbers, but you have to figure out where the decimal point goes. When multiplying decimals, the number of decimal places in the product is the sum of the decimal places in the factors. When dividing by decimals, move the decimal point in the dividend the same number of places to the right as you move the decimal point in the divisor. Then place the decimal point in the quotient above the decimal point in the dividend.

3.3.2 Self Check Solutions

Self Check B

Divide: 25.095 ÷ 0.5.

This problem can be set up as 250.95 ÷ 5; the quotient is 50.19.

Self Check C

Divide. 0.045 ÷ 100

There are two zeros in the divisor (100), so to find the quotient, take the dividend (0.045) and move the decimal point two places to the left. The quotient is 0.00045.

3.3.3 Estimation with Decimals

Learning Objective(s)
1 Estimate the answer to a decimal problem.

Introduction

Being able to **estimate** your answer is a very useful skill. Not only will it help you decide if your answer is reasonable when doing homework problems or answering test questions, it can prove to be very helpful in everyday life. When shopping, you can estimate how much money you have spent, the tip for a restaurant bill, or the price of an item on sale. By **rounding** and then doing a quick calculation, you will at least know if you are close to the exact answer.

Estimating with Decimals

Objective 1

Consider this problem. Stewart wanted to buy a DVD home theater system that cost $345.23. He also wanted a universal remote priced at $32.90. He used a calculator to add the costs and the sum that he got was $674.23. He was surprised!

Whether Stewart can afford to spend $674.23 or not is not really the problem here. Rather, the problem is whether the system and remote would total $674.23. Round both item costs to the tens place: $345.23 is about $350, and $32.90 is about $30. Is $350 + $30 close to $674.23? Of course not!

Estimating an answer is a good skill to have. Even when using a calculator, you can get an incorrect answer by accidentally pressing a wrong button. When decimals are involved, it's very easy to put the decimal point in the wrong place, and then your numbers can be drastically wrong.

Example

Problem	**Hakim wrote checks for $64.20, $47.89, and $95.80. Estimate the total of all three checks.**	
		To estimate the total, first round each of the check values. You want to round to the nearest $10 in this example.
	6**4**.20 → 60 4**7**.89 → 50 9**5**.80 → 100	Since 4 < 5, round to 60. Since 7 > 5, round up to 50. Since 5 = 5, round up to 100.
	60 + 50 + 100 = 210	Add the estimates to find the estimated total.
Answer	The total estimate for the three checks is $210.	

You can use estimation to see if you have enough money for a purchase. In this case, it is best to round all the numbers up to make sure that you have enough money.

Example		
Problem	**Sherry has $50 and wants to buy CDs that cost $11.50 each. About how many CDs can she buy?**	
		Round $11.50 to the nearest whole number.
	11.50 → 12.00	Since you want to make sure that Sherry has enough money, round up to 12.00 or 12.
	50 ÷ 12 = 4 R2	Divide. The amount of the remainder is not important.
Answer	Sherry can buy about 4 CDs with $50.	

You will generally estimate when you compute the amount of tip to leave when you eat at a restaurant. Recall that an easy way to compute the tip is to double the tax. You can probably do this in your head, if you estimate this product by rounding to the nearest $1. You can round up if the service is good or round down if not.

Example		
Problem	**After a delicious meal at a restaurant, the bill for two is $45.36, which includes tax of $3.74. The service was very good. How much tip would you leave if you follow the rule to double the amount of tax?**	
	3.74 → 4.00	Round up to $4.00, or $4, as the service was good.
	4 • 2 = 8	Multiply the rounded number by 2 to double this number.
Answer	The tip for good service would be $8.	

When using rounding in addition and subtraction problems, you usually round all numbers to the same place value—this makes adding or subtracting a bit easier. However, when you use rounding to help you multiply or divide numbers, it’s usually better to round the numbers so they each have only one or two digits that are not 0. This is shown in the example below.

Example		
Problem	**Jin is building a model ship based on a real one. Each length of the model is 0.017 times the actual length of the ship. The real ship is 132 feet long. Estimate the length of the model, then use a calculator to find the actual length of the model.**	
		The scale of the model is 0.017. It wouldn't make sense to round this value to a whole number or even tenths, but you can round it to hundredths.
	0.017 → **0.02**	Since 7 > 5, round up to 0.020 or just 0.02.
		You will round 132 to the tens to make it easier to work with.
	132 → **130**	Since 2 < 5, round to 130.
	130 • 0.02	Multiply 130 • 2 = 260. Then place the decimal point.
	130 • 2 = 260 130 • 0.02 = 2.60	
Answer	The model length is about 2.6 ft, or exactly 2.244 ft.	Use a calculator to find the exact length. The estimate is fairly accurate!

Self Check A

Evelyn is purchasing 287 ceramic tiles for her new kitchen. Each one costs $0.21. Which of the following is the most accurate estimate for the cost of purchasing the tiles for her kitchen?

Summary

Estimation is useful when you don't need an exact answer. It also lets you check to be sure an exact answer is close to being correct. Estimating with decimals works just the same as estimating with whole numbers. When rounding the values to be added, subtracted, multiplied, or divided, it helps to round to numbers that are easy to work with.

3.3.3 Self Check Solutions

Self Check A

Evelyn is purchasing 287 ceramic tiles for her new kitchen. Each one costs $0.21. Which of the following is the most accurate estimate for the cost of purchasing the tiles for her kitchen?

287 rounds to 300, and 0.21 rounds to 0.20, to arrive at 300 • 0.20 = 60 or $60.

3.4.1 Convert Percents, Decimals, and Fractions

Learning Objective(s)

1 Describe the meaning of percent.

2 Represent a number as a decimal, percent, and fraction.

Introduction

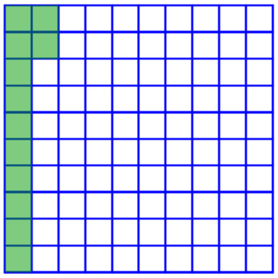
Three common formats for numbers are fractions, decimals, and percents. Percents are often used to communicate a *relative* amount. You have probably seen them used for discounts, where the percent of discount can apply to different prices. Percents are also used when discussing taxes and interest rates on savings and loans.

The Meaning of Percent

Objective **1**

A **percent** is a ratio of a number to 100. *Per cent* means "per 100," or "how many out of 100." You use the symbol % after a number to indicate percent.

Notice that 12 of the 100 squares in the grid below have been shaded green. This represents 12 percent (12 per 100).

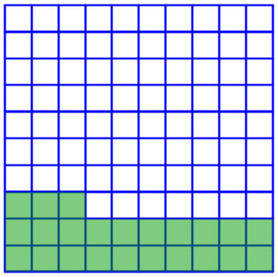


$$12\% = 12 \text{ percent} = 12 \text{ parts out of } 100 = \frac{12}{100}$$

How many of the squares in the grid above are unshaded? Since 12 are shaded and there are a total of 100 squares, 88 are unshaded. The unshaded portion of the whole grid is 88 parts out of 100, or 88% of the grid. Notice that the shaded and unshaded portions together make 100% of the grid (100 out of 100 squares).

Example

Problem **What percent of the grid is shaded?**



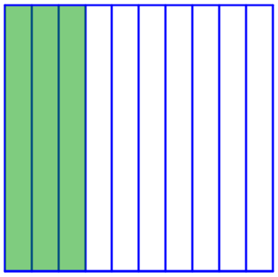
The grid is divided into 100 smaller squares, with 10 squares in each row.

23 squares out of 100 squares are shaded.

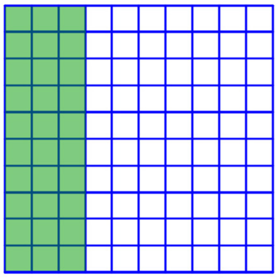
Answer 23% of the grid is shaded.

Example

Problem **What percent of the large square is shaded?**



The grid is divided into 10 rectangles. For percents, you need to look at 100 equal-sized parts of the whole. You can divide each of the 10 rectangles into 10 pieces, giving 100 parts.

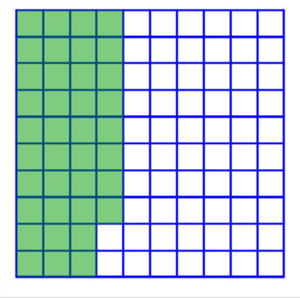


30 small squares out of 100 are shaded.

Answer 30% of the large square is shaded.

Self Check A

What percent of this grid is shaded?



Rewriting Percents, Decimals, and Fractions

Objective **2**

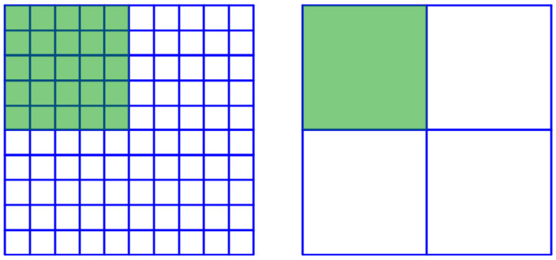
It is often helpful to change the format of a number. For example, you may find it easier to add decimals than to add fractions. If you can write the fractions as decimals, you can add them as decimals. Then you can rewrite your decimal sum as a fraction, if necessary.

Percents can be written as fractions and decimals in very few steps.

Example

Problem	**Write 25% as a simplified fraction and as a decimal.**	
Write as a fraction.	$25\% = \frac{25}{100}$	Since % means “out of 100,” 25% means 25 out of 100. You write this as a fraction, using 100 as the denominator.
	$\frac{25}{100} = \frac{25 \div 25}{100 \div 25} = \frac{1}{4}$	Simplify the fraction by dividing the numerator and denominator by the common factor 25.
Write as a decimal.	$25\% = \frac{25}{100} = 0.25$	You can also just move the decimal point in the whole number 25 two places to the left to get 0.25.
Answer	$25\% = \frac{1}{4} = 0.25$	

Notice in the diagram below that 25% of a grid is also $\frac{1}{4}$ of the grid, as you found in the example.



Notice that in the previous example, rewriting a percent as a decimal takes just a shift of the decimal point. You can use fractions to understand why this is the case. Any percentage x can be represented as the fraction $\frac{x}{100}$, and any fraction $\frac{x}{100}$ can be written as a decimal by moving the decimal point in x two places to the left. For example, 81% can be written as $\frac{81}{100}$, and dividing 81 by 100 results in 0.81. People often skip over the intermediary fraction step and just convert a percent to a decimal by moving the decimal point two places to the left.

In the same way, rewriting a decimal as a percent (or as a fraction) requires few steps.

Example

Problem	**Write 0.6 as a percent and as a simplified fraction.**	
Write as a percent.	0.6 = 0.60 = 60%	Write 0.6 as 0.60, which is 60 hundredths. 60 hundredths is 60 percent. You can also move the decimal point two places to the *right* to find the percent equivalent.
Write as a fraction.	$0.6 = \frac{6}{10}$	To write 0.6 as a fraction, you read the decimal, 6 tenths, and write 6 tenths in fraction form.
	$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$	Simplify the fraction by dividing the numerator and denominator by 2, a common factor.
Answer	$0.6 = 60\% = \frac{3}{5}$	

In this example, the percent is not a whole number. You can handle this in the same way, but it’s usually easier to convert the percent to a decimal and then convert the decimal to a fraction.

Example

Problem	**Write 5.6% as a decimal and as a simplified fraction.**	
Write as a decimal.	$5.6\% = 0.056$	Move the decimal point two places to the left. In this case, insert a 0 in front of the 5 (05.6) in order to be able to move the decimal to the left two places.
Write as a fraction.	$0.056 = \dfrac{56}{1{,}000}$	Write the fraction as you would read the decimal. The last digit is in the *thousandths* place, so the denominator is 1,000.
	$\dfrac{56}{1{,}000} = \dfrac{56 \div 8}{1{,}000 \div 8} = \dfrac{7}{125}$	Simplify the fraction by dividing the numerator and denominator by 8, a common factor.
Answer	$5.6\% = \dfrac{7}{125} = 0.056$	

Self Check B

Write 0.645 as a percent and as a simplified fraction.

In order to write a fraction as a decimal or a percent, you can write the fraction as an equivalent fraction with a denominator of 10 (or any other power of 10 such as 100 or 1,000), which can be then converted to a decimal and then a percent.

Example

Problem	**Write $\dfrac{3}{4}$ as a decimal and as a percent.**	
Write as a decimal.	$\dfrac{3}{4} = \dfrac{3 \cdot 25}{4 \cdot 25} = \dfrac{75}{100}$	Find an equivalent fraction with 10, 100, 1,000, or other power of 10 in the denominator. Since 100 is a multiple of 4, you can multiply 4 by 25 to get 100. Multiply both the numerator and the denominator by 25.

	$\frac{75}{100} = 0.75$	Write the fraction as a decimal with the 5 in the hundredths place.
Write as a percent.	0.75 = 75%	To write the decimal as a percent, move the decimal point two places to the *right*.
Answer	$\frac{3}{4} = 0.75 = 75\%$	

If it is difficult to find an equivalent fraction with a denominator of 10, 100, 1,000, and so on, you can always divide the numerator by the denominator to find the decimal equivalent.

Example		
Problem	**Write $\frac{3}{8}$ as a decimal and as a percent.**	
Write as a decimal.	$\frac{3}{8} = 3 \div 8$ $\begin{array}{r} 0.375 \\ 8\overline{)3.000} \end{array}$	Divide the numerator by the denominator. 3 ÷ 8 = 0.375.
Write as a percent.	0.375 = 37.5%	To write the decimal as a percent, move the decimal point two places to the *right*.
Answer	$\frac{3}{8} = 0.375 = 37.5\%$	

Self Check C

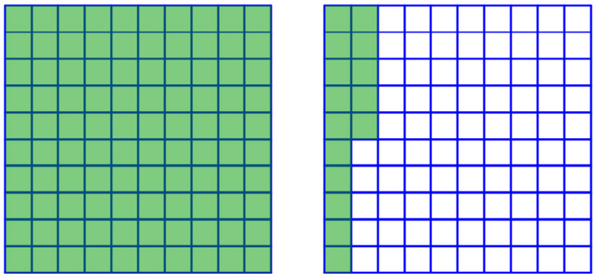
Write $\frac{4}{5}$ as a decimal and as a percent.

Mixed Numbers

All the previous examples involve fractions and decimals less than 1, so all of the percents you have seen so far have been less than 100%.

Percents greater than 100% are possible as well. Percents more than 100% are used to describe situations where there is more than one whole (fractions and decimals greater than 1 are used for the same reason).

In the diagram below, 115% is shaded. Each grid is considered a whole, and you need two grids for 115%.



Expressed as a decimal, the percent 115% is 1.15; as a fraction, it is $1\frac{15}{100}$, or $1\frac{3}{20}$. Notice that you can still convert among percents, fractions, and decimals when the quantity is greater than one whole.

Numbers greater than one that include a fractional part can be written as the sum of a whole number and the fractional part. For instance, the mixed number $3\frac{1}{4}$ is the sum of the whole number 3 and the fraction $\frac{1}{4}$. $3\frac{1}{4} = 3 + \frac{1}{4}$.

Example

Problem	**Write $2\frac{7}{8}$ as a decimal and as a percent.**	
	$2\frac{7}{8}=2+\frac{7}{8}$	Write the mixed fraction as 2 wholes plus the fractional part.
Write as a decimal.	$\frac{7}{8}=7\div 8$ $\begin{array}{r}0.875\\8\overline{)7.000}\end{array}$	Write the fractional part as a decimal by dividing the numerator by the denominator. 7 ÷ 8 = 0.875.
	$2+0.875=2.875$	Add 2 to the decimal.
Write as a percent.	2.875 = 287.5%	Now you can move the decimal point two places to the *right* to write the decimal as a percent.
Answer	$2\frac{7}{8}$ = 2.875 = 287.5%	

Note that a whole number can be written as a percent. 100% means one whole; so two wholes would be 200%.

Example

Problem	**Write 375% as a decimal and as a simplified fraction.**	
Write as a decimal.	$375\% = 3.75$	Move the decimal point two places to the left. Note that there is a whole number along with the decimal as the percent is more than 100%.
Write as a fraction.	$3.75 = 3 + 0.75$	Write the decimal as a sum of the whole number and the fractional part.
	$0.75 = \frac{75}{100}$	Write the decimal part as a fraction.
	$\frac{75}{100} = \frac{75 \div 25}{100 \div 25} = \frac{3}{4}$	Simplify the fraction by dividing the numerator and denominator by a common factor of 25.
	$3 + \frac{3}{4} = 3\frac{3}{4}$	Add the whole number part to the fraction.
Answer	$375\% = 3.75 = 3\frac{3}{4}$	

Self Check D

Write 4.12 as a percent and as a simplified fraction.

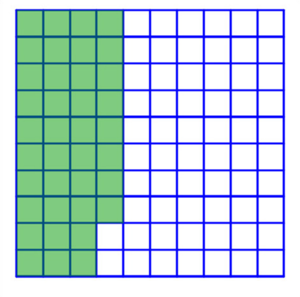
Summary

Percents are a common way to represent fractional amounts, just as decimals and fractions are. Any number that can be written as a decimal, fraction, or percent can also be written using the other two representations.

3.4.1 Self Check Solutions

Self Check A

What percent of this grid is shaded?



Three full columns of 10 squares are shaded, plus another 8 squares from the next column. So, there are 30 + 8, or 38, squares shaded out of the 100 squares in the large square. This means 38% of the large square is shaded.

Self Check B

Write 0.645 as a percent and as a simplified fraction.

$0.645 = 64.5\% = \frac{129}{200}$.

Self Check C

Write $\frac{4}{5}$ as a decimal and as a percent.

$\frac{4}{5} = 0.8 = 80\%$.

Self Check D

Write 4.12 as a percent and as a simplified fraction.

4.12 equals 412%, and the simplified form of $4\frac{12}{100}$ is $4\frac{3}{25}$.

3.4.2 Finding a Percent of a Whole

Learning Objective(s)

1 Find a percent of a whole..

Introduction

A percent, like a fraction, usually represents a portion of a whole. If the whole amount, we often want to find a portion of that whole.

Find a Percent of Whole

Objective 1

When working with fractions, if we knew a gas tank held 14 gallons, and wanted to know how many gallons were in $\frac{1}{4}$ of a tank, we would find $\frac{1}{4}$ of 14 gallons by multiplying:

$$\frac{1}{4}\bullet 14=\frac{1}{4}\bullet\frac{14}{1}=\frac{14}{4}=3\frac{2}{4}=3\frac{1}{2}\text{ gallons}$$

Likewise, if we wanted to find 25% of 14 gallons, we could find this by multiplying, but first we would need to convert the 25% to a decimal:

25% of 14 gallons = $0.25\bullet 14=3.5$ gallons

Finding a Percent of a Whole

To find a percent of a whole, multiply the percent, written as a decimal, by the whole amount.

Example		
Problem	**What is 15% of $200?**	
Write as a decimal.	15% = 0.15	Move the decimal point two places to the left.
Multiply	$0.15\bullet 200$ 30	Multiply the decimal form of the percent by the whole number.
Answer	15% of $200 is $30	

Self Check A

What number is 70% of 23?

Summary

To find a percent of a whole, multiply the percent by the whole amount.

3.4.2 Self Check Solutions

Self Check A

$0.70 \bullet 23 = 16.1$

3.5 Solving Percent Problems

Learning Objective(s)

1 Identify the amount, the base, and the percent in a percent problem.

2 Find the unknown in a percent problem.

Introduction

Percents are a ratio of a number and 100. So they are easier to compare than fractions, as they always have the same denominator, 100. A store may have a 10% off sale. The amount saved is always the same portion or fraction of the price, but a higher price means more money is taken off. Interest rates on a saving account work in the same way. The more money you put in your account, the more money you get in interest. It's helpful to understand how these percents are calculated. In the last section we found a percent of a whole; in this section we will expand on that.

Objective **1**

Parts of a Percent Problem

Jeff has a coupon at the Guitar Store for 15% off any purchase of $100 or more. He wants to buy a used guitar that has a price tag of $220 on it. Jeff wonders how much money the coupon will take off the original $220 price.

Problems involving percents have any three quantities to work with: the **percent**, the **amount**, and the **base**.

The percent has the percent symbol (%) or the word "percent." In the problem above, 15% is the percent off the purchase price.

The base is the whole amount. In the problem above, the whole price of the guitar is $220, which is the base.

The amount is the number that relates to the percent. It is always part of the whole. In the problem above, the amount is unknown. Since the percent is the percent *off*, the amount will be the *amount off* of the price.

You will return to this problem a bit later. The following examples show how to identify the three parts, the percent, the base, and the amount.

Example

Problem **Identify the percent, amount, and base in this problem.**

30 is 20% of what number?

Percent: The percent is the number with the % symbol: **20%**.

Base: The base is the whole amount, which in this case is unknown.

Amount: The amount based on the percent is **30**.

Answer Percent = 20%
Amount = 30
Base = unknown

The previous problem states that 30 is a portion of another number. That means 30 is the amount. Note that this problem could be rewritten: 20% of what number is 30?

Example

Problem **Identify the percent, amount, and base in this problem.**

What percent of 30 is 3?

Percent: The percent is unknown, because the problem states “what percent?”.

Base: The base is the whole amount, so the base is 30.

Amount: The amount is a portion of the whole, which is 3 in this case..

Answer Percent = unknown
Amount = 3
Base = 30

Solving with Equations

Objective **2**

Percent problems can be solved by writing **equations.** An equation uses an equal sign (=) to show that two mathematical expressions have the same value.

Percents are fractions, and just like fractions, when finding a percent (or fraction, or portion) of another amount, you multiply.

The percent of the base is the amount.

The Percent Equation

Percent **of the** Base **is the** Amount.
Percent • Base = Amount

In the examples below, the unknown is represented by the letter *n.* The unknown can be represented by any letter or a box ☐ or even a question mark.

Example

Problem **Write an equation that represents the following problem.**
30 is 20% of what number?

20% of what number is 30?	Rewrite the problem in the form "percent of base is amount."
Percent is: 20% Base is: unknown Amount is: 30	Identify the percent, the base, and the amount.
Percent • Base = Amount $20\% \bullet n = 30$	Write the percent equation. using *n* for the base, which is the unknown value.

Answer $20\% \bullet n = 30$.

Once you have an equation, you can solve it and find the unknown value. To do this, think about the relationship between multiplication and division. Look at the pairs of multiplication and division facts below, and look for a pattern in each row.

Multiplication	Division
$2 \bullet 3 = 6$	$6 \div 2 = 3$
$8 \bullet 5 = 40$	$40 \div 8 = 5$
$7 \bullet 4 = 28$	$28 \div 7 = 4$
$6 \bullet 9 = 54$	$54 \div 6 = 9$

Multiplication and division are inverse operations. What one does to a number, the other "undoes."

When you have an equation such as $20\% \bullet n = 30$, you can divide 30 by 20% to find the unknown: $n = 30 \div 20\%$.

You can solve this by writing the percent as a decimal or fraction and then dividing.

$$n = 30 \div 20\% = 30 \div 0.20 = 150$$

Example

Problem	**What percent of 72 is 9?**	
	Percent: unknown Base: 72 Amount: 9	Identify the percent, base, and amount.
	$n \bullet 72 = 9$	Write the percent equation: Percent • Base = Amount. Use *n* for the unknown (percent).
	$n = 9 \div 72$	Divide to undo the multiplication of *n* times 72.
	$\begin{array}{r} 0.125 \\ 72\overline{)9.000} \end{array}$	Divide 9 by 72 to find the value for *n*, the unknown.
	$n = 0.125$ $n = 12.5\%$	Move the decimal point two places to the right to write the decimal as a percent.
Answer	12.5% of 72 is 9.	

You can estimate to see if the answer is reasonable. Use 10% and 20%, numbers close to 12.5%, to see if they get you close to the answer.

$$10\% \text{ of } 72 = 0.1 \bullet 72 = 7.2$$
$$20\% \text{ of } 72 = 0.2 \bullet 72 = 14.4$$

Notice that 9 is between 7.2 and 14.4, so 12.5% is reasonable since it is between 10% and 20%.

Example

Problem	**What is 110% of 24?**	
	Percent: 110% Base: 24 Amount: unknown	Identify the percent, the base, and the amount.
	$110\% \bullet 24 = n$	Write the percent equation. Percent • Base = Amount. The amount is unknown, so use *n*.
	$1.10 \bullet 24 = n$	Write the percent as a decimal by moving the decimal point two places to the left.
	$1.10 \bullet 24 = 26.4 = n$	Multiply 24 by 1.10 or 1.1.
Answer	26.4 is 110% of 24.	

This problem is a little easier to estimate. 100% of 24 is 24. And 110% is a little bit more than 24. So, 26.4 is a reasonable answer.

Self Check A
18 is what percent of 48?

Self Check B
18 is 125% of what number?

Summary

Percent problems have three parts: the percent, the base (or whole), and the amount. Any of those parts may be the unknown value to be found. To solve percent problems, you can use the equation, Percent • Base = Amount, and solve for the unknown numbers.

3.5 Self Check Solutions

Self Check A

18 is what percent of 48?

The equation for this problem is $n \bullet 48 = 18$. The corresponding division is $18 \div 48$, so $n = 0.375$. Rewriting this decimal as a percent gives 37.5%.

Self Check B

18 is 125% of what number?

125% written as a decimal is 1.25.
The equation for this problem is $n \bullet 1.25 = 18$. Dividing $18 \div 1.25$ gives $n = 14.4$.

3.6.1 Solving Percent Applications

Learning Objective(s)

1 Solve applications involving percents.
2 Solve applications involving simple interest.

Introduction

Percents have a wide variety of applications to everyday life, showing up regularly in taxes, discounts, markups, and interest rates.

Solving Applications with Percents

Objective **1**

Let's go back to the problem that was posed at the beginning of the last section. You can now solve this problem as shown in the following example.

Example

Problem **Jeff has a coupon at the Guitar Store for 15% off any purchase of $100 or more. He wants to buy a used guitar that has a price tag of $220 on it. Jeff wonders how much money the coupon will take off of the $220 original price.**

How much is 15% of $220?	Simplify the problems by eliminating extra words.
Percent: 15% Base: 220 Amount: n	Identify the percent, the base, and the amount.
$15\% \bullet 220 = n$	Write the percent equation. Percent • Base = Amount
$0.15 \bullet 220 = 33$	Convert 15% to 0.15, then multiply by 220. 15 % of $220 is $33.

Answer The coupon will take $33 off the original price.

You can estimate to see if the answer is reasonable. Since 15% is half way between 10% and 20%, find these numbers.

10% of 220 = $0.1 \bullet 220 = 22$
20% of 220 = $0.2 \bullet 220 = 44$

The answer, 33, is between 22 and 44. So $33 seems reasonable.

There are many other situations that involve percents. Below are just a few.

Example		
Problem	**Evelyn bought some books at the local bookstore. Her total bill was $31.50, which included 5% tax. How much did the books cost before tax?**	
	What number + 5% of that number is $31.50? 105% of what number = 31.50?	In this problem, you know that the tax of 5% is added onto the cost of the books. So if the cost of the books is 100%, the cost plus tax is 105%.
	Percent: 105% Base: *n* Amount: 31.50	Identify the percent, the base, and the amount.
	$105\% \bullet n = 31.50$	Write the percent equation. Percent • Base = Amount.
	$1.05 \bullet n = 31.50$	Convert 105% to a decimal.
	$n = 3.50 \div 1.05 = 30$	Divide to undo the multiplication of *n* times 1.05.
Answer	The books cost $30 before tax.	

Example		
Problem	**Susana worked 20 hours at her job last week. This week, she worked 35 hours. In terms of a percent, how much more did she work this week than last week?**	
	35 is what percent of 20?	Simplify the problem by eliminating extra words.
	Percent: *n* Base: 20 Amount: 35	Identify the percent, the base, and the amount.
	$n \bullet 20 = 35$	Write the percent equation. Percent • Base = Amount.
	$n = 35 \div 20$	Divide to undo the multiplication of *n* times 20.
	$n = 1.75 = 175\%$	Convert 1.75 to a percent.
Answer	Since 35 is 175% of 20, Susana worked 75% more this week than she did last week. (You can think of this as “Susana worked 100% of the hours she worked last week, as well as 75% more.”)	

Self Check A

A bookcase that was originally $150 is on sale for 15% off. What is the sale price?

Simple Interest

Objective **2**

When a person takes out a loan, most lenders charge interest on the loan. **Interest** is a fee or change for borrowing money, typically a percent rate charged per year. We can compute simple interest by finding the interest rate percentage of the amount borrowed, then multiply by the number of years interest is earned.

Simple Interest Equation

$$I = p \bullet r \bullet t$$

Where:
I is the **interest** paid
p is the **principal** – the original amount of money borrowed
r is the **interest rate**, a per-year rate, written as a decimal
t is the **time** of the loan, expressed in years or portions of a year

Example

Problem	**Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, with a maturity in 4 years. How much interest will you earn?**	
	Interest, I: unknown Principal, p: $1000 Rate, r: 4% = 0.04 Time, t: 2 years	Identify the information given in the problem.
	$I = 1000 \bullet 0.04 \bullet 2$	Put the information in the simple interest equation.
	$I = 80$	Multiply
Answer	You would earn $80 in interest	

Example		
Problem	**A friend asks to borrow $240, offering to repay you $250 in 1 month. What annual interest rate is this equivalent to?**	
	Interest, *I*: $10 Principal, *p*: $240 Rate, *r*: unknown Time, *t*: 1 month	Identify the information given in the problem. Here your friend is paying back $10 more than he borrowed, so that is the interest paid.
	$1 \text{ month} = \frac{1}{12} \text{ year}$	Convert the time to years.
	$10 = 240 \bullet r \bullet \frac{1}{12}$	Put the information in the simple interest equation.
	$10 = r \bullet 240 \bullet \frac{1}{12}$ $10 = r \bullet \frac{240}{12}$ $10 = r \bullet 20$	Regroup and simplify.
	$r = 10 \div 20 = 0.50$	Divide to undo the multiplication.
	$0.50 = 50\%$	Rewrite as a percent.
Answer	This is equivalent to a 50% annual interest rate.	

Self Check B

Find the interest on $5000 borrowed at 6% for 4 years.

Summary

When solving application problems with percents, it is important to be extremely careful in identifying the percent, whole, and amount in the problem. Once those are identified, use the percent equation to solve the problem. Write your final answer back in terms of the original scenario.

3.6 Self Check Solutions

Self Check A

A bookcase that was originally \$150 is on sale for 15% off. What is the sale price?

15% is the percent, \$150 is the amount.
15% of 150 = 0.15 • 150 = \$22.5, which is how much he will save.
The sale price is \$150 - \$22.5 = \$127.50

Self Check B

Find the interest on \$5000 borrowed at 6% for 4 years.

$I = 5000 \bullet 0.06 \bullet 4 = \$1200$ in interest

3.6.2 Circle Graphs (Pie Charts)

Learning Objective(s)
1 Read and interpret data from circle graphs (pie charts).

Introduction

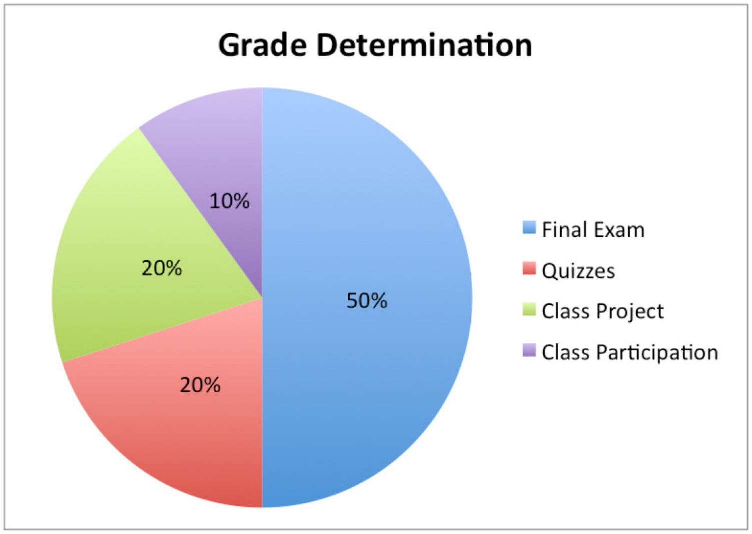
Different graphs tell different stories. While a bar graph might be appropriate for comparing some types of data, there are a number of other types of graphs that can present data in a different way. You might see them in news stories or reports, so it's helpful to know how to read and interpret them

Circle Graphs

Objective 1

Sometimes you will see categorical data presented in a **circle graph**, or **pie chart**. In these types of graphs, individual pieces of data are represented as sections of the circle (or "pieces of the pie"), corresponding to their percentage of the whole. Circle graphs are often used to show how a whole set of data is broken down into individual components.

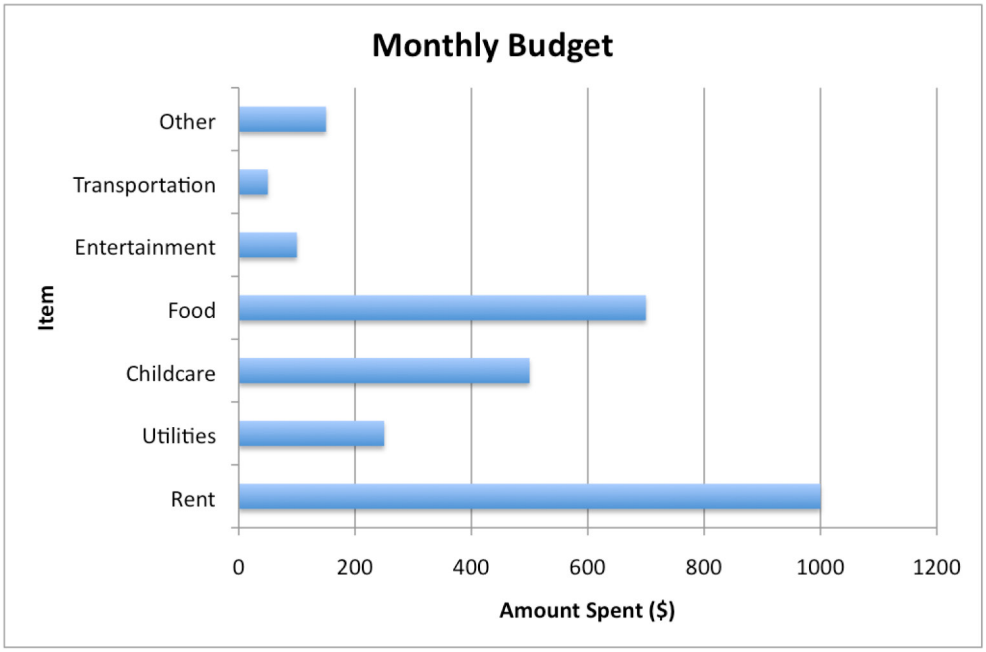
Here's an example. At the beginning of a semester, a teacher talks about how she will determine student grades. She says, "Half your grade will be based on the final exam and 20% will be determined by quizzes. A class project will also be worth 20% and class participation will count for 10%." In addition to telling the class this information, she could also create a circle graph.



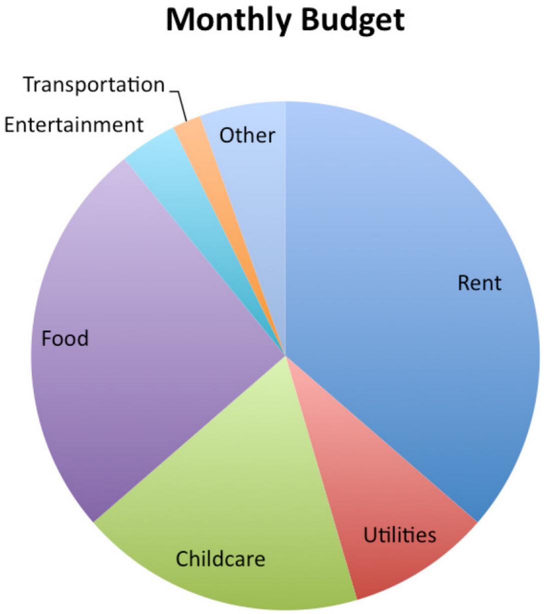
This graph is useful because it relates each part—the final exam, the quizzes, the class project, and the class participation—to the whole. It is easy to see that students in this class had better study for the final exam!

Because circle graphs relate individual parts and a whole, they are often used for budgets and other financial purposes. A sample family budget follows. It has been graphed two ways: first using a bar graph, and then using a circle graph. Each representation illustrates the information a little differently.

The bar graph shows the amounts of money spent on each item during one month. Using this data, you could figure out how much the family needs to earn every month to make this budget work.



The bar graph above focuses on the amount spent for each category. The circle graph to the right shows how each piece of the budget relates to the other pieces of the budget. This makes it easier to see where the greatest amounts of money are going, and how much of the whole budget these pieces take up. Rent and food are the greatest expenses here, with childcare also taking up a sizeable portion.

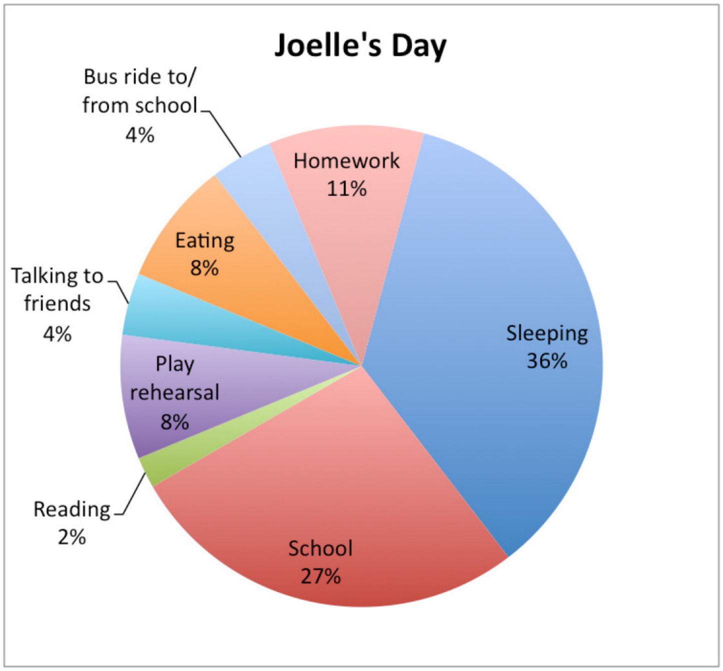


If you look closely at the circle graph, you can see that the sections for food, childcare, and utilities take up almost exactly half of the circle—this means that these three items represent half the budget! This kind of analysis is harder to do with bar graphs because each item is represented as its own entity, and is not part of a larger whole.

Circle graphs often show the relationship of each piece to the whole using percentages, as in the next example.

Example

Problem **The circle graph below shows how Joelle spent her day. Did she spend more time sleeping or doing school-related work (school, homework, and play rehearsal)?**



Sleeping: 36%

School-related: 27% + 8% + 11% = 46%

Look at the circle graph. The section labeled “Sleeping” is a little larger than the section named “School” (and notice that the percentage of time sleeping is greater than the percentage of time at school!) “Homework” and “Play rehearsal” are both smaller, but when the percentages of time are added to “School,” they add up to a larger portion of the day.

Answer Joelle spent more time doing school-related work.

Example

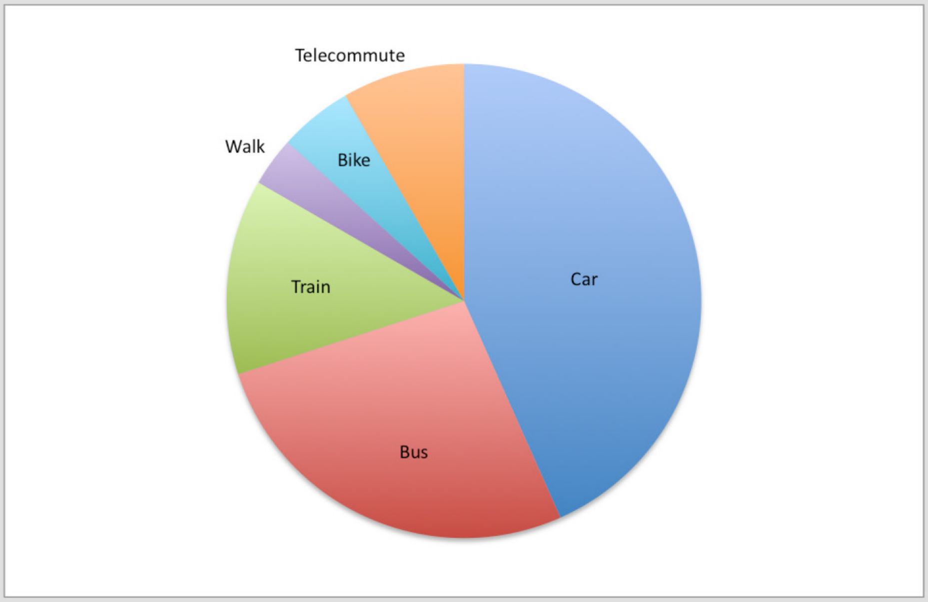
Problem **Using the graph from the previous example, how many hours does Joelle spend on homework?**

Homework: 11%	Identify how the percent from the circle graph
11% of 24 hours $0.11 \cdot 24$ 2.64 hours	Find 11% of the total. In this case, there are 24 hours total in the day

Answer Joelle spends about 2.64 hours on homework.

Self Check A

The graph below shows data about how people in one company commute to work each day.



Which statement is true?

A) Everyone takes a car, bus, or train to work.
B) Taking the bus is more popular than walking or biking.
C) More people take the train than take the bus.
D) Telecommuting is the least popular method of commuting to work.

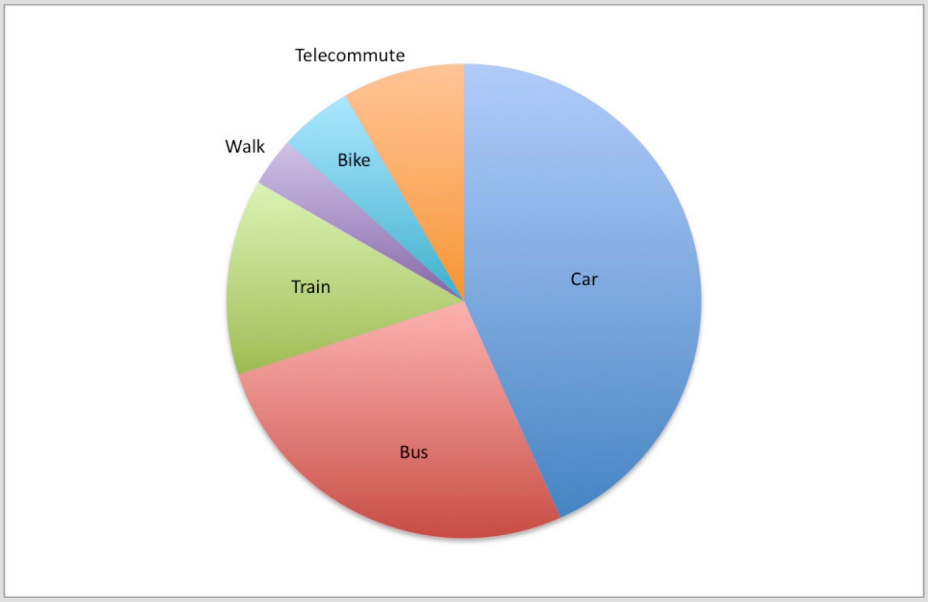
Summary

Circle graphs show how a set of data is divided up into sections, and they help the viewer visualize how each section relates to the whole. By contrast, line graphs are usually used to relate continuous data over a period of time. A third type of graph, the stem-and-leaf plot, provides another way to organize quantitative data. Stem-and-leaf plots are useful for getting a quick picture of the smallest and largest values, clusters, and gaps of the data within a set.

3.6.2 Self Check Solutions

Self Check A

The graph below shows data about how people in one company commute to work each day.



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B) Taking the bus is more popular than walking or biking.
C) More people take the train than take the bus.
D) Telecommuting is the least popular method of commuting to work.

B) Taking the bus is more popular than walking or biking.
The graph shows that about one-fourth of the company takes the bus to work, but only a small portion of people walk or bike.

4.1 Ratios and Rates

Learning Objective(s)

1 Write ratios and rates as fractions in simplest form.

2 Find unit rates.

3 Find unit prices.

Introduction

Ratios are used to compare amounts or quantities or describe a relationship between two amounts or quantities. For example, a ratio might be used to describe the cost of a month's rent as compared to the income earned in one month. You may also use a ratio to compare the number of elephants to the total number of animals in a zoo, or the amount of calories per serving in two different brands of ice cream.

Rates are a special type of ratio used to describe a relationship between different units of measure, such as speed, wages, or prices. A car can be described as traveling 60 miles per hour; a landscaper might earn $35 per lawn mowed; gas may be sold at $3 per gallon.

Ratios

Objective 1

Ratios compare quantities using division. This means that you can set up a ratio between two quantities as a division expression between those same two quantities.

Here is an example. If you have a platter containing 10 sugar cookies and 20 chocolate chip cookies, you can compare the cookies using a ratio.

The ratio of sugar cookies to chocolate chip cookies is:

$$\frac{\text{sugar cookies}}{\text{chocolate chip cookies}}=\frac{10}{20}$$

The ratio of chocolate chip cookies to sugar cookies is:

$$\frac{\text{chocolate chip cookies}}{\text{sugar cookies}}=\frac{20}{10}$$

You can write the ratio using words, a fraction, and also using a colon as shown below.

ratio of sugar cookies to
chocolate chip cookies

10 to 20

$$\frac{10}{20}$$

10 **:** 20

Some people think about this ratio as: "For every 10 sugar cookies I have, I have 20 chocolate chip cookies."

You can also simplify the ratio just as you simplify a fraction.

$$\frac{10}{20} = \frac{10 \div 10}{20 \div 10} = \frac{1}{2}$$

So we can also say that:

ratio of sugar cookies to
chocolate chip cookies

1 to 2

$$\frac{1}{2}$$

1 **:** 2

How to Write a Ratio

A ratio can be written in three different ways:

- with the word "to": 3 to 4.
- as a fraction: $\frac{3}{4}$.
- with a colon: 3 **:** 4.

A ratio is simplified if it is equivalent to a fraction that has been simplified.

Below are two more examples that illustrate how to compare quantities using a ratio, and how to express the ratio in simplified form.

Example

Problem	**A basketball player takes 50 jump shots during a practice. She makes 28 of them. What is the ratio of shots made to shots taken? Simplify the ratio.**	
	$\frac{\text{shots made}}{\text{shots taken}}$	Identify the relationship.
	$\frac{28}{50}$	Express the two quantities in fraction form.
	$\frac{28 \div 2}{50 \div 2} = \frac{14}{25}$	Simplify the fraction to express the ratio in simplest form.
	14 **:** 25 14 to 25	Consider the two other ways to write a ratio. You'll want to express your answer in a particular format if required.
Answer	The ratio of shots made to shots taken is $\frac{14}{25}$, 14 **:** 25, or 14 to 25.	

Often, one quantity in the ratio is greater than the second quantity. You do not have to write the ratio so that the lesser quantity comes first; the important thing is to keep the relationship consistent.

Example

Problem	**Paul is comparing the amount of calories in personal pizza to the calories in a sub sandwich. The pizza has 600 calories, while the sandwich has 450 calories. Write a ratio that represents the amount of calories in the pizza compared to the calories in the sandwich.**	
	$\frac{\text{calories in the pizza}}{\text{calories in the sandwich}}$	Identify the relationship.
	$\frac{600}{450}$	Write a ratio comparing calories.
	$\frac{600 \div 150}{450 \div 150} = \frac{4}{3}$	Simplify the ratio. 600 and 450 have a common factor of 150.
Answer	The ratio of calories in the pizza to calories in the sandwich is $\frac{3}{2}$, 3 **:** 2, or 3 to 2.	

Ratios can compare a *part to a part* or a *part to a whole*. Consider the example below that describes guests at a party.

Example

Problem **Luisa invites a group of friends to a party. Including Luisa, there are a total of 22 people, 10 of whom are women.**

Which is greater: the ratio of women to men at the party, or the ratio of women to the total number of people present?

$\frac{\text{number of women}}{\text{number of men}}$	Identify the first relationship.
$\frac{10}{12}$	Write a ratio comparing women to men. Since there are 22 people and 10 are women, 12 must be men.
$\frac{10 \div 2}{12 \div 2} = \frac{5}{6}$	Simplify the ratio. 10 and 12 have a common factor of 2; the ratio of women to men at the party is $\frac{5}{6}$.
$\frac{\text{number of women}}{\text{total number of people}}$	Identify the next relationship.
$\frac{10}{22}$	Write a ratio comparing the number of women to the total number of people at the party.
$\frac{10 \div 2}{22 \div 2} = \frac{5}{11}$	Simplify the ratio.10 and 22 have a common factor of 2.
$\frac{5 \bullet 11}{6 \bullet 11} = \frac{55}{66}$ $\frac{5 \bullet 6}{11 \bullet 6} = \frac{30}{66}$	Rewrite each fraction with a common denominator, 66.
$\frac{5}{6} > \frac{5}{11}$	Since $\frac{55}{66} > \frac{30}{66}$.

Answer The ratio of women to men at the party, $\frac{5}{6}$, is greater than the ratio of women to the total number of people, $\frac{5}{11}$.

Self Check A
A poll at Forrester University found that 4,000 out of 6,000 students are unmarried. Find the ratio of unmarried to married students. Express as a simplified ratio.

Rates

A rate is a ratio that compares two different quantities that have different units of measure. A rate is a comparison that provides information such as dollars per hour, feet per second, miles per hour, and dollars per quart, for example. The word "per" usually indicates you are dealing with a rate. Rates can be written using words, using a colon, or as a fraction. It is important that you know which quantities are being compared.

For example, an employer wants to rent 6 buses to transport a group of 300 people on a company outing. The rate to describe the relationship can be written using words, using a colon, or as a fraction; and you must include the units.

six buses per 300 people

6 buses **:** 300 people

$$\frac{6\text{ buses}}{300\text{ people}}$$

As with ratios, this rate can be expressed in simplest form by simplifying the fraction.

$$\frac{6\text{ buses} \div 6}{300\text{ people} \div 6} = \frac{1\text{ bus}}{50\text{ people}}$$

This fraction means that the rate of buses to people is 6 to 300 or, simplified, 1 bus for every 50 people.

Example

Problem **Write the rate as a simplified fraction: 8 phone lines for 36 employees.**

$$\frac{8\text{ phone lines}}{36\text{ employees}}$$ Write as a fraction.

$$\frac{8\text{ phone lines} \div 4}{36\text{ employees} \div 4} = \frac{2\text{ phone lines}}{9\text{ employees}}$$ Simplify the fraction by using the common factor of 4.

Answer The rate of phone lines for employees can be expressed as $\frac{2\text{ phone lines}}{9\text{ employees}}$.

Example		
Problem	**Write the rate as a simplified fraction: 6 flight attendants for 200 passengers.**	
	$\frac{\text{6 flight attendants}}{\text{200 passengers}}$	Write as a fraction.
	$\frac{\text{6 flight attendants} \div 2}{\text{200 passengers} \div 2} = \frac{\text{3 flight attendants}}{\text{100 passengers}}$	Simplify the fraction by using the common factor of 2.
Answer	The rate of flight attendants and passengers is $\frac{\text{3 flight attendants}}{\text{100 passengers}}$.	

Self Check B

Anyla rides her bike 18 blocks in 20 minutes. Express her rate as a simplified fraction.

Finding Unit Rates

Objective 2

A **unit rate** compares a quantity to one unit of measure. You often see the speed at which an object is traveling in terms of its unit rate.

For example, if you wanted to describe the speed of a boy riding his bike—and you had the measurement of the distance he traveled in miles in 2 hours—you would most likely express the speed by describing the distance traveled in *one* hour. This is a unit rate; it gives the distance traveled per one hour. The denominator of a unit rate will always be one.

Consider the example of a car that travels 300 miles in 5 hours. To find the unit rate, you find the number of miles traveled in ***one*** hour.

$$\frac{\text{300 miles} \div 5}{\text{5 hours} \div 5} = \frac{\text{60 miles}}{\text{1 hour}}$$

A common way to write this unit rate is 60 miles *per* hour.

Example

Problem **A crowded subway train has 375 passengers distributed evenly among 5 cars. What is the unit rate of passengers per subway car?**

$$\frac{\text{passengers}}{\text{subway cars}}$$

Identify the relationship.

$$\frac{375 \text{ passengers}}{5 \text{ subway cars}}$$

Write the rate as a fraction.

$$\frac{375 \text{ passengers} \div 5}{5 \text{ subway cars} \div 5} = \frac{75 \text{ passengers}}{1 \text{ subway car}}$$

Express the fraction with 1 in the denominator to find the number of passengers in one subway car.

Answer The unit rate of the subway car is 75 riders per subway car.

Finding Unit Prices

Objective 3

A **unit price** is a unit rate that expresses the price of something. The unit price always describes the price of *one* unit, so that you can easily compare prices.

You may have noticed that grocery shelves are marked with the unit price (as well as the total price) of each product. This unit price makes it easy for shoppers to compare the prices of competing brands and different package sizes.

Consider the two containers of blueberries shown below. It might be difficult to decide which is the better buy just by looking at the prices; the container on the left is cheaper, but you also get fewer blueberries. A better indicator of value is the price *per single ounce* of blueberries for each container.

Unit Price
25¢ per oz.

Total Price
$2.00
8 oz.

Unit Prices
20¢ per oz.

Total Price
$2.40
12 oz.

Look at the unit prices—the container on the right is actually a better deal, since the price per ounce is *lower* than the unit price of the container on the left. You pay more money for the larger container of blueberries, but you also get more blueberries than you would with the smaller container. Put simply, the container on the right is a better value than the container on the left.

So, how do you find the unit price?

Imagine a shopper wanted to use unit prices to compare a 3-pack of tissue for $4.98 to a single box of tissue priced at $1.60. Which is the better deal?

Find the unit price of the 3-pack: $\frac{\$4.98}{3 \text{ boxes}}$

Since the price given is for 3 boxes, divide both the numerator and the denominator by 3 to get the price of 1 box, the unit price. The unit price is $1.66 per box.

The unit price of the 3-pack is $1.66 per box; compare this to the price of a single box at $1.60. Surprisingly, the 3-pack has a higher unit price! Purchasing the single box is the better value.

Like rates, unit prices are often described with the word “per.” Sometimes, a slanted line / is used to mean “per.” The price of the tissue might be written $1.60/box, which is read “$1.60 per box.”

Example		
Problem	**3 pounds of sirloin tips cost $21. What is the unit price per pound?**	
	$\frac{\$21.00}{3 \text{ pounds}}$	Write a rate to represent the cost per number of pounds.
	$\frac{\$21.00 \div 3}{3 \text{ pounds} \div 3} = \frac{\$7.00}{1 \text{ pound}}$	Express the fraction with 1 in the denominator by dividing both the numerator and the denominator by 3.
Answer	The unit price of the sirloin tips is $7.00/pound.	

The following example shows how to use unit price to compare two products and determine which has the lower price.

Example	
Problem	**Sami is trying to decide between two brands of crackers. Which brand has the lower unit price?** **Brand A: $1.12 for 8 ounces** **Brand B: $1.56 for 12 ounces**

Brand A

$$\frac{\$1.12}{8\text{ ounces}}$$

Write a rate to represent the cost per ounce for Brand A.

$$\frac{\$1.12 \div 8}{8\text{ ounces} \div 8} = \frac{\$0.14}{1\text{ ounce}}$$

Find the unit price of Brand A by dividing both the numerator and the denominator by 8.

Brand B

$$\frac{\$1.56}{12\text{ ounces}}$$

Write a rate to represent the cost per ounce for Brand B.

$$\frac{\$1.56 \div 12}{12\text{ ounces} \div 12} = \frac{\$0.13}{1\text{ ounce}}$$

Find the unit price of Brand B by dividing both the numerator and the denominator by 12.

$$\frac{\$0.14}{1\text{ ounce}} > \frac{\$0.13}{1\text{ ounce}}$$

Compare unit prices.

Answer The unit price of Brand A crackers is 14 cents/ounce and the unit price of Brand B is 13 cents/ounce. Brand B has a lower unit price and represents the better value.

Self Check C

A shopper is comparing two packages of rice at the grocery store. A 10-pound package costs $9.89 and a 2-pound package costs $1.90. Which package has the lower unit price to the nearest cent? What is its unit price?

Summary

Ratios and rates are used to compare quantities and express relationships between quantities measured in the same units of measure and in different units of measure. They both can be written as a fraction, using a colon, or using the words “to” or “per”. Since rates compare two quantities measured in different units of measurement, such as dollars per hour or sick days per year, they must include their units. A unit rate or unit price is a rate that describes the rate or price for one unit of measure.

4.1 Self Check Solutions

Self Check A

A poll at Forrester University found that 4,000 out of 6,000 students are unmarried. Find the ratio of unmarried to married students. Express as a simplified ratio.

If 4,000 students out of 6,000 are unmarried, then 2,000 must be married. The ratio of unmarried to married students can be represented as 4,000 to 2,000, or simply 2 to 1.

Self Check B

Anyla rides her bike 18 blocks in 20 minutes. Express her rate as a simplified fraction.

$$\frac{9\ blocks}{10\ minutes}$$

Anyla's trip compares quantities with different units (blocks and minutes) so it is a rate and can be written $\frac{18 \text{ blocks}}{20 \text{ minutes}}$. This fraction can be simplified by dividing both the numerator and the denominator by 2.

Self Check C

A shopper is comparing two packages of rice at the grocery store. A 10-pound package costs $9.89 and a 2-pound package costs $1.90. Which package has the lower unit price to the nearest cent? What is its unit price?

The 2-pound bag has a lower unit price of $0.95/pound.
The unit price per pound for the 2-pound bag is $1.90 ÷ 2 = $0.95. The unit price per pound for the 10-pound bag is $9.89 ÷ 10 = $0.989 which rounds to $0.99.

4.2 Proportions

Learning Objective(s)

1 Determine whether a proportion is true or false.
2 Find an unknown in a proportion.
3 Solve application problems using proportions.
4 Solve application problems using similar triangles.

Introduction

A true **proportion** is an equation that states that two **ratios** are equal. If you know one ratio in a proportion, you can use that information to find values in the other equivalent ratio. Using proportions can help you solve problems such as increasing a recipe to feed a larger crowd of people, creating a design with certain consistent features, or enlarging or reducing an image to scale.

For example, imagine you want to enlarge a 5-inch by 8-inch photograph to fit a wood frame that you purchased. If you want the shorter edge of the enlarged photo to measure 10 inches, how long does the photo have to be for the image to scale correctly? You can set up a proportion to determine the length of the enlarged photo.

Determining Whether a Proportion Is True or False

Objective 1

A proportion is usually written as two equivalent fractions. For example:

$$\frac{12\text{ inches}}{1\text{ foot}} = \frac{36\text{ inches}}{3\text{ feet}}$$

Notice that the equation has a ratio on each side of the equal sign. Each ratio compares the same units, inches and feet, and the ratios are equivalent because the units are consistent, and $\frac{12}{1}$ is equivalent to $\frac{36}{3}$.

Proportions might also compare two ratios with the same units. For example, Juanita has two different-sized containers of lemonade mix. She wants to compare them. She could set up a proportion to compare the number of ounces in each container to the number of servings of lemonade that can be made from each container.

$$\frac{40\text{ ounces}}{84\text{ ounces}} = \frac{10\text{ servings}}{21\text{ servings}}$$

Since the units for each ratio are the same, you can express the proportion without the units:

$$\frac{40}{84} = \frac{10}{21}$$

When using this type of proportion, it is important that the numerators represent the same situation – in the example above, 40 ounces for 10 servings – and the denominators represent the same situation, 84 ounces for 21 servings.

Juanita could also have set up the proportion to compare the ratios of the container sizes to the number of servings of each container.

$$\frac{40\text{ ounces}}{10\text{ servings}}=\frac{84\text{ ounces}}{21\text{ servings}}$$

Sometimes you will need to figure out whether two ratios are, in fact, a true or false proportion. Below is an example that shows the steps of determining whether a proportion is true or false.

Example

Problem	**Is the proportion true or false?** $\frac{100\text{ miles}}{4\text{ gallons}}=\frac{50\text{ miles}}{2\text{ gallons}}$	
	miles	The units are consistent across the numerators.
	gallons	The units are consistent across the denominators.
	$\frac{100\div 4}{4\div 4}=\frac{25}{1}$	Write each ratio in simplest form.
	$\frac{50\div 2}{2\div 2}=\frac{25}{1}$	
	$\frac{25}{1}=\frac{25}{1}$	Since the simplified fractions are equivalent, the proportion is true.
Answer	The proportion is true.	

Identifying True Proportions

To determine if a proportion compares equal ratios or not, you can follow these steps.

1. Check to make sure that the units in the individual ratios are consistent either vertically or horizontally. For example, $\frac{\text{miles}}{\text{hour}}=\frac{\text{miles}}{\text{hour}}$ or $\frac{\text{miles}}{\text{miles}}=\frac{\text{hour}}{\text{hour}}$ are valid setups for a proportion.
2. Express each ratio as a simplified fraction.
3. If the simplified fractions are the same, the proportion is *true*; if the fractions are different, the proportion is *false*.

Sometimes you need to create a proportion before determining whether it is true or not. An example is shown below.

Example

Problem **One office has 3 printers for 18 computers. Another office has 20 printers for 105 computers. Is the ratio of printers to computers the same in these two offices?**

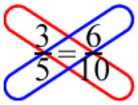
$\frac{\text{printers}}{\text{computers}} = \frac{\text{printers}}{\text{computers}}$	Identify the relationship.
$\frac{3\text{ printers}}{18\text{ computers}} = \frac{20\text{ printers}}{105\text{ computers}}$	Write ratios that describe each situation, and set them equal to each other.
printers	Check that the units in the numerators match.
computers	Check that the units in the denominators match.
$\frac{3 \div 3}{18 \div 3} = \frac{1}{6}$ $\frac{20 \div 5}{105 \div 5} = \frac{4}{21}$	Simplify each fraction and determine if they are equivalent.
$\frac{1}{6} \neq \frac{4}{21}$	Since the simplified fractions are not equal (designated by the $\neq$ sign), the proportion is not true.

Answer The ratio of printers to computers is **not** the same in these two offices.

There is another way to determine whether a proportion is true or false. This method is called “finding the cross product” or “cross multiplying”.

To cross multiply, you multiply the numerator of the first ratio in the proportion by the denominator of the other ratio. Then multiply the denominator of the first ratio by the numerator of the second ratio in the proportion. If these products are equal, the proportion is true; if these products are not equal, the proportion is not true.

This strategy for determining whether a proportion is true is called cross-multiplying because the pattern of the multiplication looks like an “x” or a criss-cross. Below is an example of finding a cross product, or cross multiplying.



In this example, you multiply 3 • 10 = 30, and then multiply 5 • 6 = 30. Both products are equal, so the proportion is true.

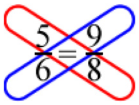
To see why this works, let’s start with a true proportion: $\frac{4}{8}=\frac{5}{10}$. If we multiplied both sides by 10, we’d get $10\bullet\frac{4}{8}=\frac{5}{10}\bullet 10$. The right side of this equation would simplify to 5, leaving $10\bullet\frac{4}{8}=5$. Now if we multiplied both sides by 8, we’d get $10\bullet\frac{4}{8}\bullet 8=5\bullet 8$, and the left side would simplify to $10\bullet 4=5\bullet 8$. Notice this is the same equation we would get by cross-multiplying, so cross-multiplying is just a quick way to do these operations.

Below is another example of determining if a proportion is true or false by using cross products.

Example

Problem **Is the proportion true or false?**

$$\frac{5}{6}=\frac{9}{8}$$



Identify the cross product relationship.

5 • 8 = 40
6 • 9 = 54

Use cross products to determine if the proportion is true or false.

$40 \neq 54$

Since the products are not equal, the proportion is false.

Answer The proportion is false.

Self Check A

Is the proportion $\frac{3}{5}=\frac{24}{40}$ true or false?

Finding an Unknown Quantity in a Proportion

Objective **2**

If you know that the relationship between quantities is proportional, you can use proportions to find missing quantities. Below is an example.

Example

Problem **Solve for the unknown quantity, *n*.**

$$\frac{n}{4} = \frac{25}{20}$$

$20 \cdot n = 4 \cdot 25$ Cross multiply.

$20n = 100$ You are looking for a number that when you multiply it by 20 you get 100.

$$20\overline{)100}^{\,5}$$

You can find this value by dividing 100 by 20.

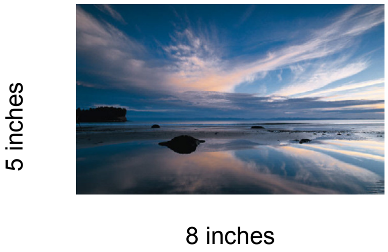
$n = 5$

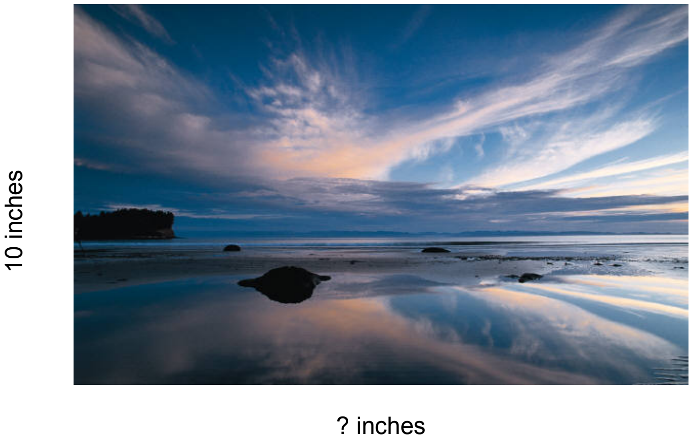
Answer $n = 5$

Self Check B

Solve for the unknown quantity, *x*. $\frac{15}{x} = \frac{6}{10}$

Now back to the original example. Imagine you want to enlarge a 5-inch by 8-inch photograph to make the length 10 inches and keep the proportion of the width to length the same. You can set up a proportion to determine the width of the enlarged photo.





Example

Problem **Find the length of a photograph whose width is 10 inches and whose proportions are the same as a 5- inch by 8-inch photograph.**

$\frac{\text{width}}{\text{length}}$	Determine the relationship.
Original photo: $\frac{5 \text{ inches wide}}{8 \text{ inches long}}$	Write a ratio that compares the length to the width of each photograph.
Enlarged photo: $\frac{10 \text{ inches wide}}{n \text{ inches long}}$	Use a letter to represent the quantity that is not known (the width of the enlarged photo).
$\frac{5}{8}=\frac{10}{n}$	Write a proportion that states that the two ratios are equal.
$5 \cdot n = 8 \cdot 10$	Cross multiply.
$5n = 80$	You are looking for a number that when it is multiplied by 5 will give you 80.
$\frac{5n}{5}=\frac{80}{5}$ $n=\frac{80}{5}$	Divide both sides by 5 to isolate the variable. $\begin{array}{r} 16 \\ 5\overline{)80} \end{array}$
$n = 16$	

Answer The length of the enlarged photograph is 16 inches.

Solving Application Problems Using Proportions

Objective **3**

Setting up and solving a proportion is a helpful strategy for solving a variety of proportional reasoning problems. In these problems, it is always important to determine what the unknown value is, and then identify a proportional relationship that you can use to solve for the unknown value. Below are some examples.

Example

Problem **Among a species of tropical birds, 30 out of every 50 birds are female. If a certain bird sanctuary has a population of 1,150 of these birds, how many of them would you expect to be female?**

Let x = the number of female birds in the sanctuary.

Determine the unknown item: the number of female birds in the sanctuary. Assign a letter to this unknown quantity.

$$\frac{30\text{ female birds}}{50\text{ birds}}=\frac{x\text{ female birds in sanctuary}}{1{,}150\text{ birds in sanctuary}}$$

Set up a proportion setting the ratios equal.

$$\frac{30\div 10}{50\div 10}=\frac{3}{5}$$

Simplify the ratio on the left to make the upcoming cross multiplication easier.

$$\frac{3}{5}=\frac{x}{1{,}150}$$

$$3\cdot 1{,}150=5\cdot x$$
$$3{,}450=5x$$

Cross multiply.

$$\begin{array}{r} 690 \\ 5\overline{)3{,}450} \end{array}$$

What number when multiplied by 5 gives a product of 3,450? You can find this value by dividing 3,450 by 5.

x = 690 birds

Answer You would expect 690 birds in the sanctuary to be female.

Example

Problem **It takes Sandra 1 hour to word process 4 pages. At this rate, how long will she take to complete 27 pages?**

$$\frac{4\text{ pages}}{1\text{ hour}}=\frac{27\text{ pages}}{x\text{ hours}}$$

Set up a proportion comparing the pages she types and the time it takes to type them.

$$4\cdot x=1\cdot 27$$

Cross multiply.

$$4x=27$$

You are looking for a number that when it is multiplied by 4 will give you 27.

$$\begin{array}{r} 6.75 \\ 4\overline{)27.00} \end{array}$$

You can find this value by dividing 27 by 4.

x = 6.75 hours

Answer It will take Sandra 6.75 hours to complete 27 pages.

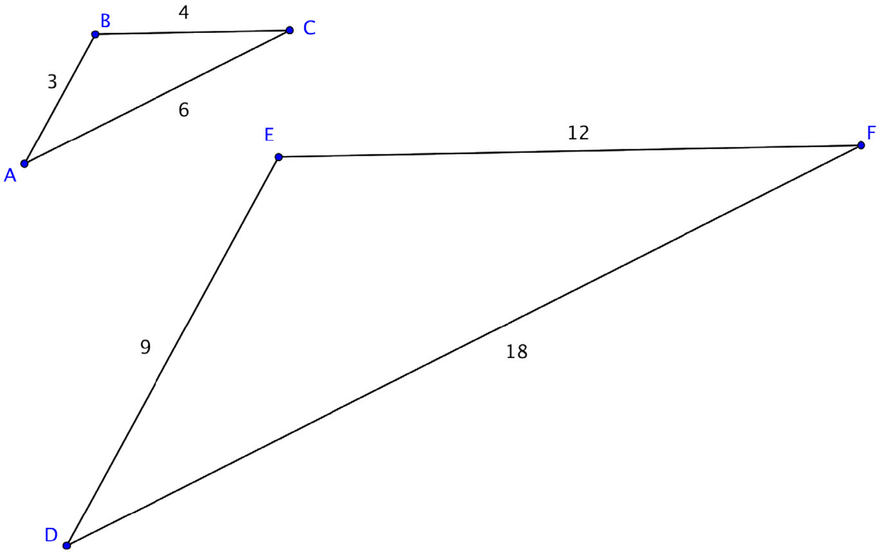
Self Check C

A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

Solving Application Problems Using Similar Triangles

Objective **4**

In the photograph problem from earlier, we created an enlargement of the picture, and both the width and height scaled proportionally. We would call the two rectangles **similar**. With triangles, we say two triangles are **similar triangles** if the ratios of the pairs of corresponding sides are equal sides. Consider the two triangles below.

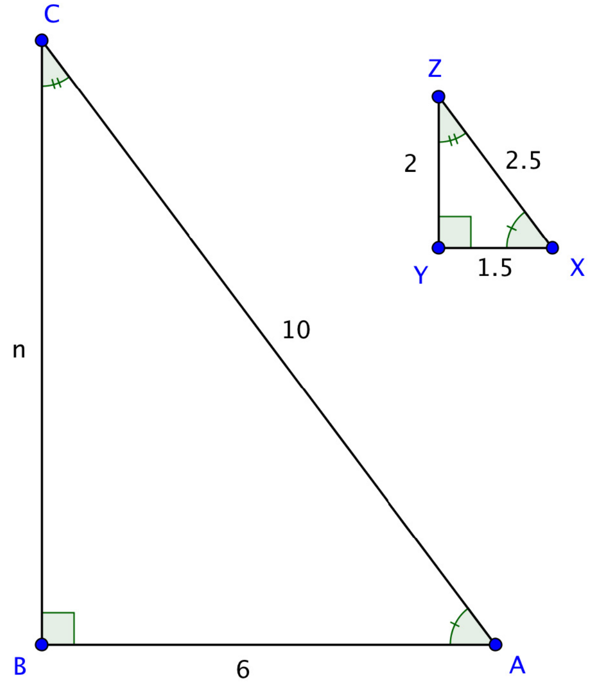


We see that side AB corresponds with side DE and so on, and we can see that each of the ratios of corresponding sides are equal: $\frac{9}{3}=\frac{12}{4}=\frac{18}{6}$, so these triangles are similar. If two triangles have the same angles, then they will also be similar.

You can find the missing measurements in a triangle if you know some measurements of a similar triangle. Let's look at an example.

Example

Problem **ΔABC and ΔXYZ are similar triangles. What is the length of side *BC*?**



$$\frac{BC}{YZ}=\frac{AB}{XY}$$

In similar triangles, the ratios of corresponding sides are proportional. Set up a proportion of two ratios, one that includes the missing side.

$$\frac{n}{2}=\frac{6}{1.5}$$

Substitute in the known side lengths for the side names in the ratio. Let the unknown side length be *n.*

$$2 \bullet 6 = 1.5 \bullet n$$

Solve for *n* using cross multiplication.

$$12 = 1.5n$$

$$8 = n$$

Answer The missing length of side *BC* is 8 units.

This process is fairly straightforward—but be careful that your ratios represent corresponding sides, recalling that corresponding sides are opposite corresponding angles.

Applying knowledge of triangles, similarity, and congruence can be very useful for solving problems in real life. Just as you can solve for missing lengths of a triangle drawn on a page, you can use triangles to find unknown distances between locations or objects.

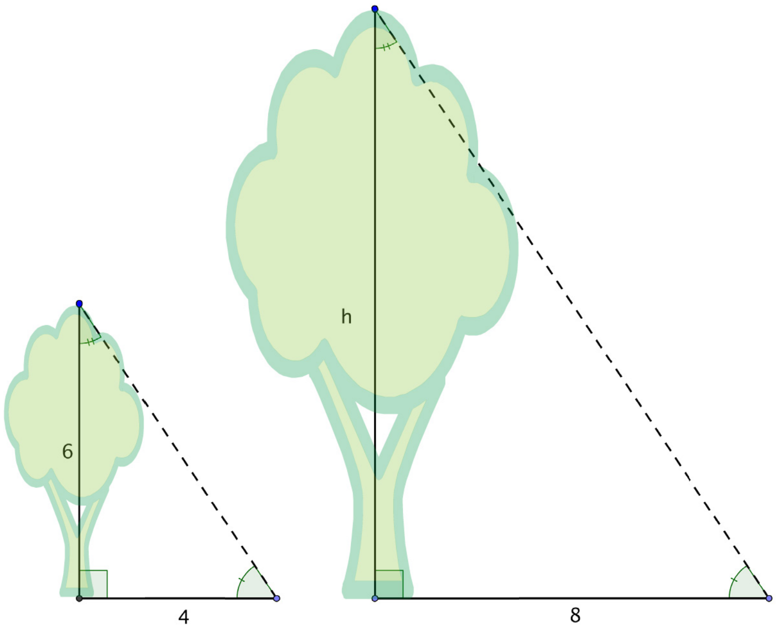
Let’s consider the example of two trees and their shadows. Suppose the sun is shining down on two trees, one that is 6 feet tall and the other whose height is unknown. By measuring the length of each shadow on the ground, you can use triangle similarity to find the unknown height of the second tree.

First, let’s figure out where the triangles are in this situation! The trees themselves create one pair of corresponding sides. The shadows cast on the ground are another pair of corresponding sides. The third side of these imaginary similar triangles runs from the top of each tree to the tip of its shadow on the ground. This is the hypotenuse of the triangle.

If you know that the trees and their shadows form similar triangles, you can set up a proportion to find the height of the tree.

Example

Problem **When the sun is at a certain angle in the sky, a 6-foot tree will cast a 4-foot shadow. How tall is a tree that casts an 8-foot shadow?**



$$\frac{\text{Tree 1}}{\text{Tree 2}} = \frac{\text{Shadow 1}}{\text{Shadow 2}}$$

The angle measurements are the same, so the triangles are similar triangles. Since they are similar triangles, you can use proportions to find the size of the missing side.

Set up a proportion comparing the heights of the trees and the lengths of their shadows.

$$\frac{6}{h} = \frac{4}{8}$$

Substitute in the known lengths. Call the missing tree height *h.*

$$6 \cdot 8 = 4h$$ Solve for *h* using cross-multiplication.

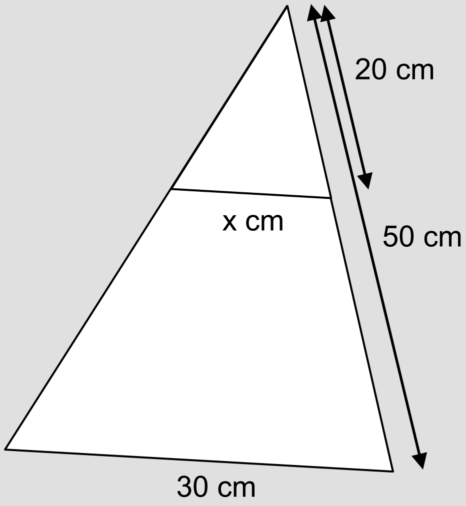
$$48 = 4h$$

$$12 = h$$

Answer The tree is 12 feet tall.

Self Check D

Find the unknown side.



Summary

A proportion is an equation comparing two ratios. If the ratios are equivalent, the proportion is true. If not, the proportion is false. Finding a cross product is another method for determining whether a proportion is true or false. Cross multiplying is also helpful for finding an unknown quantity in a proportional relationship. Setting up and solving proportions is a skill that is useful for solving a variety of problems.

4.2 Self Check Solutions

Self Check A

Is the proportion $\frac{3}{5} = \frac{24}{40}$ true or false?

True

Using cross products, you find that 3 • 40 = 120 and 5 • 24 = 120, so the cross products are equal and the proportion is true.

Self Check B

Solve for the unknown quantity, x. $\frac{15}{x} = \frac{6}{10}$

Cross-multiplying, you get the equation $6x = 150$. Dividing, you find $x = 25$.

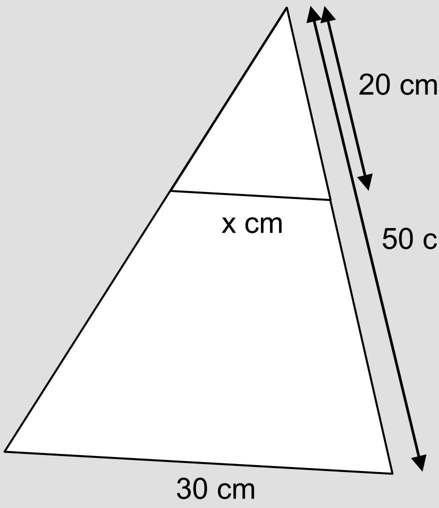
Self Check C

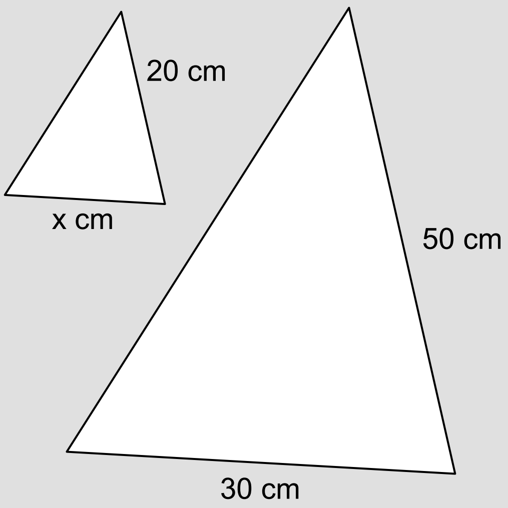
A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

Setting up the proportion $\frac{2\text{ inches}}{5\text{ miles}} = \frac{20\text{ inches}}{x}$, you find that $x = 50$ miles.

Self Check D

Find the unknown side.





To see the similar triangles, it may be helpful to split apart the picture, as shown to the right above. Setting up the proportion $\frac{20\text{ cm}}{50\ cm} = \frac{x\text{ cm}}{30\text{ cm}}$, you find $x = 12$ cm.

4.3 Volumes of Solids

Learning Objective(s)

1 Identify geometric solids.
2 Find the volume of geometric solids.
3 Find the volume of a composite geometric solid.

Introduction

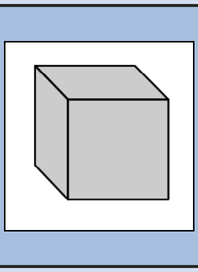
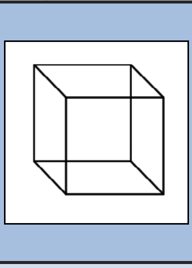
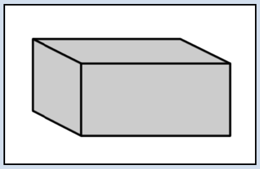
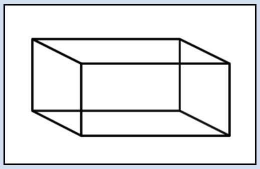
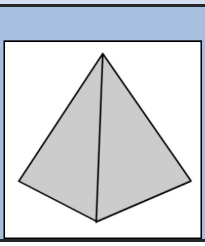
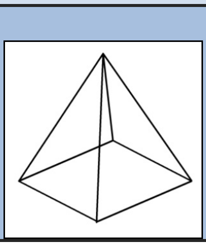
Living in a two-dimensional world would be pretty boring. Thankfully, all of the physical objects that you see and use every day—computers, phones, cars, shoes—exist in three dimensions. They all have length, width, and height. (Even very thin objects like a piece of paper are three-dimensional. The thickness of a piece of paper may be a fraction of a millimeter, but it does exist.)

In the world of geometry, it is common to see three-dimensional figures. In mathematics, a flat side of a three-dimensional figure is called a **face**. **Polyhedrons** are shapes that have four or more faces, each one being a polygon. These include cubes, prisms, and pyramids. Sometimes you may even see single figures that are composites of two of these figures. Let's take a look at some common polyhedrons.

Identifying Solids

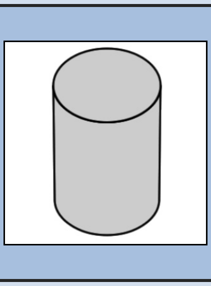
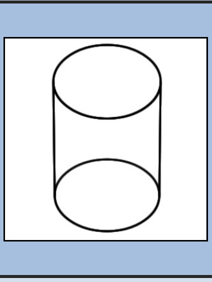
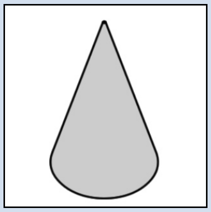
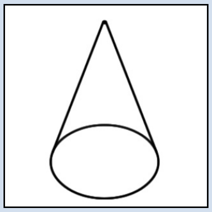
Objective 1

The first set of solids contains rectangular bases. Have a look at the table below, which shows each figure in both solid and transparent form.

Name	Definition	Solid Form	Transparent Form
Cube	A six-sided polyhedron that has congruent squares as faces.		
Rectangular prism	A polyhedron that has three pairs of congruent, rectangular, parallel faces.		
Pyramid	A polyhedron with a polygonal base and a collection of triangular faces that meet at a point.		

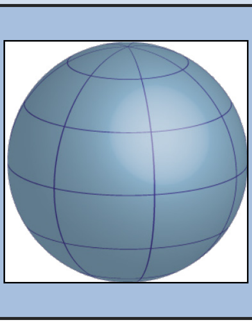
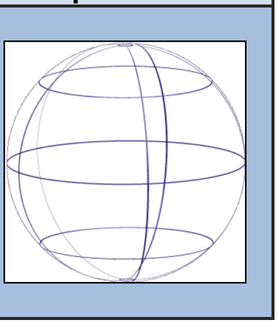
Notice the different names that are used for these figures. A **cube** is different than a square, although they are sometimes confused with each other—a cube has three dimensions, while a square only has two. Likewise, you would describe a shoebox as a **rectangular prism** (not simply a rectangle), and the ancient **pyramids** of Egypt as…well, as pyramids (not triangles)!

In this next set of solids, each figure has a circular base.

Name	**Definition**	**Solid Form**	**Transparent Form**
Cylinder	A solid figure with a pair of circular, parallel bases and a round, smooth face between them.		
Cone	A solid figure with a single circular base and a round, smooth face that diminishes to a single point.		

Take a moment to compare a pyramid and a **cone**. Notice that a pyramid has a rectangular base and flat, triangular faces; a cone has a circular base and a smooth, rounded body.

Finally, let's look at a shape that is unique: a **sphere**.

Name	**Definition**	**Solid Form**	**Transparent Form**
Sphere	A solid, round figure where every point on the surface is the same distance from the center.		

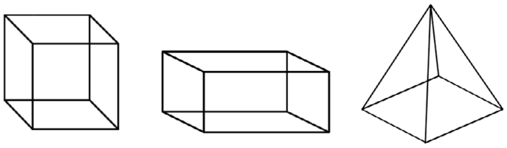
There are many spherical objects all around you—soccer balls, tennis balls, and baseballs being three common items. While they may not be perfectly spherical, they are generally referred to as spheres.

Example

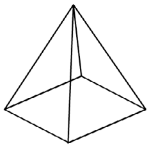
Problem **A three-dimensional figure has the following properties:**

- **It has a rectangular base.**
- **It has four triangular faces.**

What kind of a solid is it?



A rectangular base indicates that it must be a cube, rectangular prism, or pyramid.



Since the faces are triangular, it must be a pyramid.

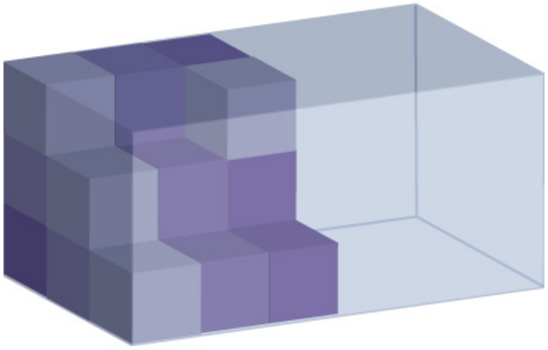
Answer The solid is a pyramid.

Volume

Objective **2**

Recall that perimeter measures one dimension (length), and area measures two dimensions (length and width). To measure the amount of space a three-dimensional figure takes up, you use another measurement called **volume**.

To visualize what “volume” measures, look back at the transparent image of the rectangular prism mentioned earlier (or just think of an empty shoebox). Imagine stacking identical cubes inside that box so that there are no gaps between any of the cubes. Imagine filling up the entire box in this manner. If you counted the number of cubes that fit inside that rectangular prism, you would have its volume.

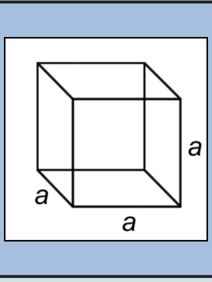
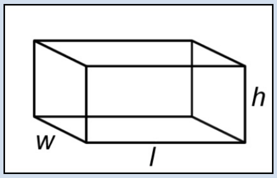
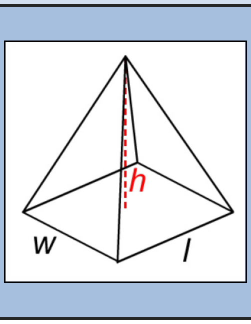


Volume is measured in cubic units. The shoebox illustrated above may be measured in cubic inches (usually represented as in^3 or inches^3), while the Great Pyramid of Egypt would be more appropriately measured in cubic meters (m^3 or meters^3).

To find the volume of a geometric solid, you could create a transparent version of the solid, create a bunch of 1x1x1 cubes, and then stack them carefully inside. However, that would take a long time! A much easier way to find the volume is to become familiar with some geometric formulas, and to use those instead.

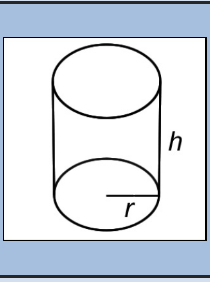
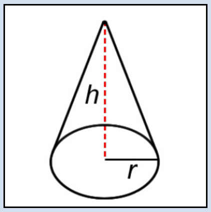
Let's go through the geometric solids once more and list the volume formula for each.

As you look through the list below, you may notice that some of the volume formulas look similar to their area formulas. To find the volume of a rectangular prism, you find the area of the base and then multiply that by the height.

Name	Transparent Form	Volume Formula
Cube		$V = a \bullet a \bullet a = a^3$ a = the length of one side
Rectangular prism		$V = l \bullet w \bullet h$ l = length w = width h = height
Pyramid		$V = \frac{l \bullet w \bullet h}{3}$ l = length w = width h = height

Remember that all cubes are rectangular prisms, so the formula for finding the volume of a cube is the area of the base of the cube times the height.

Now let’s look at solids that have a circular base.

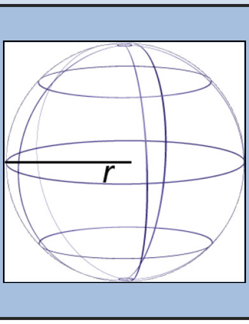
Name	Transparent Form	Volume Formula
Cylinder		$V = \pi \bullet r^2 \bullet h$ r = radius h = height
Cone		$V = \frac{\pi \bullet r^2 \bullet h}{3}$ r = radius h = height

Here you see the number π again.

The volume of a **cylinder** is the area of its base, πr^2, times its height, h.

Compare the formula for the volume of a cone ($V = \frac{\pi \bullet r^2 \bullet h}{3}$) with the formula for the volume of a pyramid ($V = \frac{l \bullet w \bullet h}{3}$). The numerator of the cone formula is the volume formula for a cylinder, and the numerator of the pyramid formula is the volume formula for a rectangular prism. Then divide each by 3 to find the volume of the cone and the pyramid. Looking for patterns and similarities in the formulas can help you remember which formula refers to a given solid.

Finally, the formula for a sphere is provided below. Notice that the radius is cubed, not squared and that the quantity πr^3 is multiplied by $\frac{4}{3}$.

Name	Wireframe Form	Volume Formula
Sphere		$V = \frac{4}{3}\pi r^3$ r = radius

Applying the Formulas

You know how to identify the solids, and you also know the volume formulas for these solids. To calculate the actual volume of a given shape, all you need to do is substitute the solid's dimensions into the formula and calculate.

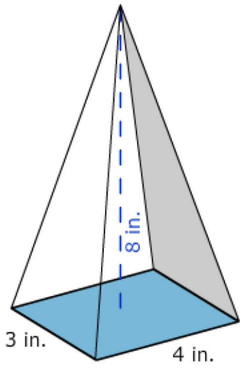
In the examples below, notice that cubic units (meters^3, inches^3, feet^3) are used.

Example

Problem	**Find the volume of a cube with side lengths of 6 meters.**	
	$V = a \cdot a \cdot a = a^3$	Identify the proper formula to use.
	a = side length	
	$V = 6 \cdot 6 \cdot 6 = 6^3$	Substitute a = 6 into the formula.
	$6 \cdot 6 \cdot 6 = 216$	Calculate the volume.
Answer	Volume = 216 meters^3	

Example

Problem **Find the volume of the shape shown below.**



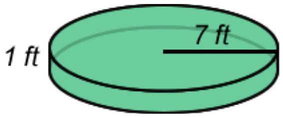
	Pyramid.	Identify the shape. It has a rectangular base and rises to a point, so it is a pyramid.
	$V = \frac{l \cdot w \cdot h}{3}$	Identify the proper formula to use.

	l = length = 4 w = width = 3 h = height = 8 $V = \frac{4 \cdot 3 \cdot 8}{3}$	Use the image to identify the dimensions. Then substitute $l = 4$, $w = 3$, and $h = 8$ into the formula.
	$V = \frac{96}{3}$ $= 32$	Calculate the volume.
Answer	The volume of the pyramid is 32 inches^3.	

Example

Problem **Find the volume of the shape shown below.**

Use 3.14 for π, and round the answer to the nearest hundredth.



	Cylinder.	Identify the shape. It has a circular base and has uniform thickness (or height), so it is a cylinder.
	$V = \pi \cdot r^2 \cdot h$	Identify the proper formula to use.
	$V = \pi \cdot 7^2 \cdot 1$	Use the image to identify the dimensions. Then substitute $r = 7$ and $h = 1$ into the formula.
	$V = \pi \cdot 49 \cdot 1$ $= 49\pi$ ≈ 153.86	Calculate the volume, using 3.14 as an approximation for π.
Answer	The volume is 49π or approximately 153.86 feet^3.	

Self Check A

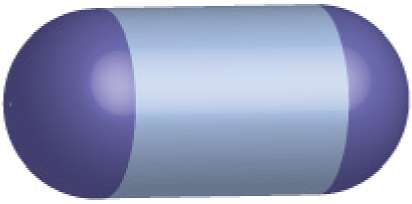
Find the volume of a rectangular prism that is 8 inches long, 3 inches wide, and 10 inches tall.

Composite Solids

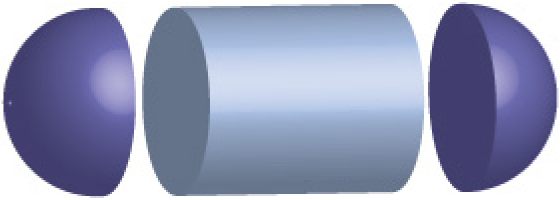
Objective **3**

Composite geometric solids are made from two or more geometric solids. You can find the volume of these solids as well, as long as you are able to figure out the individual solids that make up the composite shape.

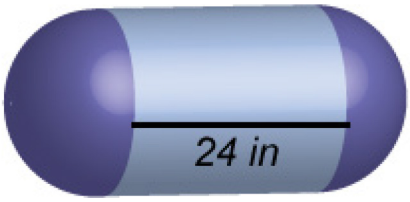
Look at the image of a capsule below. Each end is a half-sphere. You can find the volume of the solid by taking it apart. What solids can you break this shape into?

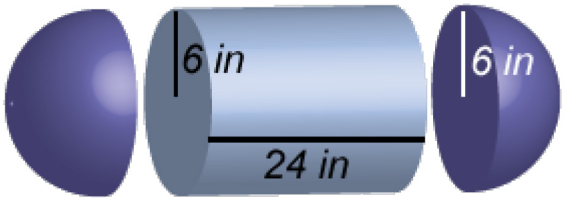


You can break it into a cylinder and two half-spheres.



Two half-spheres form a whole one, so if you know the volume formulas for a cylinder and a sphere, you can find the volume of this capsule.

Example	
Problem	**If the radius of the spherical ends is 6 inches, find the volume of the solid below. Use 3.14 for π. Round your final answer to the nearest whole number.** 



Identify the composite solids. This capsule can be thought of as a cylinder with a half-sphere on each end.

Volume of a cylinder: $\pi \cdot r^2 \cdot h$

Volume of a sphere: $\frac{4}{3}\pi r^3$

Identify the proper formulas to use.

Volume of a cylinder: $\pi \cdot 6^2 \cdot 24$

Volume of a sphere: $\frac{4}{3}\pi \cdot 6^3$

Substitute the dimensions into the formulas.

The height of a cylinder refers to the section between the two circular bases. This dimension is given as 24 inches, so *h* = 24.

The radius of the sphere is 6 inches. You can use *r* = 6 in both formulas.

Volume of the cylinder:

$$\begin{aligned} V &= \pi \cdot 36 \cdot 24 \\ &= 864 \cdot \pi \\ &\approx 2712.96 \end{aligned}$$

Calculate the volume of the cylinder and the sphere.

Volume of the sphere:

$$\begin{aligned} V &= \frac{4}{3}\pi \cdot 216 \\ &= 288 \cdot \pi \\ &\approx 904.32 \end{aligned}$$

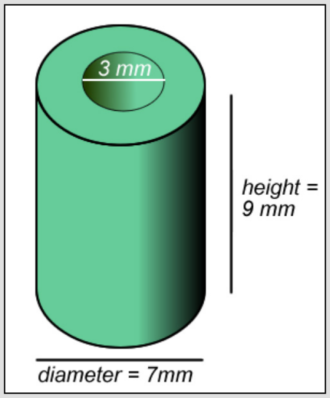
Volume of capsule: Add the volumes.

$$2712.96 + 904.32 \approx 3617.28$$

Answer The volume of the capsule is $1{,}152\pi$ or approximately 3617 inches^3.

Self Check B

A machine takes a solid cylinder with a height of 9 mm and a diameter of 7 mm, and bores a hole all the way through it. The hole that it creates has a diameter of 3 mm. Find the volume of the solid.



Summary

Three-dimensional solids have length, width, and height. You use a measurement called volume to figure out the amount of space that these solids take up. To find the volume of a specific geometric solid, you can use a volume formula that is specific to that solid. Sometimes, you will encounter composite geometric solids. These are solids that combine two or more basic solids. To find the volume of these, identify the simpler solids that make up the composite figure, find the volumes of those solids, and combine them as needed.

4.3 Self Check Solutions

Self Check A

Find the volume of a rectangular prism that is 8 inches long, 3 inches wide, and 10 inches tall.

240 inches^3

To find the volume of the rectangular prism, use the formula $V = l \bullet w \bullet h$, and then substitute in the values for the length, width, and height. 8 inches • 3 inches • 10 inches = 240 inches^3.

Self Check B

A machine takes a solid cylinder with a height of 9 mm and a diameter of 7 mm, and bores a hole all the way through it. The hole that it creates has a diameter of 3 mm. Find the volume of the solid.

You find the volume of the entire cylinder by multiplying $\pi \bullet 3.5^2 \bullet 9$, then subtract the empty cylinder in the middle, which is found by multiplying $\pi \bullet 1.5^2 \bullet 9$.

$(\pi \bullet 3.5^2 \bullet 9) - (\pi \bullet 1.5^2 \bullet 9) \approx 282.6\,\text{mm}^3$

4.4.1 U.S. Measurement - Length

Learning Objective(s)

1 Define units of length and convert from one to another.
2 Perform arithmetic calculations on units of length.
3 Solve application problems involving units of length.

Introduction

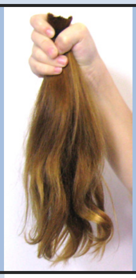
Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the **metric system** and the **U.S. customary measurement system**. This topic addresses the measurement of length using the U.S. customary measurement system.

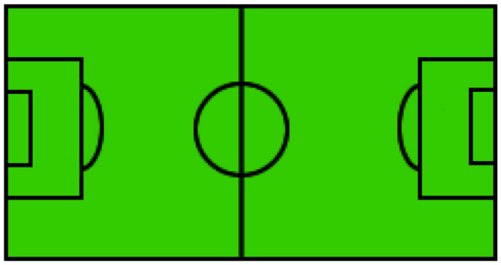
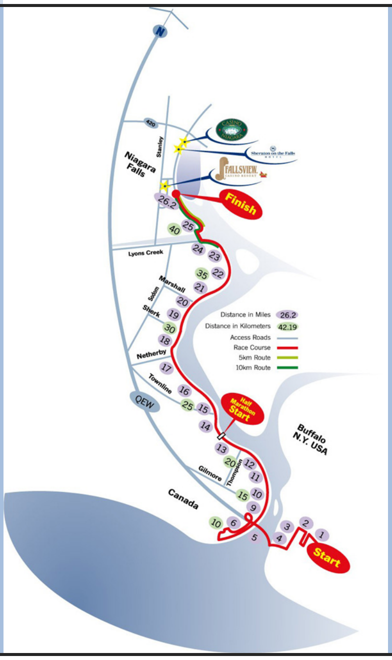
Suppose you want to purchase tubing for a project, and you see two signs in a hardware store: *$1.88 for 2 feet* of tubing and *$5.49 for 3 yards* of tubing. If both types of tubing will work equally well for your project, which is the better price? You need to know about two **units of measurement**, yards and feet, in order to determine the answer.

Units of Length

Objective 1

Length is the distance from one end of an object to the other end, or from one object to another. For example, the length of a letter-sized piece of paper is 11 inches. The system for measuring length in the United States is based on the four customary units of length: **inch**, **foot**, **yard**, and **mile**. Below are examples to show measurement in each of these units.

Unit	Description	Image
Inch/Inches	Some people donate their hair to be made into wigs for cancer patients who have lost hair as a result of treatment. One company requires hair donations to be at least 8 inches long.	
	Frame size of a bike: the distance from the center of the crank to the top of the seat tube. Frame size is usually measured in inches. This frame is 16 inches.	

Foot/Feet	Rugs are typically sold in standard lengths. One typical size is a rug that is 8 feet wide and 11 feet long. This is often described as an 8 by 11 rug.	
Yard/Yards	Soccer fields vary some in their size. An official field can be any length between 100 and 130 yards.	
Mile/Miles	A marathon is 26.2 miles long. One marathon route is shown in the map to the right.	

You can use any of these four U.S. customary measurement units to describe the length of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the length of a rug in feet rather than miles, and to describe a marathon in miles rather than inches.

You may need to convert between units of measurement. For example, you might want to express your height using feet and inches (5 feet 4 inches) or using only inches (64 inches). You need to know the unit equivalents in order to make these conversions between units.

The table below shows equivalents and conversion factors for the four customary units of measurement of length.

Unit Equivalents	Conversion Factors (longer to shorter units of measurement)	Conversion Factors (shorter to longer units of measurement)
1 foot = 12 inches	$\frac{12\text{ inches}}{1\text{ foot}}$	$\frac{1\text{ foot}}{12\text{ inches}}$
1 yard = 3 feet	$\frac{3\text{ feet}}{1\text{ yard}}$	$\frac{1\text{ yard}}{3\text{ feet}}$
1 mile = 5,280 feet	$\frac{5{,}280\text{ feet}}{1\text{ mile}}$	$\frac{1\text{ mile}}{5{,}280\text{ feet}}$

Note that each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all since it is the same as multiplying by 1; it just changes the units that you are using to measure.

Converting Between Units of Length

You can use the conversion factors to convert a measurement, such as feet, to another type of measurement, such as inches.

Note that there are many more inches for a measurement than there are feet for the same measurement, as feet is a longer unit of measurement. You could use the conversion factor $\frac{12\text{ inches}}{1\text{ foot}}$.

If a length is measured in feet, and you'd like to convert the length to yards, you can think, "I am converting from a shorter unit to a longer one, so the length in yards will be less than the length in feet." You could use the conversion factor $\frac{1\text{ yard}}{3\text{ feet}}$.

If a distance is measured in miles, and you want to know how many feet it is, you can think, "I am converting from a longer unit of measurement to a shorter one, so the number of feet would be greater than the number of miles." You could use the conversion factor $\frac{5{,}280\text{ feet}}{1\text{ mile}}$.

You can use the **factor label method** (also known as **dimensional analysis**) to convert a length from one unit of measure to another using the conversion factors. In the factor label method, you multiply by unit fractions to convert a measurement from one unit to another. Study the example below to see how the factor label method can be used to convert $3\frac{1}{2}$ feet into an equivalent number of inches.

Example

Problem **How many inches are in $3\frac{1}{2}$ feet?**

$$3\frac{1}{2}\text{feet} = \_\_\_ \text{ inches}$$

Begin by reasoning about your answer. Since a foot is longer than an inch, this means the answer would be greater than $3\frac{1}{2}$.

$$3\frac{1}{2}\text{feet} \bullet \frac{12\text{ inches}}{1\text{ foot}} = \_\_\_ \text{ inches}$$

Find the conversion factor that compares inches and feet, with “inches” in the numerator, and multiply.

$$\frac{7\text{ feet}}{2} \bullet \frac{12\text{ inches}}{1\text{ foot}} = \_\_\_ \text{ inches}$$

Rewrite the mixed number as an improper fraction before multiplying.

$$\frac{7\ \cancel{\text{feet}}}{2} \bullet \frac{12\text{ inches}}{1\ \cancel{\text{foot}}} = \_\_\_ \text{ inches}$$

You can cancel similar units when they appear in the numerator *and* the denominator. So here, cancel the similar units “feet” and “foot.” This eliminates this unit from the problem.

$$\frac{7}{2} \bullet \frac{12\text{ inches}}{1} = \_\_\_ \text{ inches}$$

$$\frac{7 \bullet 12\text{ inches}}{2 \bullet 1} = \_\_\_ \text{ inches}$$

Rewrite as multiplication of numerators and denominators.

$$\frac{84\text{ inches}}{2} = \_\_\_ \text{ inches}$$

Multiply.

$$\frac{84\text{ inches}}{2} = 42\text{ inches}$$

Divide.

Answer There are 42 inches in $3\frac{1}{2}$ feet.

Notice that by using the factor label method you can cancel the units out of the problem, just as if they were numbers. You can only cancel if the unit being cancelled is in both the numerator and denominator of the fractions you are multiplying.

In the problem above, you cancelled *feet* and *foot* leaving you with *inches*, which is what you were trying to find.

$$\frac{7\ \cancel{\text{feet}}}{2} \bullet \frac{12 \text{ inches}}{1\ \cancel{\text{foot}}} = \_\_\_ \text{ inches}$$

What if you had used the wrong conversion factor?

$$\frac{7 \text{ feet}}{2} \bullet \frac{1 \text{ foot}}{12 \text{ inches}} =$$

You could not cancel the feet because the unit is not the same in *both* the numerator and the denominator. So if you complete the computation, you would still have both feet and inches in the answer and no conversion would take place.

Here is another example of a length conversion using the factor label method.

	Example	
Problem	**How many yards is 7 feet?**	
	$7 \text{ feet} = \_\_\_ \text{ yards}$	Start by reasoning about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. So your answer will be less than 7.
	$7 \text{ feet} \bullet \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_ \text{ yards}$	Find the conversion factor that compares feet and yards, with yards in the numerator.
	$\frac{7 \text{ feet}}{1} \bullet \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_ \text{ yards}$	Rewrite the whole number as a fraction in order to multiply.
	$\frac{7\ \cancel{\text{feet}}}{1} \bullet \frac{1 \text{ yard}}{3\ \cancel{\text{feet}}} = \_\_\_ \text{ yards}$	Cancel the similar units “feet” and “feet” leaving only yards.
	$\frac{7}{1} \bullet \frac{1 \text{ yard}}{3} = \_\_\_ \text{ yards}$	
	$\frac{7 \bullet 1 \text{ yard}}{1 \bullet 3} = \_\_\_ \text{ yards}$	Multiply.
	$\frac{7 \text{ yards}}{3} = 2\frac{1}{3} \text{ yards}$	Divide, and write as a mixed number.
Answer	7 feet equals $2\frac{1}{3}$ yards.	

Note that if the units do not cancel to give you the answer you are trying to find, you may not have used the correct conversion factor.

Self Check A

How many feet are in $2\frac{1}{2}$ miles?

Applying Unit Conversions

Objective **2, 3**

There are times when you will need to perform computations on measurements that are given in different units. For example, consider the tubing problem given earlier. You must decide which of the two options is a better price, and you have to compare prices given in different unit measurements.

In order to compare, you need to convert the measurements into one single, common unit of measurement. To be sure you have made the computation accurately, think about whether the unit you are converting to is smaller or larger than the number you have. Its relative size will tell you whether the number you are trying to find is greater or lesser than the given number.

Example	
Problem	**An interior decorator needs border trim for a home she is wallpapering. She needs 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room. How many yards of border trim does she need?**
15 feet + 30 feet + 26 feet = 71 feet	You need to find the total length of border trim that is needed for all three rooms in the house. Since the measurements for each room are given in feet, you can add the numbers.
71 feet = ___ yards	How many yards is 71 feet? Reason about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. Expect your answer to be less than 71.
$\frac{71\text{ feet}}{1} \cdot \frac{1\text{ yard}}{3\text{ feet}} = \_\_\_$ yards	Use the conversion factor $\frac{1\text{ yard}}{3\text{ feet}}$.
$\frac{71\ \cancel{\text{feet}}}{1} \cdot \frac{1\text{ yard}}{3\ \cancel{\text{feet}}} = \_\_\_$ yards	Since "feet" is in the numerator and denominator, you can cancel this unit.

$$\frac{71}{1}\cdot\frac{1\text{ yard}}{3}=\_\_\_\text{ yards}$$

$$\frac{71\bullet 1\text{ yard}}{1\bullet 3}=\_\_\_\text{ yards}$$ Multiply.

$$\frac{71\text{ yards}}{3}=\_\_\_\text{ yards}$$

$$\frac{71\text{ yards}}{3}=23\frac{2}{3}\text{ yards}$$ Divide, and write as a mixed number.

Answer The interior decorator needs $23\frac{2}{3}$ yards of border trim.

The next example uses the factor label method to solve a problem that requires converting from miles to feet.

Example

Problem **Two runners were comparing how much they had trained earlier that day. Jo said, "According to my pedometer, I ran 8.3 miles." Alex said, "That's a little more than what I ran. I ran 8.1 miles." How many more feet did Jo run than Alex?**

$$8.3\text{ miles} - 8.1\text{ miles} = 0.2\text{ mile}$$

$$0.2\text{ mile} = \frac{2}{10}\text{ mile}$$

You need to find the difference between the distance Jo ran and the distance Alex ran. Since both distances are given in the same unit, you can subtract and keep the unit the same.

$$\frac{2}{10}\text{ mile} = \_\_\_\text{ feet}$$

Since the problem asks for the difference in *feet*, you must convert from miles to feet. How many feet is 0.2 mile? Reason about the size of your answer. Since a mile is longer than a foot, the distance when expressed as feet will be a number greater than 0.2.

$$\frac{2\text{ mile}}{10}\bullet\frac{5{,}280\text{ feet}}{1\text{ mile}} = \_\_\_\text{ feet}$$

Use the conversion factor $\frac{5{,}280\text{ feet}}{1\text{ mile}}$.

$$\frac{2\ \cancel{\text{mile}}}{10}\bullet\frac{5{,}280\text{ feet}}{1\ \cancel{\text{mile}}} = \_\_\_\text{ feet}$$

	$\frac{2}{10} \bullet \frac{5{,}280 \text{ feet}}{1} = \underline{\quad} \text{ feet}$	
	$\frac{2 \bullet 5{,}280 \text{ feet}}{10 \bullet 1} = \underline{\quad} \text{ feet}$	Multiply.
	$\frac{10{,}560 \text{ feet}}{10} = \underline{\quad} \text{ feet}$	
	$\frac{10{,}560 \text{ feet}}{10} = 1{,}056 \text{ feet}$	Divide.
Answer	Jo ran 1,056 feet further than Alex.	

Now let's revisit the question from earlier.

Example		
Problem	**You are walking through a hardware store and notice two sales on tubing.** ▪ **3 yards of Tubing A costs $5.49.** ▪ **Tubing B sells for $1.88 for 2 feet.** **Either tubing is acceptable for your project. Which tubing is less expensive?**	
	Tubing A 3 yards = $5.49	Find the unit price for each tubing. This will make it easier to compare.
	$\frac{\$5.49 \div 3}{3 \text{ yards} \div 3} = \frac{\$1.83}{1 \text{ yard}}$	Find the cost per yard of Tubing A by dividing the cost of 3 yards of the tubing by 3.
	Tubing B 2 feet = $1.88 $\frac{\$1.88 \div 2}{2 \text{ feet} \div 2} = \frac{\$0.94}{1 \text{ foot}}$	Tubing B is sold by the foot. Find the cost per foot by dividing $1.88 by 2 feet.
	$\frac{\$0.94}{1 \text{ foot}} \bullet \frac{3 \text{ feet}}{1 \text{ yard}} = \frac{\$ \underline{\quad}}{\underline{\quad} \text{ yard}}$	To compare the prices, you need to have the same unit of measure.

$\frac{\$0.94}{1\ \cancel{\text{foot}}} \cdot \frac{3\ \cancel{\text{feet}}}{1\ \text{yard}} = \frac{\$2.82}{1\ \text{yard}}$	Use the conversion factor $\frac{3\ \text{feet}}{1\ \text{yard}}$, cancel and multiply.
\$2.82 per yard	
Tubing A: \$1.83 per yard Tubing B: \$2.82 per yard	Compare prices for 1 yard of each tubing.

Answer Tubing A is less expensive than Tubing B.

In the problem above, you could also have found the price per foot for each kind of tubing and compared the unit prices of each per foot.

Self Check B

A fence company is measuring a rectangular area in order to install a fence around its perimeter. If the length of the rectangular area is 130 yards and the width is 75 feet, what is the total length of the distance to be fenced?

Summary

The four basic units of measurement that are used in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths.

You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

4.4.1 Self Check Solutions

Self Check A

How many feet are in $2\frac{1}{2}$ miles?

There are 5,280 feet in a mile, so multiply $2\frac{1}{2}$ by 5,280 to get 13,200 feet.

Self Check B

A fence company is measuring a rectangular area in order to install a fence around its perimeter. If the length of the rectangular area is 130 yards and the width is 75 feet, what is the total length of the distance to be fenced?

130 yards is equivalent to 390 feet. To find the perimeter, add length + length + width + width: 390 feet + 390 feet + 75 feet + 75 feet = 930 feet.

4.4.2 U.S. Measurement - Weight

Learning Objective(s)

1 Define units of weight and convert from one to another.
2 Perform arithmetic calculations on units of weight.
3 Solve application problems involving units of weight.

Introduction

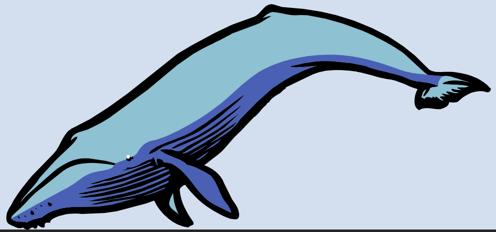
When you mention how heavy or light an object is, you are referring to its weight. In the U.S. customary system of measurement, weight is measured in ounces, pounds, and tons. Like other units of measurement, you can convert between these units and you sometimes need to do this to solve problems.

The grocery store sells a 36 ounce canister of ground coffee for $14, and sells bulk coffee for $9 per pound. Which is the better deal? To answer this question, you need to understand the relationship between ounces and pounds.

Units of Weight

Objective **1**

You often use the word **weight** to describe how heavy or light an object or person is. Weight is measured in the U.S. customary system using three units: ounces, pounds, and tons. An **ounce** is the smallest unit for measuring weight, a **pound** is a larger unit, and a **ton** is the largest unit.

Whales are some of the largest animals in the world. Some species can reach weights of up to 200 tons--that's equal to 400,000 pounds.	
Meat is a product that is typically sold by the pound. One pound of ground beef makes about four hamburger patties.	
Ounces are used to measure lighter objects. A stack of 11 pennies is equal to about one ounce.	

You can use any of the customary measurement units to describe the weight of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the weight of a human being in pounds rather than tons. It makes more sense to describe the weight of a car in tons rather than ounces.

1 pound = 16 ounces

1 ton = 2,000 pounds

Converting Between Units of Weight

Four ounces is a typical serving size of meat. Since meat is sold by the pound, you might want to convert the weight of a package of meat from pounds to ounces in order to determine how many servings are contained in a package of meat.

The weight capacity of a truck is often provided in tons. You might need to convert pounds into tons if you are trying to determine whether a truck can safely transport a big shipment of heavy materials.

The table below shows the unit conversions and conversion factors that are used to make conversions between customary units of weight.

Unit Equivalents	**Conversion Factors (heavier to lighter units of measurement)**	**Conversion Factors (lighter to heavier units of measurement)**
1 pound = 16 ounces	$\frac{16\text{ ounces}}{1\text{ pound}}$	$\frac{1\text{ pound}}{16\text{ ounces}}$
1 ton = 2000 pounds	$\frac{2000\text{ pounds}}{1\text{ ton}}$	$\frac{1\text{ ton}}{2000\text{ pounds}}$

You can use the *factor label method* to convert one customary unit of weight to another customary unit of weight. This method uses conversion factors, which allow you to “cancel” units to end up with your desired unit of measurement.

Each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all, since it is the same as multiplying by 1. It just changes the units that you are using to measure it in.

Two examples illustrating the factor label method are shown below.

Example

Problem

How many ounces are in $2\frac{1}{4}$ pounds?

$$2\frac{1}{4}\text{pounds} = \_\_\_ \text{ ounces}$$

Begin by reasoning about your answer. Since a pound is heavier than an ounce, expect your answer to be a number greater than $2\frac{1}{4}$.

$$2\frac{1}{4}\text{ pounds} \bullet \frac{16\text{ ounces}}{1\text{ pound}} = \_\_\_ \text{ ounces}$$

Multiply by the conversion factor that relates ounces and pounds: $\frac{16\text{ ounces}}{1\text{ pound}}$.

$$\frac{9\text{ pounds}}{4} \bullet \frac{16\text{ ounces}}{1\text{ pound}} = \_\_\_ \text{ ounces}$$

Write the mixed number as an improper fraction.

$$\frac{9\ \cancel{\text{pounds}}}{4} \bullet \frac{16\text{ ounces}}{1\ \cancel{\text{pound}}} = \_\_\_ \text{ ounces}$$

The common unit “pound” can be cancelled because it appears in both the numerator and denominator.

$$\frac{9}{4} \bullet \frac{16\text{ ounces}}{1} = \_\_\_ \text{ ounces}$$

Multiply and simplify.

$$\frac{9 \bullet 16\text{ ounces}}{4 \bullet 1} = \_\_\_ \text{ ounces}$$

$$\frac{144\text{ ounces}}{4} = \_\_\_ \text{ ounces}$$

$$\frac{144\text{ ounces}}{4} = 36\text{ ounces}$$

Answer

There are 36 ounces in $2\frac{1}{4}$ pounds.

Example

Problem **How many tons is 6,500 pounds?**

$$6{,}500 \text{ pounds} = \_\_\_ \text{ tons}$$

Begin by reasoning about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 6,500.

$$6{,}500 \text{ pounds} \cdot \frac{1 \text{ ton}}{2{,}000 \text{ pounds}} = \_\_\_ \text{ tons}$$

$$\frac{6{,}500 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2{,}000 \text{ pounds}} = \_\_\_ \text{ tons}$$

Multiply by the conversion factor that relates tons to pounds: $\frac{1 \text{ ton}}{2{,}000 \text{ pounds}}$.

$$\frac{6{,}500 \ \cancel{\text{pounds}}}{1} \cdot \frac{1 \text{ ton}}{2{,}000 \ \cancel{\text{pounds}}} = \_\_\_ \text{ tons}$$

Apply the Factor Label method.

$$\frac{6{,}500}{1} \cdot \frac{1 \text{ ton}}{2{,}000} = \_\_\_ \text{ tons}$$

Multiply and simplify.

$$\frac{6{,}500 \text{ tons}}{2{,}000} = 3\frac{1}{4} \text{ tons}$$

Answer 6,500 pounds is equal to $3\frac{1}{4}$ tons.

Self Check A
How many pounds is 72 ounces?

Applying Unit Conversions

Objective 2, 3

There are times when you need to perform calculations on measurements that are given in different units. To solve these problems, you need to convert one of the measurements to the same unit of measurement as the other measurement.

Think about whether the unit you are converting to is smaller or larger than the unit you are converting from. This will help you be sure that you are making the right computation. You can use the factor label method to make the conversion from one unit to another.

Here is an example of a problem that requires converting between units.

Example

Problem **A municipal trash facility allows a person to throw away a maximum of 30 pounds of trash per week. Last week, 140 people threw away the maximum allowable trash. How many tons of trash did this equal?**

$$140 \bullet 30 \text{ pounds} = 4{,}200 \text{ pounds}$$

Determine the total trash for the week expressed in pounds.
If 140 people each throw away 30 pounds, you can find the total by multiplying.

$$4{,}200 \text{ pounds} = \_\_\_ \text{ tons}$$

Then convert 4,200 pounds to tons. Reason about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 4,200.

$$\frac{4{,}200 \text{ pounds}}{1} \bullet \frac{1 \text{ ton}}{2{,}000 \text{ pounds}} = \_\_\_ \text{ tons}$$

Find the conversion factor appropriate for the situation: $\frac{1 \text{ ton}}{2{,}000 \text{ pounds}}$.

$$\frac{4{,}200 \ \cancel{\text{pounds}}}{1} \bullet \frac{1 \text{ ton}}{2{,}000 \ \cancel{\text{pounds}}} = \_\_\_ \text{ tons}$$

$$\frac{4{,}200}{1} \bullet \frac{1 \text{ ton}}{2{,}000} = \_\_\_ \text{ tons}$$

$$\frac{4{,}200 \bullet 1 \text{ ton}}{1 \bullet 2{,}000} = \_\_\_ \text{ tons}$$

Multiply and simplify.

$$\frac{4{,}200 \text{ ton}}{2{,}000} = \_\_\_ \text{ tons}$$

$$\frac{4{,}200 \text{ ton}}{2{,}000} = 2\frac{1}{10} \text{ tons}$$

Answer The total amount of trash generated is $2\frac{1}{10}$ tons.

Let's revisit the coffee price problem that was posed earlier. We can use unit conversion to solve this problem.

Example

Problem **The grocery store sells a 36 ounce canister of ground coffee for $14, and sells bulk coffee for $7 per pound. Which is the better deal?**

$$36 \text{ ounces} = \_\_\_ \text{ pounds}$$

Since canister pricing is for ounces, convert the weight of the canister to pounds.

$$\frac{36 \text{ ounces}}{1} \bullet \frac{1 \text{ pound}}{16 \text{ ounces}} = \_\_\_ \text{ pound}$$

First use the factor label method to convert ounces to pounds.

$$\frac{36 \cancel{\text{ ounces}}}{1} \bullet \frac{1 \text{ pound}}{\cancel{\text{ounces}}} = \_\_\_ \text{ pound}$$

$$\frac{36}{1} \bullet \frac{1 \text{ pound}}{16} = 2\frac{1}{4} \text{ pounds}$$

$$\frac{\$14}{2\frac{1}{4} \text{ pounds}}$$

Now calculate the price per pound by dividing.

$$\frac{\$14}{2\frac{1}{4} \text{ pounds}} \approx \$6.22 \text{ per pound}$$

Answer The canister is a better deal at $6.22 per pound.

Self Check B

The average weight of a northern bluefin tuna is 1,800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern bluefin tuna?

Summary

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2,000 pounds. While an object's weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people.

Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

4.4.2 Self Check Solutions

Self Check A

How many pounds is 72 ounces?

There are 16 ounces in one pound, so 72 ounces • $\frac{1 \text{ pound}}{16 \text{ ounces}} = 4\frac{1}{2}\text{pounds}$.

Self Check B

The average weight of a northern bluefin tuna is 1,800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern bluefin tuna?

$2\frac{1}{2}$ tons = 5,000 pounds. 5,000 pounds – 1,800 pounds = 3,200 pounds.

4.4.3 U.S. Measurement - Capacity

Learning Objective(s)

1 Define units of capacity and convert from one to another.
2 Perform arithmetic calculations on units of capacity.
3 Solve application problems involving units of capacity.

Introduction

Capacity is the amount of liquid (or other pourable substance) that an object can hold when it's full. When a liquid, such as milk, is being described in gallons or quarts, this is a measure of capacity.

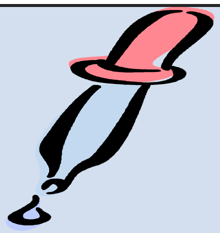
Understanding units of capacity can help you solve problems like this: Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How many cups of soup did they have all together?

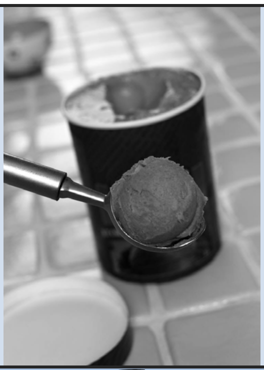
Units of Capacity

Objective 1

There are five main units for measuring capacity in the U.S. customary measurement system. The smallest unit of measurement is a **fluid ounce**. "Ounce" is also used as a measure of weight, so it is important to use the word "fluid" with ounce when you are talking about capacity. Sometimes the prefix "fluid" is not used when it is clear from the context that the measurement is capacity, not weight.

The other units of capacity in the customary system are the **cup**, **pint**, **quart**, and **gallon**. The table below describes each unit of capacity and provides an example to illustrate the size of the unit of measurement.

Fluid Ounce A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62°F weighs about one ounce. The amount of liquid medicine is often measured in fluid ounces.	
Cup A unit equal to 8 fluid ounces. The capacity of a standard measuring cup is one cup.	

Pint A unit equal to 16 fluid ounces, or 2 cups. The capacity of a carton of ice cream is often measured in pints.	
Quart A unit equal to 32 fluid ounces, or 4 cups. You often see quarts of milk being sold in the supermarket.	
Gallon A unit equal to 4 quarts, or 128 fluid ounces. When you fill up your car with gasoline, the price of gas is often listed in dollars per gallon.	

You can use any of these five measurement units to describe the capacity of an object, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the capacity of a swimming pool in gallons and the capacity of an expensive perfume in fluid ounces.

Sometimes you will need to convert between units of measurement. For example, you might want to express 5 gallons of lemonade in cups if you are trying to determine how many 8-fluid ounce servings the amount of lemonade would yield.

The table below shows some of the most common equivalents and conversion factors for the five customary units of measurement of capacity.

Unit Equivalents	Conversion Factors (heavier to lighter units of measurement)	Conversion Factors (lighter to heavier units of measurement)
1 cup = 8 fluid ounces	$\frac{1\text{ cup}}{8\text{ fluid ounces}}$	$\frac{8\text{ fluid ounces}}{1\text{ cup}}$
1 pint = 2 cups	$\frac{1\text{ pint}}{2\text{ cups}}$	$\frac{2\text{ cups}}{1\text{ pint}}$
1 quart = 2 pints	$\frac{1\text{ quart}}{2\text{ pints}}$	$\frac{2\text{ pints}}{1\text{ quart}}$
1 quart = 4 cups	$\frac{1\text{ quart}}{4\text{ cups}}$	$\frac{4\text{ cups}}{1\text{ quart}}$
1 gallon = 4 quarts	$\frac{1\text{ gallon}}{4\text{ quarts}}$	$\frac{4\text{ quarts}}{1\text{ gallon}}$
1 gallon = 16 cups	$\frac{1\text{ gallon}}{16\text{ cups}}$	$\frac{16\text{ cups}}{1\text{ gallon}}$

Converting Between Units of Capacity

Objective 2

As with converting units of length and weight, you can use the factor label method to convert from one unit of capacity to another. An example of this method is shown below.

Example

Problem

How many pints is $2\frac{3}{4}$ gallons?

$2\frac{3}{4}$ gallons =___ pints

Begin by reasoning about your answer. Since a gallon is larger than a pint, expect the answer in pints to be a number greater than $2\frac{3}{4}$.

$\frac{11\text{ gallons}}{4}\bullet\frac{4\text{ quarts}}{1\text{ gallon}}\bullet\frac{2\text{ pints}}{1\text{ quart}}=$___ pints

The table above does not contain a conversion factor for gallons and pints, so you cannot convert it in one step. However, you can use quarts as an intermediate unit, as shown here.

$$\frac{11\ \cancel{\text{gallons}}}{4} \bullet \frac{4\ \cancel{\text{quarts}}}{1\ \cancel{\text{gallon}}} \bullet \frac{2\ \text{pints}}{1\ \cancel{\text{quart}}} = \_\_\_\ \text{pints}$$

Set up the equation so that two sets of labels cancel—gallons and quarts.

$$\frac{11}{4} \bullet \frac{4}{1} \bullet \frac{2\ \text{pints}}{1} = \_\_\_\ \text{pints}$$

$$\frac{11 \bullet 4 \bullet 2\ \text{pints}}{4 \bullet 1 \bullet 1} = \_\_\_\ \text{pints}$$

Multiply and simplify.

$$\frac{88\ \text{pints}}{4} = 22\ \text{pints}$$

Answer $2\frac{3}{4}$ gallons is 22 pints.

Example

Problem **How many gallons is 32 fluid ounces?**

32 fluid ounces = ___ gallons

Begin by reasoning about your answer. Since gallons is a larger unit than fluid ounces, expect the answer to be less than 32.

$$\frac{32\ \text{fl oz}}{1} \bullet \frac{1\ \text{cup}}{8\ \text{fl oz}} \bullet \frac{1\ \text{pt}}{2\ \text{cups}} \bullet \frac{1\ \text{qt}}{2\ \text{pt}} \bullet \frac{1\ \text{gal}}{4\ \text{qt}} = \_\_\_\ \text{gal}$$

The table above does not contain a conversion factor for gallons and fluid ounces, so you cannot convert it in one step. Use a series of intermediate units, as shown here.

$$\frac{32\ \cancel{\text{fl oz}}}{1} \bullet \frac{1\ \cancel{\text{cup}}}{8\ \cancel{\text{fl oz}}} \bullet \frac{1\ \cancel{\text{pt}}}{2\ \cancel{\text{cups}}} \bullet \frac{1\ \cancel{\text{qt}}}{2\ \cancel{\text{pt}}} \bullet \frac{1\ \text{gal}}{4\ \cancel{\text{qt}}} = \_\_\_\ \text{gal}$$

Cancel units that appear in both the numerator and denominator.

$$\frac{32}{1} \bullet \frac{1}{8} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1\ \text{gal}}{4} = \_\_\_\ \text{gal}$$

$$\frac{32 \bullet 1 \bullet 1 \bullet 1 \bullet 1\ \text{gal}}{1 \bullet 8 \bullet 2 \bullet 2 \bullet 4} = \_\_\_\ \text{gal}$$

Multiply and simplify.

$$\frac{32\text{ gal}}{128} = \frac{1}{4}\text{ gal}$$

Answer 32 fluid ounces is the same as $\frac{1}{4}$ gallon.

Self Check A
Find the sum of 4 gallons and 2 pints. Express your answer in cups.

Applying Unit Conversions

Objective 3

There are times when you will need to combine measurements that are given in different units. In order to do this, you need to convert first so that the units are the same.

Consider the situation posed earlier in this topic.

Example

Problem	**Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How much soup did they have total?**	
	1 quart + 3 pints + 9 cups	Since the problem asks for the total amount of soup, you must add the three quantities. Before adding, you must convert the quantities to the same unit. The problem does not require a particular unit, so you can choose. Cups might be the easiest computation.
	1 quart = 4 cups	This is given in the table of equivalents.
	$\frac{3\text{ pints}}{1} \bullet \frac{2\text{ cups}}{1\text{ pint}} = \underline{\quad}\text{ cups}$	Use the factor label method to convert pints to cups.
	$\frac{3\ \cancel{\text{pints}}}{1} \bullet \frac{2\text{ cups}}{1\ \cancel{\text{pint}}} = 6\text{ cups}$	
	4 cups + 6 cups + 9 cups = 19 cups	Add the 3 quantities.
Answer	There are 19 cups of soup for the dinner.	

Example

Problem	**Natasha is making lemonade to bring to the beach. She has two containers. One holds one gallon and the other holds 2 quarts. If she fills both containers, how many cups of lemonade will she have?**	
	1 gallon + 2 quarts = ___ cups	This problem requires you to find the sum of the capacity of each container and then convert that sum to cups.
	4 quarts + 2 quarts = 6 quarts	First, find the sum in quarts. 1 gallon is equal to 4 quarts.
	$\frac{6\text{ quarts}}{1} \bullet \frac{2\text{ pints}}{1\text{ quart}} \bullet \frac{2\text{ cups}}{1\text{ pint}} = \_\_\_\text{ cups}$	Since the problem asks for the capacity in cups, convert 6 quarts to cups.
	$\frac{6\ \cancel{\text{quarts}}}{1} \bullet \frac{2\ \cancel{\text{pints}}}{1\ \cancel{\text{quart}}} \bullet \frac{2\text{ cups}}{1\ \cancel{\text{pint}}} = \_\_\_\text{ cups}$	Cancel units that appear in both the numerator and denominator.
	6 • 2 • 2 = 24 cups	Multiply.
Answer	Natasha will have 24 cups of lemonade.	

Another way to work the problem above would be to first change 1 gallon to 16 cups and change 2 quarts to 8 cups. Then add: 16 + 8 = 24 cups.

Self Check B

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2-gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

Summary

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

4.4.3 Self Check Solutions

Self Check A

Find the sum of 4 gallons and 2 pints. Express your answer in cups.

Each gallon has 16 cups, so 4 • 16 = 64 will give you the number of cups in 4 gallons. Each pint has 2 cups, so 2 • 2 = 4 will give you the number of cups in 2 pints. 64 + 4 = 68 cups.

Self Check B

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2-gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

5 quarts = 5 • 4 cups = 20 cups, so 24 cups of chili will not fit into the 5-quart pot. 2 gallons = 32 cups, so 24 cups of chili will fit in this pot.

4.5.1 The Metric System

Learning Objective(s)

1 Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and volume.

2 Define the metric prefixes and use them to perform basic conversions among metric units.

Introduction

Objective 1

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like “kilometer,” “liter,” and “milligram” to measure the length, volume, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of “10” is not present in the U.S. customary system—there are 12 inches in a foot, and 3 feet in a yard…and 5,280 feet in a mile!

So, what if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

What is Metric?

The metric system uses units such as **meter**, **liter**, and **gram** to measure length, liquid volume, and mass, just as the U.S. customary system uses feet, quarts, and ounces to measure these.

In addition to the difference in the basic units, the metric system is based on 10s, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word “meter” is part of all of these units.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1,000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of “getting bigger or smaller by 10” is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard, and 16 ounces equals 1 pound.

Length, Mass, and Volume

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.

Length	**Mass**	**Volume**
basic units		
meter	gram	liter
other units you may see		
kilometer	kilogram	dekaliter
centimeter	centigram	centiliter
millimeter	milligram	milliliter

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram.

Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word "mass" is used here instead of "weight." In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an object's weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the object's mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeably—but it is worth noting that there is a difference!

Finally, the basic metric unit of volume is the liter. A liter is slightly larger than a quart.

		
The handle of a shovel is about 1 meter.	A paperclip weighs about 1 gram.	A medium-sized container of milk is about 1 liter.

Though it is rarely necessary to convert between the customary and metric systems, sometimes it helps to have a mental image of how large or small some units are. The table below shows the relationship between some common units in both systems.

	Common Measurements in Customary and Metric Systems
Length	1 centimeter is a little less than half an inch.
	1.6 kilometers is about 1 mile.
	1 meter is about 3 inches longer than 1 yard.
Mass	1 kilogram is a little more than 2 pounds.
	28 grams is about the same as 1 ounce.
Volume	1 liter is a little more than 1 quart.
	4 liters is a little more than 1 gallon.

Prefixes in the Metric System

Objective **2**

The metric system is a base 10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the **prefix**. To tell whether the unit is measuring length, mass, or volume, you look at the base.

Prefixes in the Metric System						
kilo-	*hecto-*	*deka-*	meter gram liter	*deci-*	*centi-*	*milli-*
1,000 times **larger** than base unit	100 times **larger** than base unit	10 times **larger** than base unit	base units	10 times **smaller** than base unit	100 times **smaller** than base unit	1,000 times **smaller** than base unit

Using this table as a reference, you can see the following:

- A kilogram is 1,000 times larger than one gram (so 1 kilogram = 1,000 grams).
- A centimeter is 100 times smaller than one meter (so 1 meter = 100 centimeters).
- A dekaliter is 10 times larger than one liter (so 1 dekaliter = 10 liters).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

Measuring Mass in the Metric System						
kilogram (kg)	hectogram (hg)	dekagram (dag)	gram (g)	decigram (dg)	centigram (cg)	milligram (mg)
1,000 grams	100 grams	10 grams	gram	0.1 gram	0.01 gram	0.001 gram

Since the prefixes remain constant through the metric system, you could create similar charts for length and volume. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or volume (liter).

Self Check A

Which of the following sets of three units are all metric measurements of length?

A) inch, foot, yard

B) kilometer, centimeter, millimeter

C) kilogram, gram, centigram

D) kilometer, foot, decimeter

Converting Units Up and Down the Metric Scale

Converting between metric units of measure requires knowledge of the metric prefixes and an understanding of the decimal system—that's about it.

For instance, you can figure out how many centigrams are in one dekagram by using the table above. One dekagram is larger than one centigram, so you expect that one dekagram will equal many centigrams.

In the table, each unit is 10 times larger than the one to its immediate right. This means that 1 dekagram = 10 grams; 10 grams = 100 decigrams; and 100 decigrams = 1,000 centigrams. So, 1 dekagram = 1,000 centigrams.

Example

Problem **How many milligrams are in one decigram?**

kg	hg	dag	g	dg	cg	mg
				^		^

Identify locations of milligrams and decigrams.

Decigrams (dg) are larger than milligrams (mg), so you expect there to be many mg in one dg.

					×10	×10
kg	hg	dag	g	dg	cg	mg
				^		^

→ →

Dg is 10 times larger than a cg, and a cg is 10 times larger than a mg.

Since you are going from a larger unit to a smaller unit, multiply.

$1 \text{ dg} \bullet 10 \bullet 10 = 100 \text{ mg}$

Multiply: $1 \bullet 10 \bullet 10$, to find the number of milligrams in one decigram.

Answer There are 100 milligrams (mg) in 1 decigram (dg).

Example

Problem **Convert 1 centimeter to kilometers.**

km	hm	dam	m	dm	cm	mm
^					^	

Identify locations of kilometers and centimeters.

Kilometers (km) are larger than centimeters (cm), so you expect there to be less than one km in a cm.

	÷10 ÷10 ÷10 ÷10 ÷10 km | hm | dam | m | dm | cm | mm ^ ^ ← ← ← ← ←	Cm is 10 times smaller than a dm; a dm is 10 times smaller than a m, etc. Since you are going from a smaller unit to a larger unit, divide.
	1 cm ÷ 10 ÷ 10 ÷ 10 ÷ 10 ÷ 10 = 0.00001 km	Divide: 1 ÷ 10 ÷ 10 ÷ 10 ÷ 10 ÷ 10, to find the number of kilometers in one centimeter.
Answer	1 centimeter (cm) = 0.00001 kilometers (km).	

Once you begin to understand the metric system, you can use a shortcut to convert among different metric units. The size of metric units increases tenfold as you go up the metric scale. The decimal system works the same way: a tenth is 10 times larger than a hundredth; a hundredth is 10 times larger than a thousandth, etc. By applying what you know about decimals to the metric system, converting among units is as simple as moving decimal points.

Here is the first problem from above: How many milligrams are in one decigram? You can recreate the order of the metric units as shown below:

$$kg \quad hg \quad dag \quad g \quad d\underbrace{g \quad c}_{1}\underbrace{g \quad m}_{2}g$$

This question asks you to start with 1 decigram and convert that to milligrams. As shown above, milligrams is two places to the right of decigrams. You can just move the decimal point two places to the right to convert decigrams to milligrams: $1\,dg = 1\underbrace{0}_{1}\underbrace{0}_{2}.\,mg$.

The same method works when you are converting from a smaller to a larger unit, as in the problem: Convert 1 centimeter to kilometers.

$$k\underbrace{m \quad h}_{5}\underbrace{m \quad d}_{4}\underbrace{am \quad}_{3}\underbrace{m \quad d}_{2}\underbrace{m \quad c}_{1}m \quad mm$$

Note that instead of moving to the right, you are now moving to the left—so the decimal point must do the same: $1\,cm = 0.\underbrace{0}_{5}\underbrace{0}_{4}\underbrace{0}_{3}\underbrace{0}_{2}\underbrace{1}_{1}\,km$.

Self Check B

How many milliliters are in 1 liter?

Self Check C

Convert 3,085 milligrams to grams.

Factor Label Method

There is yet another method that you can use to convert metric measurements—the **factor label method**. You used this method when you were converting measurement units within the U.S. customary system.

The factor label method works the same in the metric system; it relies on the use of unit fractions and the cancelling of intermediate units. The table below shows some of the **unit equivalents** and **unit fractions** for length in the metric system. (You should notice that all of the unit fractions contain a factor of 10. Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.)

Also, notice that two new prefixes have been added here: mega- (which is very big) and micro- (which is very small).

Unit Equivalents	Conversion Factors	
1 meter = 1,000,000 micrometers	$\frac{1\,m}{1{,}000{,}000\,\mu m}$	$\frac{1{,}000{,}000\,\mu m}{1\,m}$
1 meter = 1,000 millimeters	$\frac{1\,m}{1{,}000\,mm}$	$\frac{1{,}000\,mm}{1\,m}$
1 meter = 100 centimeters	$\frac{1\,m}{100\,cm}$	$\frac{100\,cm}{1\,m}$
1 meter = 10 decimeters	$\frac{1\,m}{10\,dm}$	$\frac{10\,dm}{1\,m}$
1 dekameter = 10 meters	$\frac{1\,dam}{10\,m}$	$\frac{10\,m}{1\,dam}$
1 hectometer = 100 meters	$\frac{1\,hm}{100\,m}$	$\frac{100\,m}{1\,hm}$
1 kilometer = 1,000 meters	$\frac{1\,km}{1{,}000\,m}$	$\frac{1{,}000\,m}{1\,km}$
1 megameter = 1,000,000 meters	$\frac{1\,Mm}{1{,}000{,}000\,m}$	$\frac{1{,}000{,}000\,m}{1\,Mm}$

When applying the factor label method in the metric system, be sure to check that you are not skipping over any intermediate units of measurement!

Example

Problem **Convert 7,225 centimeters to meters.**

7,225 cm = ___ m	Meters is larger than centimeters, so you expect your answer to be less than 7,225.
$\frac{7,225\ cm}{1} \bullet \frac{1\,m}{100\ cm} = \_\_\_ m$	Using the factor label method, write 7,225 cm as a fraction and use unit fractions to convert it to m.
$\frac{7,225\ \cancel{cm}}{1} \bullet \frac{1\,m}{100\ \cancel{cm}} = \_\_\_ m$	Cancel similar units, multiply, and simplify.
$\frac{7,225}{1} \bullet \frac{1\,m}{100} = \frac{7,225}{100} m$	
$\frac{7,225m}{100} = 72.25m$	

Answer 7,225 centimeters = 72.25 meters

Self Check D

Convert 32.5 kilometers to meters.

Now that you have seen how to convert among metric measurements in multiple ways, let's revisit the problem posed earlier.

Example

Problem **If you have a prescription for 5,000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?**

5,000 mg = ___ g? Need to convert mg to g.

$$\frac{5,000\ mg}{1} \bullet \frac{1\,g}{1,000\ mg} = \_\_\_ g$$

$$\frac{5,000\ \cancel{mg}}{1} \bullet \frac{1\,g}{1,000\ \cancel{mg}} = \_\_\_ g$$

$$\frac{5{,}000 \bullet 1g}{1 \bullet 1{,}000} = \frac{5{,}000g}{1{,}000}$$

$$\frac{5{,}000g}{1{,}000} = 5\ g$$

Answer 5 g = 5,000 mg, so the pharmacist did not make a mistake.

Summary

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo-, hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one.

This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

4.5.1 Self Check Solutions

Self Check A

Which of the following sets of three units are all metric measurements of length?

kilometer, centimeter, millimeter
Correct. All of these measurements are from the metric system. You can tell they are measurements of length because they all contain the word “meter.”

Self Check B

How many milliliters are in 1 liter?

There are 10 milliliters in a centiliter, 10 centiliters in a deciliter, and 10 deciliters in a liter. Multiply: 10 • 10 • 10, to find the number of milliliters in a liter, 1,000.

Self Check C

Convert 3,085 milligrams to grams.

One gram is 1,000 times larger than a milligram, so you can move the decimal point in 3,085 three places to the left.

Self Check D

Using whichever method you prefer, convert 32.5 kilometers to meters.

32,500 meters

$\frac{32.5\ km}{1} \bullet \frac{1{,}000\ m}{1\ km} = \frac{32{,}500\ m}{1}$. The km units cancel, leaving the answer in m.

4.5.2 Using Metric Conversions to Solve Problems

Learning Objective(s)
1 Solve application problems involving metric units of length, mass, and volume.

Introduction

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

Understanding Context and Performing Conversions

Objective 1

The first step in solving any real-world problem is to understand its context. This will help you figure out what kinds of solutions are reasonable (and the problem itself may give you clues about what types of conversions are necessary). Here is an example.

Example

Problem	**Marcus bought at 2 meter board, and cut off a piece 1 meter and 35 cm long. How much board is left?**	
	2 meters – 1 meter 35 cm	To answer this question, we will need to subtract. First convert all measurements to one unit. Here we will convert to centimeters.
	$\frac{2\,m}{1} \bullet \frac{100\,cm}{1\,m} = \_\_\_\ cm$	Use the factor label method and unit fractions to convert from meters to centimeters.
	$\frac{2\,\cancel{m}}{1} \bullet \frac{100cm}{1\,\cancel{m}} = \_\_\_\ cm$	Cancel, multiply, and solve.
	$\frac{200\,cm}{1} = 200\,cm$	

	1 meter + 35 cm 100 cm + 35 cm 135 cm	Convert the 1 meter to centimeters, and combine with the additional 35 centimeters.
	200 cm – 135 cm 65 cm	Subtract the cut length from the original board length.
Answer	There is 65 cm of board left.	

An example with a different context, but still requiring conversions, is shown below.

Example		
Problem	**A faucet drips 10 ml every minute. How much water will be wasted in a week?**	
	$\frac{10\ ml}{1\text{ minute}}\bullet\frac{60\text{ minute}}{1\text{ hour}}\bullet\frac{24\text{ hours}}{1\text{ day}}\bullet\frac{7\text{ days}}{1\text{ week}}$	Start by calculating how much water will be used in a week using the factor label method to convert the time units.
	$\frac{10\ ml}{1\ \cancel{\text{minute}}}\bullet\frac{60\ \cancel{\text{minute}}}{1\ \cancel{\text{hour}}}\bullet\frac{24\ \cancel{\text{hours}}}{1\ \cancel{\text{day}}}\bullet\frac{7\ \cancel{\text{days}}}{1\text{ week}}$ $\frac{10\bullet60\bullet24\bullet7\ ml}{1\bullet1\bullet1\bullet1\text{ week}}$	Cancel, multiply and solve.
	$\frac{100800\ ml}{1\text{ week}}$	To give a more useable answer, convert this into liters.
	$\frac{100800\text{ ml}}{1\text{ week}}\bullet\frac{1\text{ L}}{1000\text{ ml}}$ $\frac{100800\ \cancel{\text{ml}}}{1\text{ week}}\bullet\frac{1\text{ L}}{1000\ \cancel{\text{ml}}}$ $\frac{100800\text{ L}}{1000\text{ week}}=100.8\frac{\text{L}}{\text{week}}$	Cancel, multiply and solve.
Answer	The faucet wastes about 100.8 liters each week.	

This problem asked for the difference between two quantities. The easiest way to find this is to convert one quantity so that both quantities are measured in the same unit, and then subtract one from the other.

Self Check A

A bread recipe calls for 600 g of flour. How many kilograms of flour would you need to make 5 loaves?

Checking your Conversions

Sometimes it is a good idea to check your conversions using a second method. This usually helps you catch any errors that you may make, such as using the wrong unit fractions or moving the decimal point the wrong way.

Example

Problem **A bottle contains 1.5 liters of a beverage. How many 250 mL servings can be made from that bottle?**

$1.5 \text{ L} \div 250 \text{ mL}$	To answer the question, you will need to divide 1.5 liters by 250 milliliters. To do this, convert both to the same unit. You could convert either measurement.
$250 \text{ mL} = \underline{\quad} \text{ L}$	Convert 250 mL to liters
$\frac{250 \text{ mL}}{1} \cdot \frac{1 \text{L}}{1000 \text{ mL}} = \underline{\quad} \text{ L}$	
$\frac{250 \text{ L}}{1000} = 0.25 \text{ L}$	
$1.5 \text{ L} \div 250 \text{ mL} = \frac{1.5 \text{ L}}{250 \text{ mL}} = \frac{1.5 \text{ L}}{0.25 \text{ L}}$	Now we can divide using the converted measurement
$\frac{1.5 \text{ L}}{0.25 \text{ L}} = 6$	

Answer The bottle holds 6 servings.

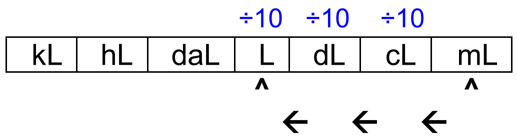
Having come up with the answer, you could also check your conversions using the quicker "move the decimal" method, shown below.

Example

Problem **A bottle contains 1.5 liters of a beverage. How many 250 mL servings can be made from that bottle?**

250 mL = ___ L

You need to convert 250 mL to liters



kL	hL	daL	L	dL	cL	mL

On the chart, L is three places to the left of mL.

250. mL → 0.250 L

Move the decimal point three places to the left in 250 mL

$$1.5\text{ L} \div 250\text{ mL} = \frac{1.5\,\text{L}}{250\,\text{mL}} = \frac{1.5\,\text{L}}{0.25\,\text{L}}$$

Now divide as was done in the last example

$$\frac{1.5\,\text{L}}{0.25\,\text{L}} = 6$$

Answer The bottle holds 6 servings.

The initial answer checks out — the bottle holds 6 servings. Checking one conversion with another method is a good practice for catching any errors in scale.

Summary

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the “move the decimal” method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

4.5.2 Self Check Solutions

Self Check A

A bread recipe calls for 600 g of flour. How many kilograms of flour would you need to make 5 loaves?

Multiplying 600 g per loaf by the 5 loaves,
600g · 5 = 3000 g
Using factor labels or the “move the decimal” method, convert this to 3 kilograms
You will need 3 kg of flour to make 5 loaves.

4.6 Temperature Scales

Learning Objective(s)

1 State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.

2 Convert from one temperature scale to the other, using conversion formulas.

Introduction

Turn on the television any morning and you will see meteorologists talking about the day's weather forecast. In addition to telling you what the weather conditions will be like (sunny, cloudy, rainy, muggy), they also tell you the day's forecast for high and low temperatures. A hot summer day may reach 100° in Philadelphia, while a cool spring day may have a low of 40° in Seattle.

If you have been to other countries, though, you may notice that meteorologists measure heat and cold differently outside of the United States. For example, a TV weatherman in San Diego may forecast a high of 89°, but a similar forecaster in Tijuana, Mexico—which is only 20 miles south—may look at the same weather pattern and say that the day's high temperature is going to be 32°. What's going on here?

The difference is that the two countries use different temperature scales. In the United States, temperatures are usually measured using the **Fahrenheit** scale, while most countries that use the metric system use the **Celsius** scale to record temperatures. Learning about the different scales—including how to convert between them—will help you figure out what the weather is going to be like, no matter which country you find yourself in.

Measuring Temperature on Two Scales

Objective 1

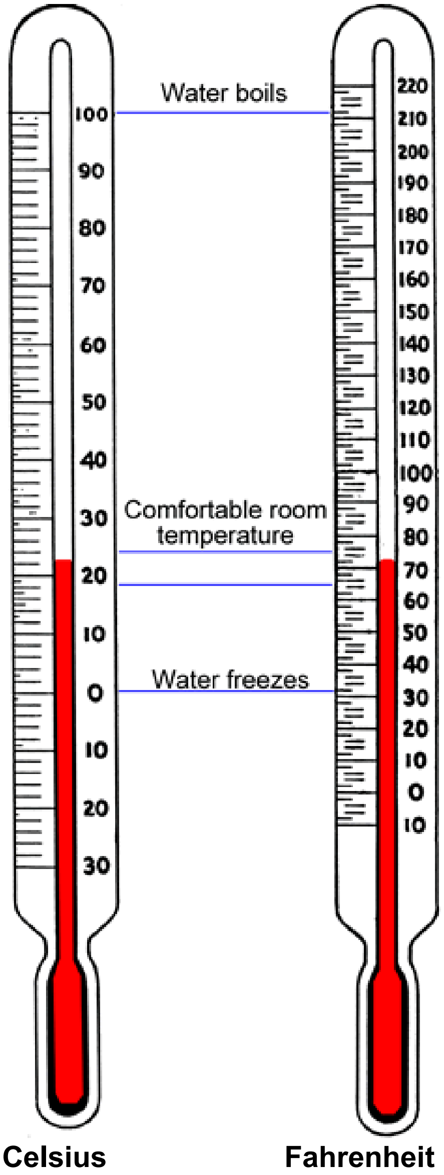
Fahrenheit and Celsius are two different scales for measuring temperature.

A thermometer measuring a temperature of 22° Celsius is shown here.

On the Celsius scale, water freezes at 0° and boils at 100°.

If the United States were to adopt the Celsius scale, forecast temperatures would rarely go below -30° or above 45°. (A temperature of -18° may be forecast for a cold winter day in Michigan, while a temperature of 43° may be predicted for a hot summer day in Arizona.)

Most office buildings maintain an indoor temperature between 18°C and 24°C to keep employees comfortable.



A thermometer measuring a temperature of 72° Fahrenheit is shown here.

On the Fahrenheit scale, water freezes at 32° and boils at 212°.

In the United States, forecast temperatures measured in Fahrenheit rarely go below -20° or above 120°. (A temperature of 0° may be forecast for a cold winter day in Michigan, while a temperature of 110° may be predicted for a hot summer day in Arizona.)

Most office buildings maintain an indoor temperature between 65°F and 75°F to keep employees comfortable.

Self Check A

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

Converting Between the Scales

Objective **2**

By looking at the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenheit measurement or vice versa. For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process as long as you use the formulas provided below.

Temperature Conversion Formulas

To convert a Fahrenheit measurement to a Celsius measurement, use this formula.

$$C = \frac{5}{9}(F - 32)$$

To convert a Celsius measurement to a Fahrenheit measurement, use this formula.

$$F = \frac{9}{5}C + 32$$

How were these formulas developed? They came from comparing the two scales. Since the freezing point is 0° in the Celsius scale and 32° on the Fahrenheit scale, we subtract 32 when converting from Fahrenheit to Celsius, and add 32 when converting from Celsius to Fahrenheit.

There is a reason for the fractions $\frac{5}{9}$ and $\frac{9}{5}$, also. There are 100 degrees between the freezing (0°) and boiling points (100°) of water on the Celsius scale and 180 degrees between the similar points (32° and 212°) on the Fahrenheit scale. Writing these two scales as a ratio, $\frac{F^\circ}{C^\circ}$, gives $\frac{180^\circ}{100^\circ} = \frac{180^\circ \div 20}{100^\circ \div 20} = \frac{9}{5}$. If you flip the ratio to be $\frac{C^\circ}{F^\circ}$, you get $\frac{100^\circ}{180^\circ} = \frac{100^\circ \div 20}{180^\circ \div 20} = \frac{5}{9}$. Notice how these fractions are used in the conversion formulas.

The example below illustrates the conversion of Celsius temperature to Fahrenheit temperature, using the boiling point of water, which is 100° C.

Example

Problem	**The boiling point of water is 100°C. What temperature does water boil at in the Fahrenheit scale?**	
	$F=\frac{9}{5}C+32$	A Celsius temperature is given. To convert it to the Fahrenheit scale, use the formula at the left.
	$F=\frac{9}{5}(100)+32$	Substitute 100 for *C* and multiply.
	$F=\frac{900}{5}+32$	
	$F=\frac{900\div 5}{5\div 5}+32$	Simplify $\frac{900}{5}$ by dividing numerator and denominator by 5.
	$F=\frac{180}{1}+32$	
	$F=212$	Add 180 + 32.
Answer	The boiling point of water is 212°F.	

Example

Problem	**Water freezes at 32°F. On the Celsius scale, what temperature is this?**	
	$C=\frac{5}{9}(F-32)$	A Fahrenheit temperature is given. To convert it to the Celsius scale, use the formula at the left.
	$C=\frac{5}{9}(32-32)$	Substitute 32 for *F* and subtract.
	$C=\frac{5}{9}(0)$	Any number multiplied by 0 is 0.
	$C=0$	
Answer	The freezing point of water is 0°C.	

The two previous problems used the conversion formulas to verify some temperature conversions that were discussed earlier—the boiling and freezing points of water. The next example shows how these formulas can be used to solve a real-world problem using different temperature scales.

Example

Problem **Two scientists are doing an experiment designed to identify the boiling point of an unknown liquid. One scientist gets a result of 120°C; the other gets a result of 250°F. Which temperature is higher and by how much?**

What is the difference between 120°C and 250°F?	One temperature is given in °C, and the other is given in °F. To find the difference between them, we need to measure them on the same scale.
$F=\frac{9}{5}C+32$	Use the conversion formula to convert 120°C to °F. (You could convert 250°F to °C instead; this is explained in the text after this example.)
$F=\frac{9}{5}(120)+32$	Substitute 120 for *C*.
$F=\frac{1080}{5}+32$	Multiply.
$F=\frac{1080\div 5}{5\div 5}+32$	Simplify $\frac{1080}{5}$ by dividing numerator and denominator by 5.
$F=\frac{216}{1}+32$	Add 216 + 32.
$F=248$	You have found that 120°C = 248°F.
250°F – 248°F = 2°F	To find the difference between 248°*F* and 250°F, subtract.

Answer 250°F is the higher temperature by 2°F.

You could have converted 250°F to °C instead, and then found the difference in the two measurements. (Had you done it this way, you would have found that 250°F = 121.1°C, and that 121.1°C is 1.1°C higher than 120°C.) Whichever way you choose, it is important to compare the temperature measurements within the same scale, and to apply the conversion formulas accurately.

Self Check B

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26°C. What is the average temperature in degrees Fahrenheit?

Summary

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale.

4.6 Self Check Solutions

Self Check A

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

Fahrenheit

Water boils at 212° on the Fahrenheit scale, so a measurement of 132° on a Fahrenheit scale is legitimate for hot (but non-boiling) water.

Self Check B

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26°C. What is the average temperature in degrees Fahrenheit?

Tatiana can find the Fahrenheit equivalent by solving the equation $F=\frac{9}{5}(26)+32$. The result is 78.8°F, which rounds to 79°F.

5.1.1 Integers

Learning Objective(s)
1 Locate integers on a number line.
2 Find the absolute value of a given number.
3 Find the opposite of a given number.

Introduction

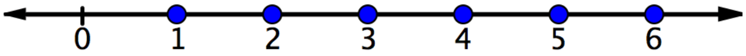
You've worked with numbers on a number line. You know how to graph numbers like 0, 1, 2, 3, etc. on the number line. There are other kinds of numbers that can be graphed on the number line, too**.** Let's see what they look like and where they are located on the number line.

Natural Numbers and Whole Numbers

In mathematics, it's sometimes helpful to talk about groups of things, which are called **sets**. Numbers can be grouped into sets, and a particular number can belong to more than one set.

You probably are familiar with the set of **natural numbers**, which are also called the **counting numbers**. These are the numbers 1, 2, 3, and so on—the numbers we use when *counting.*

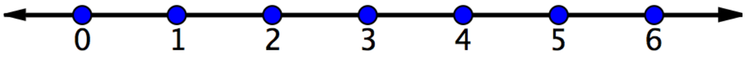
The following illustration shows the natural numbers graphed on a number line.



The number line continues in both directions. The set of natural numbers only continues to the right, so you can include 6, 7, and so on, all the way up into the hundreds, thousands, and beyond. You can only show so much on one picture!

When 0 is added to the set of 1, 2, 3, and so on, it forms the set of **whole numbers**. These are called "whole" because they have no fractional parts. (A trick to help you remember which are *natural* numbers and which are *whole* numbers is to think of a "hole," which can be represented by 0. The whole ("hole") numbers include 0, the natural numbers do not.)

The following illustration shows the whole numbers graphed on the number line.



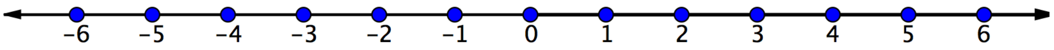
Integers

Objective **1**

When you work with something like temperature, you sometimes want to use numbers that are less than zero, which are called **negative numbers**. Negative numbers are written using a negative sign in front, such as −1, −5, and −30. These are read "negative one," "negative five," and "negative thirty." (The negative sign should not be read as "minus"; *minus* means subtraction.)

The numbers greater than 0 are called **positive numbers** and can be written with or without the “+” sign. Notice that 0 is neither positive nor negative!

Integers are the numbers: …, −3, −2, −1, 0, 1, 2, 3, …. Notice that all of the whole numbers are also integers. The following illustration shows the integers graphed on the number line. The integers include zero and continue to the right and to the left.



Self Check A

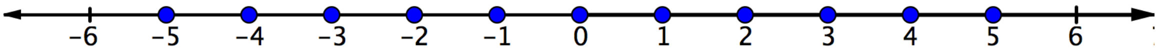
The number 0 belongs to which of the following sets of numbers?

natural numbers
whole numbers
integers

Absolute Value and the Number Line

Objective **2**

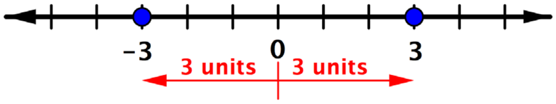
The number line below shows all the integers between and including −5 and 5. Notice that the positive integers go to the right: 1, 2, 3, and so on. The negative integers go to the left: −1, −2, −3, and so on.



The distance between a number’s place on the number line and 0 is called the number’s **absolute value**. To write the absolute value of a number, use short vertical lines (|) on either side of the number. For example, the absolute value of −3 is written |−3|.

Notice that distance is always positive or 0.

|−3| = 3, as −3 is 3 units away from 0 and |3| = 3, as 3 is 3 units away from 0.



Here are some other examples.

$$|0| = 0$$

$$|-23| = 23$$

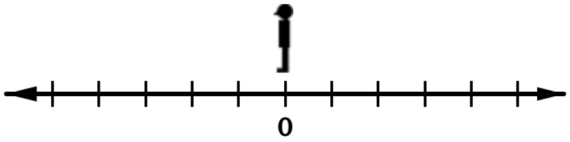
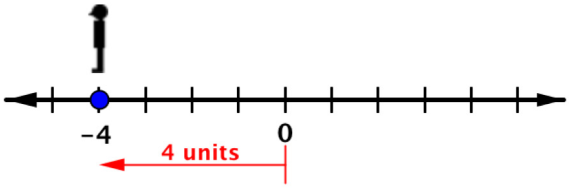
$$|6| = 6$$

$$|817| = 817$$

$$|-3{,}000| = 3{,}000$$

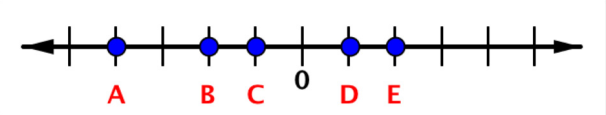
Example		
Problem	**Find |-7|**	
Answer	|−7|= 7	Since −7 is 7 units from 0, the absolute value is 7.

To locate an integer on the number line, imagine standing on the number line at 0. If the number is 0, you're there. If the number is positive, face to the right—numbers greater than 0. If the number is negative, face to the left—numbers less than 0. Then, move forward the number of units equal to the absolute value of the number.

Example		
Problem	**Find −4 on the number line. Then determine |−4|.**	
		Imagine standing at 0. Since −4 is negative, face to the left.
		Move 4 units from 0 in the negative direction. Draw a dot on the number line at that location, which is −4.
Answer	|−4| = 4	Direction moved does not affect absolute value, only the distance moved.

Self Check B

Which point represents −2 on this number line?



Opposites

Objective 3

You may have noticed that, except for 0, the integers come in pairs of positive and negative numbers: 1 and −1, 3 and −3, 72 and −72, and so on. Each number is the **opposite** of the other number in the pair: 72 is the opposite of −72, and −72 is the opposite of 72.

A number and its opposite are the same distance from 0, so they have the same absolute value.

$$|72| = 72, \text{ and } |-72| = 72$$

The set of **integers** are all the whole numbers and their opposites.

Self Check C

What is the opposite of −29?

Summary

Some numbers are natural numbers (1, 2, 3, ...) or whole numbers (0, 1, 2, 3, ...). Whole numbers are also integers. There are other integers that are the opposites of the whole numbers (−1, −2, −3, ...). These negative numbers lie to the left of 0 on the number line. Integers are the whole numbers and their opposites. The absolute value of a number is its distance to 0 on the number line. Absolute values are always positive or 0.

5.1.1 Self Check Solutions

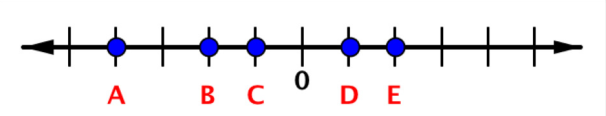
Self Check A

The number 0 belongs to which of the following sets of numbers?

Both whole numbers and integers include 0, but the natural numbers do not.

Self Check B

Which point represents −2 on this number line?



Point B is 2 units to the left of 0, so it represents −2.

Self Check C

What is the opposite of −29?

Answer: The opposite of −29 is 29.

5.1.2 Rational and Real Numbers

Learning Objective(s)

1 Identify the subset(s) of the real numbers that a given number belongs to.
2 Locate points on a number line.
3 Compare rational numbers.
4 Identify rational and irrational numbers.

Introduction

You've worked with fractions and decimals, like 3.8 and $21\frac{2}{3}$. These numbers can be found between the integer numbers on a number line. There are other numbers that can be found on a number line, too. When you include all the numbers that can be put on a number line, you have the real number line. Let's dig deeper into the number line and see what those numbers look like. Let's take a closer look to see where these numbers fall on the number line.

Rational Numbers

Objective **1, 2**

The fraction $\frac{16}{3}$, mixed number $5\frac{1}{3}$, and decimal 5.33… (or $5.\overline{3}$) all represent the same number. This number belongs to a set of numbers that mathematicians call **rational numbers**. Rational numbers are numbers that can be written as a ratio of two integers. Regardless of the form used, $5.\overline{3}$ is rational because this number *can* be written as the ratio of 16 over 3, or $\frac{16}{3}$.

Examples of rational numbers include the following.

0.5, as it can be written as $\frac{1}{2}$

$2\frac{3}{4}$, as it can be written as $\frac{11}{4}$

−1.6, as it can be written as $-1\frac{6}{10}=\frac{-16}{10}$

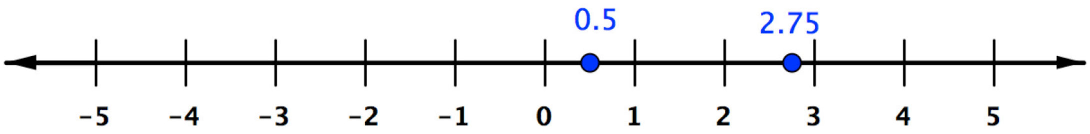
4, as it can be written as $\frac{4}{1}$

-10, as it can be written as $\frac{-10}{1}$

All of these numbers can be written as the ratio of two integers.

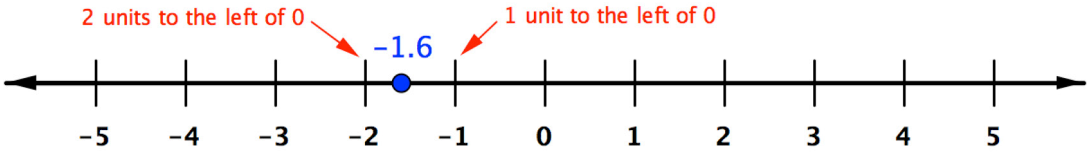
You can locate these points on the number line.

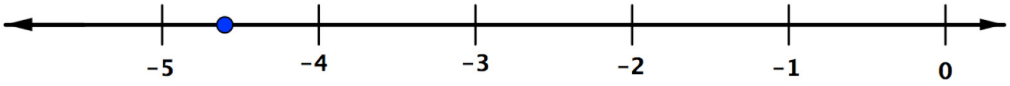
In the following illustration, points are shown for 0.5 or $\frac{1}{2}$, and for 2.75 or $2\frac{3}{4} = \frac{11}{4}$.



As you have seen, rational numbers can be negative. Each positive rational number has an opposite. The opposite of $5.\overline{3}$ is $-5.\overline{3}$, for example.

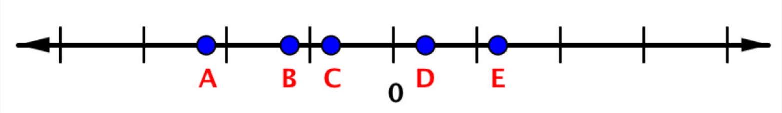
Be careful when placing **negative numbers** on a number line. The negative sign means the number is to the left of 0, and the absolute value of the number is the distance from 0. So to place −1.6 on a number line, you would find a point that is |−1.6| or 1.6 units to the left of 0. This is more than 1 unit away, but less than 2.



Example	
Problem	**Place $-\frac{23}{5}$ on a number line.**
	It's helpful to first write this improper fraction as a mixed number: 23 divided by 5 is 4 with a remainder of 3, so $-\frac{23}{5}$ is $-4\frac{3}{5}$. Since the number is negative, you can think of it as moving $4\frac{3}{5}$ units to the *left* of 0. $-4\frac{3}{5}$ will be between −4 and −5.
Answer	

Self Check A

Which of the following points represents $-1\frac{1}{4}$?



Comparing Rational Numbers

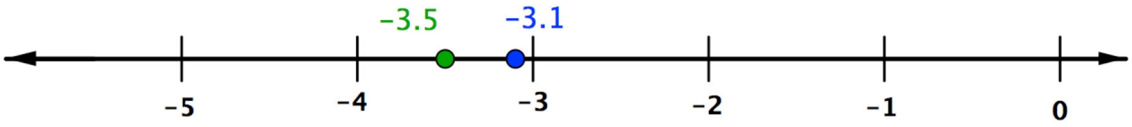
Objective 3

When two **whole numbers** are graphed on a number line, the number to the right on the number line is always greater than the number on the left.

The same is true when comparing two **integers** or rational numbers. The number to the right on the number line is always greater than the one on the left.

Here are some examples.

Numbers to Compare	**Comparison**	**Symbolic Expression**
−2 and −3	−2 is greater than −3 because −2 is to the right of −3	$-2 > -3$ or $-3 < -2$
2 and 3	3 is greater than 2 because 3 is to the right of 2	$3 > 2$ or $2 < 3$
−3.5 and −3.1	−3.1 is greater than −3.5 because −3.1 is to the right of −3.5 (see below)	$-3.1 > -3.5$ or $-3.5 < -3.1$



Self Check B

Which of the following are true?

i. $-4.1 > 3.2$
ii. $-3.2 > -4.1$
iii. $3.2 > 4.1$
iv. $-4.6 < -4.1$

Irrational and Real Numbers

Objective 4

There are also numbers that are not rational. **Irrational numbers** cannot be written as the ratio of two integers.

Any square root of a number that is not a perfect square, for example $\sqrt{2}$, is irrational. Irrational numbers are most commonly written in one of three ways: as a root (such as a square root), using a special symbol (such as π), or as a nonrepeating, nonterminating decimal.

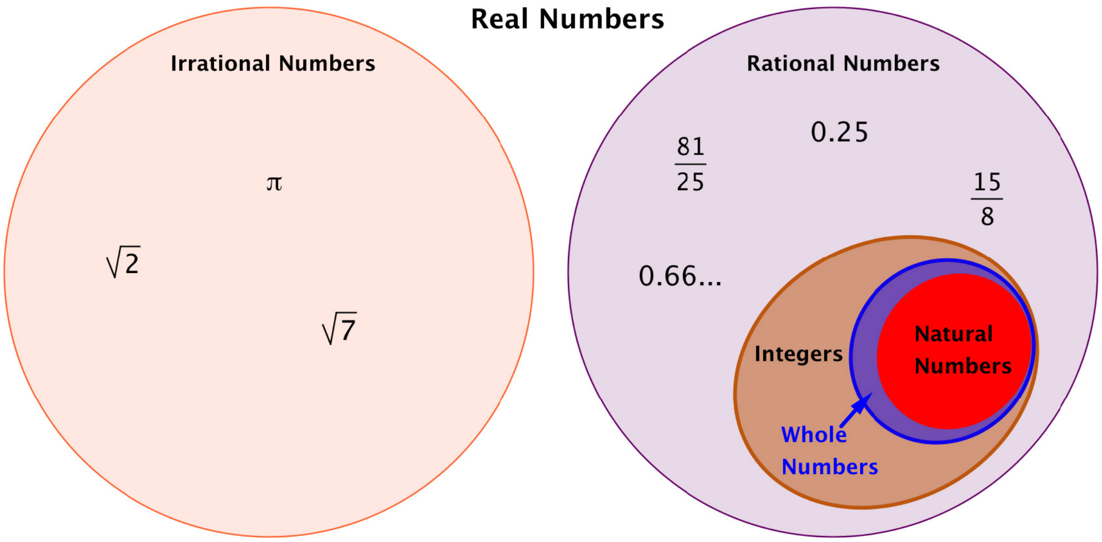
Numbers with a decimal part can either be **terminating decimals** or **nonterminating decimals**. Terminating means the digits stop eventually (although you can always write 0s at the end). For example, 1.3 is terminating, because there's a last digit. The decimal form of $\frac{1}{4}$ is 0.25. Terminating decimals are always rational.

Nonterminating decimals have digits (other than 0) that continue forever. For example, consider the decimal form of $\frac{1}{3}$, which is 0.3333…. The 3s continue indefinitely. Or the decimal form of $\frac{1}{11}$, which is 0.090909…: the sequence "09" continues forever.

In addition to being nonterminating, these two numbers are also **repeating decimals**. Their decimal parts are made of a number or sequence of numbers that repeats again and again. A **nonrepeating decimal** has digits that never form a repeating pattern. The value of $\sqrt{2}$, for example, is 1.414213562…. No matter how far you carry out the numbers, the digits will never repeat a previous sequence.

If a number is terminating *or* repeating, it must be rational; if it is both nonterminating *and* nonrepeating, the number is irrational.

Type of Decimal	Rational or Irrational	Examples
Terminating	Rational	0.25 (or $\frac{1}{4}$) 1.3 (or $\frac{13}{10}$)
Nonterminating and Repeating	Rational	0.66… (or $\frac{2}{3}$) 3.242424… (or) $\frac{321}{99} = \frac{107}{33}$
Nonterminating and Nonrepeating	Irrational	π (or 3.14159…) $\sqrt{7}$ (or 2.6457…)



Example	
Problem	**Is −82.91 rational or irrational?**
Answer	−82.91 is rational, because it is a *terminating* decimal.

The **set** of **real numbers** is made by combining the set of rational numbers and the set of irrational numbers. The real numbers include **natural numbers** or **counting numbers**, whole numbers, integers, rational numbers (fractions and repeating or terminating decimals), and irrational numbers. The set of real numbers is all the numbers that have a location on the number line.

Sets of Numbers	
Natural numbers	1, 2, 3, …
Whole numbers	0, 1, 2, 3, …
Integers	…, −3, −2, −1, 0, 1, 2, 3, …
Rational numbers	numbers that can be written as a ratio of two integers—rational numbers are terminating or repeating when written in decimal form
Irrational numbers	numbers than cannot be written as a ratio of two integers—irrational numbers are nonterminating and nonrepeating when written in decimal form
Real numbers	any number that is rational or irrational

Example		
Problem	**What sets of numbers does 32 belong to?**	
Answer	The number 32 belongs to all these sets of numbers: Natural numbers Whole numbers Integers Rational numbers Real numbers	Every natural or counting number belongs to all of these sets!

Example		
Problem	**What sets of numbers does $382.\overline{3}$ belong to?**	
Answer	$382.\overline{3}$ belongs to these sets of numbers**:** Rational numbers Real numbers	The number is rational because it's a repeating decimal. It's equal to $382\frac{1}{3}$ or $\frac{1,147}{3}$, or $382.\overline{3}$.

Example		
Problem	**What sets of numbers does $-\sqrt{5}$ belong to?**	
Answer	$-\sqrt{5}$ belongs to these sets of numbers: Irrational numbers Real numbers	The number is irrational because it can't be written as a ratio of two integers. Square roots that aren't perfect squares are always irrational.

Self Check C

Which of the following sets does $\frac{-33}{5}$ belong to?

whole numbers
integers
rational numbers
irrational numbers
real numbers

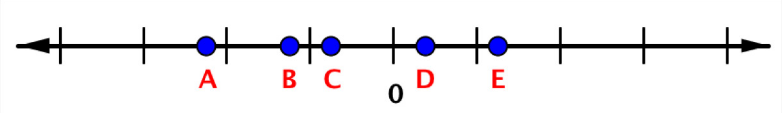
Summary

The set of real numbers is all numbers that can be shown on a number line. This includes natural or counting numbers, whole numbers, and integers. It also includes rational numbers, which are numbers that can be written as a ratio of two integers, and irrational numbers, which cannot be written as a the ratio of two integers. When comparing two numbers, the one with the greater value would appear on the number line to the right of the other one.

5.1.2 Self Check Solutions

Self Check A

Which of the following points represents $-1\frac{1}{4}$?



B.

Negative numbers are to the left of 0, and $-1\frac{1}{4}$ should be 1.25 units to the left. Point B is the only point that’s more than 1 unit and less than 2 units to the left of 0.

Self Check B

Which of the following are true?

i. $-4.1 > 3.2$
ii. $-3.2 > -4.1$
iii. $3.2 > 4.1$
iv. $-4.6 < -4.1$

ii and iv
−3.2 is to the right of −4.1, so $-3.2 > -4.1$. Also, −4.6 is to the left of −4.1, so $-4.6 < -4.1$.

Self Check C

Which of the following sets does $\frac{-33}{5}$ belong to?

rational and real numbers
The number is between integers, so it can't be an integer or a whole number. It's written as a ratio of two integers, so it's a rational number and not irrational. All rational numbers are real numbers, so this number is rational and real.

5.2.1 Adding Integers

Learning Objective(s)
1 Add two or more integers with the same sign.
2 Add two or more integers with different signs.

Introduction

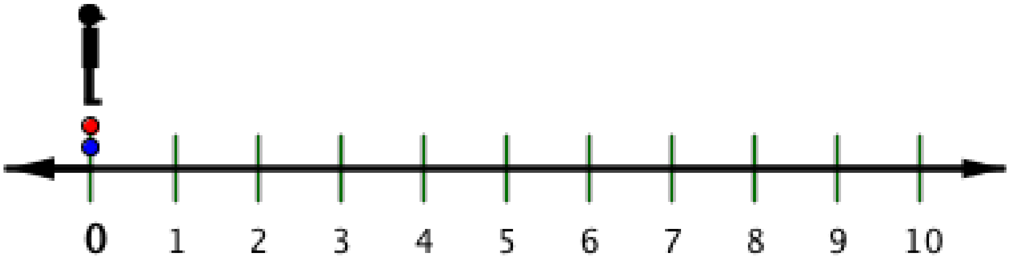
On an extremely cold day, the temperature may be −10. If the temperature rises 8 degrees, how will you find the new temperature? Knowing how to add integers is important here and in much of algebra.

Adding Integers with the Same Signs

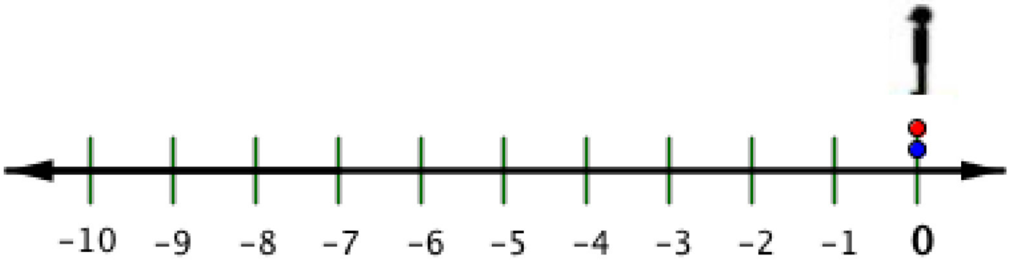
Objective 1

Since positive **integers** are the same as natural numbers, adding two positive integers is the same as adding two natural numbers.

To add integers on the number line, you move forward, and you face right (the positive direction) when you add a positive number.



As with positive numbers, to add negative integers on the number line, you move forward, but you face left (the negative direction) when you add a negative number.



In both cases, the total number of units moved is the total distance moved. Since the distance of a number from 0 is the absolute value of that number, then the absolute value of the sum of the integers is the sum of the absolute values of the **addends**.

When both numbers are negative, you move left in a negative direction, and the sum is negative. When both numbers are positive, you move right in a positive direction, and the sum is positive.

To add two numbers with the same sign (both positive or both negative):

- Add their absolute values and give the sum the same sign.

Example	
Problem	**Find −23 + (−16).**
	Both addends have the same sign (negative). So, add their absolute values: |−23| = 23 and |−16| = 16. The sum of those numbers is 23 + 16 = 39. Since both addends are negative, the sum is negative.
Answer	−23 + (−16) = −39

With more than two addends that have the same sign, use the same process with all addends.

Example	
Problem	**Find −27 + (−138) + (−55).**
	All addends have the same sign (negative). So, add their absolute values: |−27| = 27, |−138| = 138, and |−55| = 55. The sum of those numbers is 27 + 138 + 55 = 220. Since all addends are negative, the sum is negative.
Answer	−27 + (−138) + (−55) = −220

Self Check A

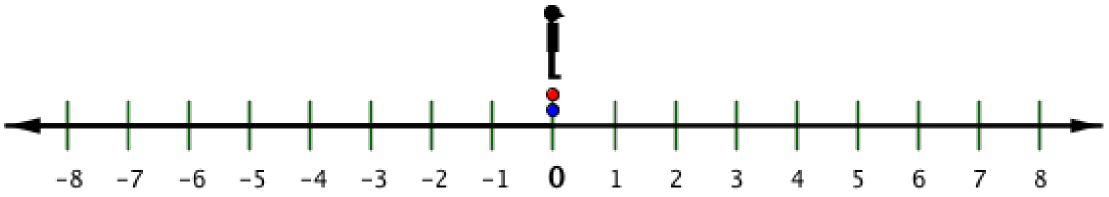
Find −32 + (−14).

Adding Integers with Different Signs

Objective **2**

Consider what happens when the addends have different signs, like in the temperature problem in the introduction. If it's −10 degrees, and then the temperature rises 8 degrees, the new temperature is −10 + 8. How can you calculate the new temperature?

Using the number line below, you move forward to add, just as before. Face and move in a positive direction (right) to add a positive number, and move forward in a negative direction (left) to add a negative number.

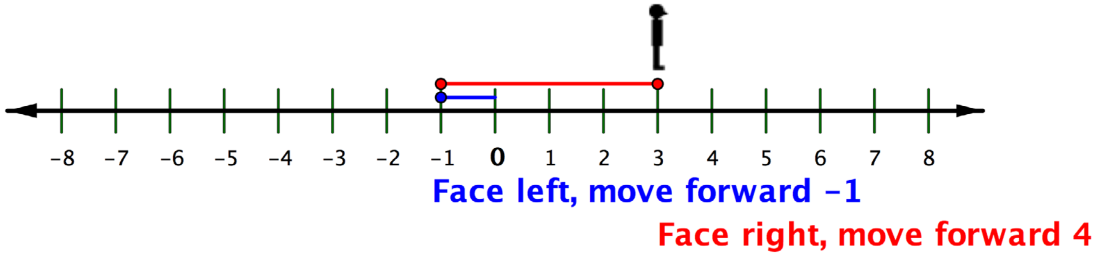


See if you can find a rule for adding numbers without using the number line. Notice that when you add a positive integer and a negative integer, you move forward in the positive (right) direction to the first number, and then move forward in the negative (left) direction to add the negative integer.

Since the distances overlap, the absolute value of the sum is the difference of their distances. So to add a positive number and a negative number, you subtract their absolute values (their distances from 0.)

What is the sign of the sum? It's pretty easy to figure out. If you moved further to the right than you did to the left, you ended to the right of 0, and the answer is positive; and if you move further to the left, the answer is negative. Let's look at the illustration below and determine the sign of the sum.

−1 + 4 = 3

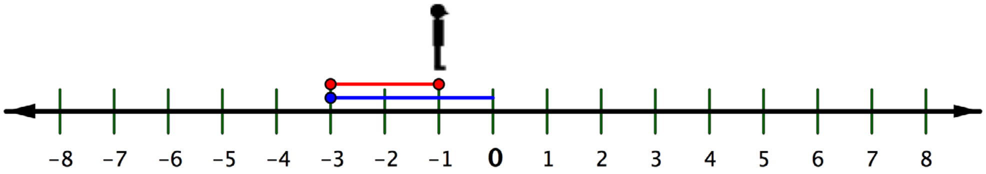


If you didn't have the number line to refer to, you can find the sum of −1 + 4 by

- subtracting the distances from zero (the absolute values) 4 – 1 = 3 and then
- applying the sign of the one furthest from zero (the largest absolute value). In this case, 4 is further from 0 than −1, so the answer is positive: −1 + 4 = 3

Look at the illustration below.

$-3 + 2 = -1$



Face left, move forward −3

Face right, move forward 2

If you didn't have the number line to refer to, you can find the sum of −3 + 2 by

- subtracting the distances from zero (the absolute values) 3 – 2 = 1 and then
- applying the sign of the one furthest from zero (the largest absolute value). In this case, |−3| > |2|, so the answer is negative: −3 + 2 = −1

To add two numbers with different signs (one positive and one negative):

- Find the *difference* of their absolute values.
- Give the sum the same sign as the number with the greater absolute value.

Note that when you find the difference of the absolute values, you always subtract the lesser absolute value from the greater one. The example below shows you how to solve the temperature question that you considered earlier.

Example	
Problem	**Find 8 + (−10).**
	The addends have different signs. So find the difference of their absolute values. |−10| = 10 and |8| = 8. The difference of the absolute values is 10 – 8 = 2. Since 10 > 8, the sum has the same sign as −10.
Answer	8 + (−10) = −2

Example

Problem **Find −22 + 37 when x = −22.**

	|−22| = 22 and |37| = 37 37 – 22 = 15	The addends have different signs. So find the difference of their absolute values. Since |37| > |−22|, the sum has the same sign as 37.
Answer	−22 + 37 = 15	

With more than two addends, you can add the first two, then the next one, and so on.

Example

Problem **Find −27 + (−138) + 55.**

		Add two at a time, starting with −27 + (−138).
	|−27| = 27 and |−138| = 138 27 + 138 = 165 −27 + −138 = −165	Since they have the same signs, you add their absolute values and use the same sign.
	−165 + 55 |−165| = 165 and |55| = 55 165 – 55 = 110	Now add −165 + 55. Since −165 and 55 have different signs, you add them by *subtracting* their absolute values.
	−165 + 55 = −110	Since 165 > 55, the sign of the final sum is the same as the sign of −165.
Answer	−27 + (−138) + 55 = −110	

Self Check B

Find 32 + (−14).

Summary

There are two cases to consider when adding integers. When the signs are the same, you add the absolute values of the addends and use the same sign. When the signs are different, you find the difference of the absolute values and use the same sign as the addend with the greater absolute value.

5.2.1 Self Check Solutions

Self Check A

Find −32 + (−14).

−46

The sum is found by first adding the absolute values of the addends:
|−32| + |−14| = 32 + 14 = 46. Then you must give the sum the same sign as the two addends, so the answer is −46.

Self Check B

Find 32 + (−14).

18

Since the addends have different signs, you must find the difference of the absolute values. |32| = 32 and |−14| = 14. The difference is 32 – 14 = 18. The sign of the sum is the same as the addend with the greater absolute value. Since |32| > |−14|, the sum is positive.

5.2.2 Adding Real Numbers

Learning Objective(s)

1 Add two or more real numbers with the same sign.
2 Add two or more real numbers with different signs.
3 Simplify by using the identity property of 0.
4 Solve application problems requiring the addition of real numbers.

Introduction

Adding real numbers follows the same rules as adding integers. The number 0 has some special attributes that are very important in algebra. Knowing how to add these numbers can be helpful in real-world situations as well as algebraic situations.

Rules for Adding Real Numbers

Objective **1, 2**

The rules for adding **integers** apply to other real numbers, including **rational numbers**.

To add two numbers with the same sign (both positive or both negative)

- *Add* their absolute values.
- Give the sum the same sign.

To add two numbers with different signs (one positive and one negative)

- Find the *difference* of their absolute values. (Note that when you find the difference of the absolute values, you always subtract the lesser absolute value from the greater one.)
- Give the sum the same sign as the number with the greater absolute value.

Remember—to add fractions, you need them to have the same denominator. This is still true when one or more of the fractions are negative.

Example

Problem **Find** $-\frac{3}{7}+\left(-\frac{6}{7}\right)+\frac{2}{7}$

$\left\|-\frac{3}{7}\right\|=\frac{3}{7}$ and $\left\|-\frac{6}{7}\right\|=\frac{6}{7}$	This problem has three **addends**. Add the first two, and then add the third.
$\frac{3}{7}+\frac{6}{7}=\frac{9}{7}$	Since the signs of the first two are the same, find the sum of the absolute values of the fractions
$-\frac{3}{7}+\left(-\frac{6}{7}\right)=-\frac{9}{7}$	Since both addends are negative, the sum is negative.

	$\left\|-\frac{9}{7}\right\| = \frac{9}{7}$ and $\left\|\frac{2}{7}\right\| = \frac{2}{7}$	Now add the third addend. The signs are different, so find the *difference* of their absolute values.
	$\frac{9}{7} - \frac{2}{7} = \frac{7}{7}$	
	$-\frac{9}{7} + \frac{2}{7} = -\frac{7}{7}$	Since $\left\|-\frac{9}{7}\right\| > \left\|\frac{2}{7}\right\|$, the sign of the final sum is the same as the sign of $-\frac{9}{7}$.
Answer	$-\frac{3}{7} + \left(-\frac{6}{7}\right) + \frac{2}{7} = -\frac{7}{7}$	

Example

Problem	**Find $-2\frac{3}{4} + \frac{7}{8}$**	
	$\left\|-2\frac{3}{4}\right\| = 2\frac{3}{4}$ and $\left\|\frac{7}{8}\right\| = \frac{7}{8}$	The signs are different, so find the *difference* of their absolute values.
	$2\frac{3}{4} - \frac{7}{8}$	
	$2\frac{3}{4} = \frac{2(4)+3}{4} = \frac{11}{4}$	First rewrite $2\frac{3}{4}$ as an improper fraction, then rewrite the fraction using a common denominator.
	$\frac{11}{4} = \frac{11 \bullet 2}{4 \bullet 2} = \frac{22}{8}$	
	$\frac{22}{8} - \frac{7}{8}$	Now substitute the rewritten fraction in the problem.
	$\frac{22}{8} - \frac{7}{8} = \frac{15}{8}$	Subtract the numerators and keep the same denominator. Simplify to lowest terms, if possible.
Answer	$-2\frac{3}{4} + \frac{7}{8} = -\frac{15}{8}$	Since $\left\|-2\frac{3}{4}\right\| > \left\|\frac{7}{8}\right\|$, the sign of the final sum is the same as the sign of $-2\frac{3}{4}$.

When you add decimals, remember to line up the decimal points so you are adding tenths to tenths, hundredths to hundredths, and so on.

Example

Problem **Find 27.832 + (−3.06).**

Since the addends have different signs, subtract their absolute values.

$$\begin{array}{r} 27.832 \\ -\ \ 3.06 \\ \hline 24.772 \end{array} \qquad |-3.06| = 3.06$$

The sum has the same sign as 27.832 whose absolute value is greater.

Answer 27.832 + (−3.06) = 24.772

Self Check A

Find −32.22 + 124.3.

Applications of Addition

Objective **3, 4**

There are many situations that use negative numbers. For example, temperatures colder than 0° are usually described using negative numbers. In golf tournaments, players' scores are often reported as a number over or under par, instead of the total number of strokes it takes to hit the ball into the hole. (*Par* is the expected number of strokes needed to complete a hole.) A number under par is negative, and a number over par is positive.

The following examples show how addition of real numbers, including negative numbers, can be useful.

Example

Problem **Boston is, on average, 7 degrees warmer than Bangor, Maine. The low temperature on one cold winter day in Bangor was −13° F. About what low temperature would you expect Boston to have on that day?**

	The phrase "7 degrees warmer" means you add 7 degrees to Bangor's temperature to estimate Boston's temperature.
Boston's temperature is −13 + 7	On that day, Bangor's low was −13°, so you add 7° to −13°
−13 + 7 = −6	Add the integers. Since one is positive and the other is negative, you find the difference of |−13| and |7|, which is 6. Since |−13| > |7|, the final sum is negative.

Answer You would expect Boston to have a temperature of −6 degrees.

Example

Problem **Before Joanne could deposit her paycheck of $802.83, she overdrew her checking account. The balance was −$201.35. What was her balance after she deposited the paycheck?**

−201.35 + 802.83 −201.35 + 802.83 = 601.83	By depositing her paycheck, Joanne is adding money to her account. The new balance is the sum of the old (−201.35) and the paycheck amount.
	Since the numbers have different signs, find the difference of −201.35. Since |802.83| > |−201.35|, the sum is positive.

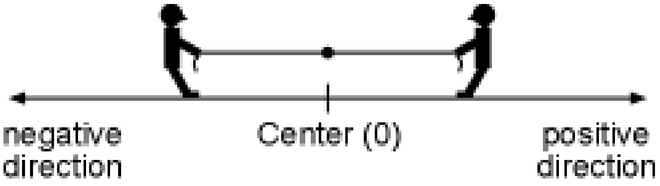
Answer The new balance is $601.48.

When forces or objects are working in opposite directions, sometimes it's helpful to assign a negative value to one and a positive value to the other. This is done often in physics and engineering, but it could also be done in other contexts, such as football or a tug-of-war.

Example

Problem **Two people are in a tug-of-war contest. They are facing each other, each holding the end of a rope. They both pull on the rope, trying to move the center toward themselves.**

Here's an illustration of this situation. The person on the right is pulling in the *positive* direction, and the person on the left is pulling in the *negative* direction.



At one point in the competition, the person on the right was pulling with 122.8 pounds of force. The person on the left was pulling with 131.3 pounds of force. The forces on the center of the rope, then, were 122.8 lbs and −131.3 lbs.

a) What was the net (total sum) force on the center of the rope?
b) In which direction was it moving?

	Net force = 122.8 +(−131.3)	The net force is the sum of the two forces on the rope.
	Net force = −8.5	To find the sum, add the difference of the absolute values of the addends. Since |−131.3| > 122.8, the sum is negative.
Answer	The net force is −8.5 lbs (or 8.5 lbs to the left). The center of the rope is moving to the left (the negative direction).	Notice that it makes sense that the rope was moving to the left, since that person was pulling with more force.

Self Check B

After Bangor reached a low temperature of −13°, the temperature rose only 4 degrees higher for the rest of the day. What was the high temperature that day?

Summary

As with integers, adding real numbers is done following two rules. When the signs are the same, you add the absolute values of the addends and use the same sign. When the signs are different, you subtract the absolute values and use the same sign as the addend with the greater absolute value.

5.2.2 Self Check Solutions

Self Check A

Find −32.22 + 124.3.

92.08

Correct. Since the addends have different signs, you must subtract their absolute values.

124.3 – 32.22 is 92.08. Since |124.3| > |−32.22|, the sum is positive.

Self Check B

After Bangor reached a low temperature of −13°, the temperature rose only 4 degrees higher for the rest of the day. What was the high temperature that day?

The temperature rose (added) 4 degrees from −13, so the high temperature is −13 + 4. Since the addends have different signs, you must find the difference of the absolute values. |−13| = 13 and |4| = 4. The difference is 13 – 4 = 9. The sign of the sum is the same as the addend with the greater absolute value. Since |−13| > |4|, the sum is −9.

5.2.3 Subtracting Real Numbers

Learning Objective(s)

1 Subtract two or more real numbers.
2 Simplify combinations that require both addition and subtraction of real numbers.
3 Solve application problems requiring subtraction of real numbers.

Introduction

Subtraction and addition are closely related. They are called **inverse operations**, because one "undoes" the other. So, just as with integers, you can rewrite subtraction as addition to subtract real numbers.

Additive Inverses

Inverse operations, such as addition and subtraction, are a key idea in algebra. Suppose you have $10 and you loan a friend $5. An hour later, she pays you back the $5 she borrowed. You are back to having $10. You could represent the transaction like this:

$10 – 5 + 5 = 10$.

This works because a number minus itself is 0.

$3 – 3 = 0$ $63.5 – 63.5 = 0$ $39{,}283 – 39{,}283 = 0$

So, adding a number and then subtracting the same number is like adding 0.

Thinking about this idea in terms of **opposite** numbers, you can also say that a number plus its opposite is also 0. Notice that each example below consists of a positive and a negative number pair added together.

$3 + (−3) = 0$ $−63.5 + 63.5 = 0$ $39{,}283 + (−39{,}283) = 0$

Two numbers are **additive inverses** if their sum is 0. Since this means the numbers are opposites (same absolute value but different signs), "additive inverse" is another, more formal term for the opposite of a number. (Note that 0 is its own additive inverse.)

Subtracting Real Numbers

Objective **1**

You can use the additive inverses or opposites to rewrite subtraction as addition. If you are adding two numbers with different signs, you find the difference between their absolute values and keep the sign of the number with the greater absolute value.

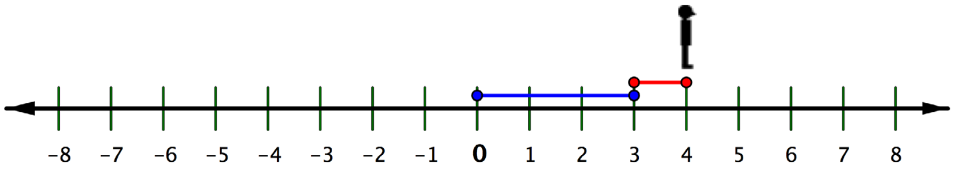
When the greater number is positive, it's easy to see the connection.

$$13 + (−7) = 13 – 7$$

Both equal 6.

Let’s see how this works. When you add positive numbers, you are moving forward, facing in a positive direction.

3 + 1 = 4

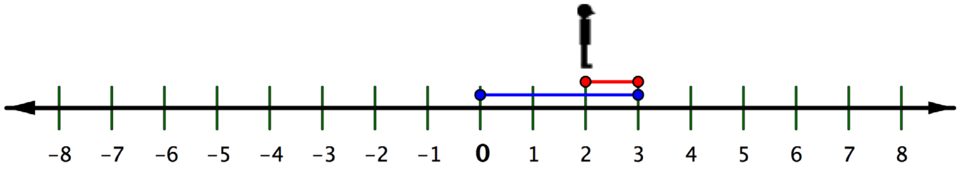


Face right, move forward 3

Face right, move forward 1

When you subtract positive numbers, you can imagine moving *backward,* but still facing in a positive direction.

3 - 1 = 2



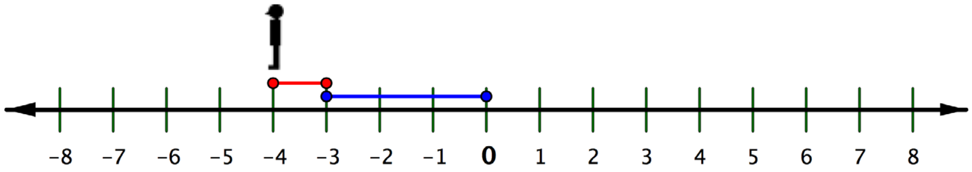
Face right, move forward 3

Face right, move backward 1

Now let's see what this means when one or more of the numbers is negative.

Recall that when you add a negative number, you move forward, but face in a negative direction (to the left).

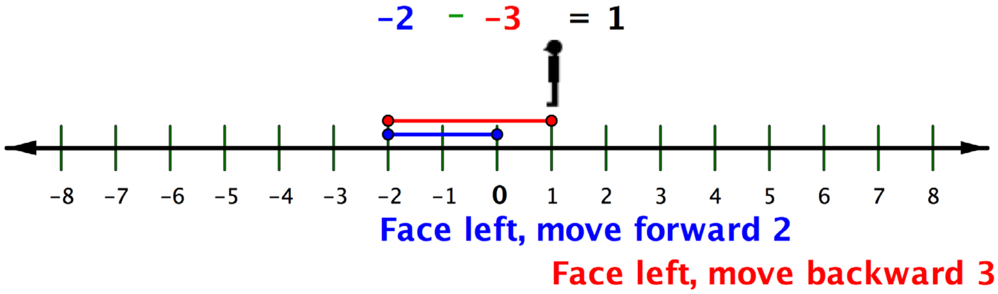
−3 + −1 = −4



Face left, move forward 3

Face left, move forward 1

How do you subtract a negative number? First face and move forward in a negative direction to the first number, −2. Then continue facing in a negative direction (to the left), but move *backward* to subtract −3.



But isn’t this the same result as if you had added positive 3 to −2? −2 + 3 = 1.

In each addition problem, you face one direction and move some distance forward. In the paired subtraction problem, you face the *opposite* direction and then move the same distance *backward.* Each gives the same result!

> To subtract a real number, you can rewrite the problem as adding the opposite (additive inverse).

Note, that while this always works, whole number subtraction is still the same. You can subtract 38 – 23 just as you have always done. Or, you could also rewrite it as 38 + (−23). Both ways you will get the same answer.

38 – 23 = 38 + (−23) = 15.

It’s your choice in these cases.

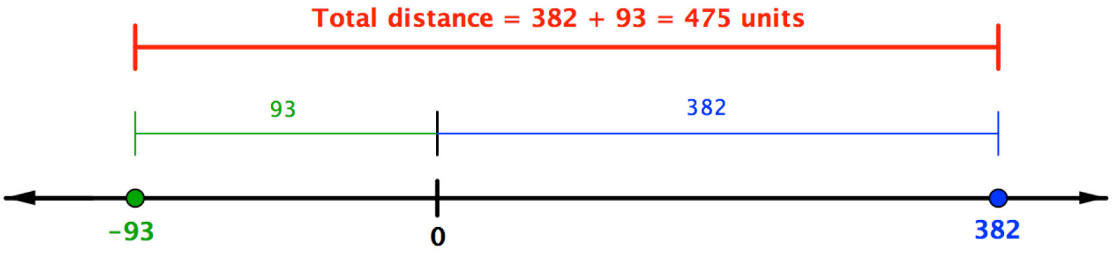
Example		
Problem	**Find 23 – 73.**	
		You can't use your usual method of subtraction, because 73 is greater than 23.
	23 + (−73)	Rewrite the subtraction as adding the opposite.
	|23| = 23 and |−73| = 73 73 – 23 = 50	The addends have different signs, so find the difference of their absolute values.
Answer	23 – 73 = −50	Since |−73| > |23|, the final answer is negative.

Example

Problem **Find 382 – (−93).**

	382 + 93	Rewrite the subtraction as adding the opposite. The opposite of −93 is 93. So, this becomes a simple addition problem.
	382 + 93 = 475	
Answer	382 – (−93) = 475	

Another way to think about subtracting is to think about the distance between the two numbers on the number line. In the example above, 382 is to the *right* of 0 by 382 units, and −93 is to the *left* of 0 by 93 units. The distance between them is the sum of their distances to 0: 382 + 93.



Example

Problem **Find $22\frac{1}{3}-\left(-\frac{3}{5}\right)$.**

	$22\frac{1}{3}+\frac{3}{5}$	Rewrite the subtraction as adding the opposite. The opposite of $-\frac{3}{5}$ is $\frac{3}{5}$.
	$22\frac{1\cdot 5}{3\cdot 5}+\frac{3\cdot 3}{5\cdot 3}=22\frac{5}{15}+\frac{9}{15}$	This is now just adding two rational numbers. Remember to find a common denominator when adding fractions. 3 and 5 have a common multiple of 15; change denominators of both fractions to 15 (and make the necessary changes in the numerator!) before adding.
	$22\frac{5}{15}+\frac{9}{15}=22\frac{14}{15}$	
Answer	$22\frac{14}{15}$	

Self Check A
Find -32.3 – (-16.3).

Adding and Subtracting More Than Two Real Numbers

Objective **2**

When you have more than two **real numbers** to add or subtract, work from left to right as you would when adding more than two whole numbers. Be sure to change subtraction to addition of the opposite when needed.

Example

Problem **Find −23 + 16 – (−32) – 4 + 6.**

−23 + 16 – (−32) – 4 + 6 **−7** – (−32) – 4 + 6	Start with −23 + 16. The addends have different signs, so find the difference and use the sign of the addend with the greater absolute value. −23 + 16 = −7.
−7 – (−32) – 4 + 6 **−7 + 32** – 4 + 6	Now you have −7 – (−32). Rewrite this subtraction as addition of the opposite. The opposite of −32 is 32, so this becomes −7 + 32, which equals 25.
25 – 4 + 6	You now have 25 – 4. You *could* rewrite this as an addition problem, but you don't need to.
21 + 6	Complete the final addition of 21 + 6.

Answer −23 + 16 – (−32) – 4 + 6 = 27

Self Check B
Find 32 – (−14) – 2 + (−82).

Applications of Subtraction

Objective **3**

Situations that use negative numbers can require subtraction as well as addition. As you saw above, sometimes subtracting two positive numbers can give a negative result. You should be sure that a negative number makes sense in the problem.

Example		
Problem	**Boston is, on average, 7 degrees warmer than Bangor, Maine. The low temperature on one cold winter day in Boston was 3°F. About what low temperature would you expect Bangor to have on that day?**	
	.	The phrase "7 degrees warmer" means you can subtract 7 degrees from Boston's temperature to estimate Bangor's temperature. (Note that you can also add 7 degrees to Bangor's temperature to estimate Boston's temperature. Be careful about which should have the greater number!)
	Bangor's temperature is 3 – 7	On that day, Boston's low was 3°, so you subtract 7° from 3°.
	3 – 7 = 3 + (−7)	Since 3 < 7, rewrite the subtraction problem as addition of the opposite. Add the numbers. Since one is positive and the other is negative, you find the difference of |−7| and |3|, which is 4. Since |−7| > |3|, the final sum is negative.
Answer	You would expect the low temperature in Bangor, Maine to be −4°F.	

Example		
Problem	**Everett paid several bills without balancing his checkbook first! When the last check he wrote was still to be deducted from his balance, Everett's account was already overdrawn. The balance was −$201.35. The final check was for $72.66, and another $25 will be subtracted as an overdraft charge. What will Everett's account balance be after that last check and the overdraft charge are deducted?**	
	−201.35 – 72.66 – 25	The new balance will be the existing balance of −$201.35, minus the check's amount and the overdraft charge.
	−201.35 – 72.66 – 25 **−201.35 + (−72.66)** – 25	Start with the first subtraction, −201.35 – 72.66. Rewrite it as the addition of the opposite of 72.66.
	−274.01 – 25	Since the addends have the same signs, the sum is the sum of their absolute values (201.35 + 72.66) with the same sign (negative).
	−274.01 + (−25)	Again, rewrite the subtraction as the addition of the opposite.
	−274.01 + (−25) = −299.01	Add, by adding the sum of their absolute values and use the same sign as both addends.
Answer	Everett’s account balance will be $−299.01.	

Example		
Problem	**One winter, Phil flew from Syracuse, NY to Orlando, FL. The temperature in Syracuse was −20°F. The temperature in Orlando was 75°F. What was the difference in temperatures between Syracuse and Orlando?**	
	75 – (−20)	To find the difference between the temperatures, you need to subtract. We subtract the ending temperature from the beginning temperature to get the change in temperature.
	75 + 20	Rewrite the subtraction as adding the opposite. The opposite of −20 is 20.
	75 + 20 = 95	There is a 95 degree difference between 75° and −20°.
Answer	The difference in temperatures is 95 degrees.	

Self Check C

Louise noticed that her bank balance was −$33.72 before her paycheck was deposited. After the check had been deposited, the balance was $822.98. No other deductions or deposits were made. How much money was she paid?

Summary

Subtracting a number is the same as adding its opposite (also called its additive inverse). To subtract you can rewrite the subtraction as adding the opposite and then use the rules for the addition of real numbers.

5.2.3 Self Check Solutions

Self Check A

Find -32.3 – (-16.3).

To subtract, change the problem to adding the opposite of −16.3, which gives −32.3 + 16.3. Then use the rules for adding two numbers with different signs. Since the difference between 32.3 and 16.3 is 16, and |−32.3| > |16.3|, the correct answer is −16.

Self Check B

Find 32 – (−14) – 2 + (−82).

To subtract 32 – (−14), write the subtraction as addition of the opposite, giving 32 + 14 = 46. Then subtract 2 to get 44, and add −82 to get −38.

Self Check C

Louise noticed that her bank balance was −$33.72 before her paycheck was deposited. After the check had been deposited, the balance was $822.98. No other deductions or deposits were made. How much money was she paid?

$856.70. The amount she was paid is the difference between the two balances: 822.98 – (−33.72). This is the same as 822.98 + 33.72, or 856.70.

5.3 Multiplying and Dividing Real Numbers

Learning Objective(s)

1 Multiply two or more real numbers.
2 Simplify by using the identity property of 1.
3 Divide real numbers.
4 Solve application problems requiring multiplication or division of real numbers.

Introduction

After addition and subtraction, the next operations you learned how to do were multiplication and division. You may recall that multiplication is a way of computing “repeated addition,” and this is true for negative numbers as well.

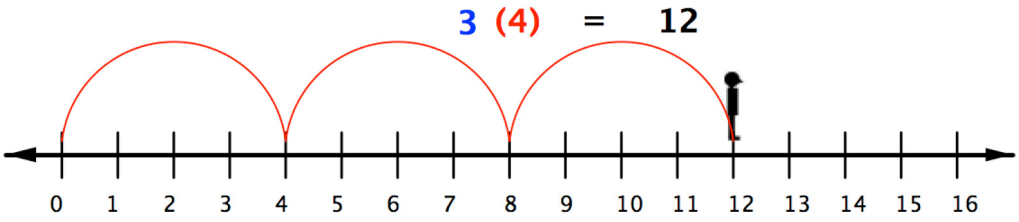
Multiplication and division are **inverse operations**, just as addition and subtraction are. You may recall that when you divide fractions, you multiply by the reciprocal.

Multiplying Real Numbers

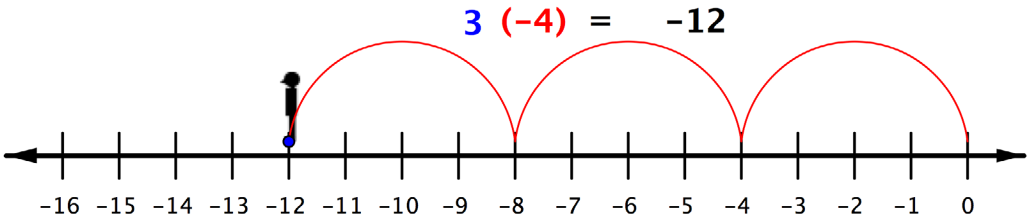
Objective **1**

Multiplying real numbers is not that different from multiplying whole numbers and positive fractions. However, you haven't learned what effect a negative sign has on the product.

With whole numbers, you can think of multiplication as repeated addition. Using the number line, you can make multiple jumps of a given size. For example, the following picture shows the product 3 • 4 as 3 jumps of 4 units each.



So to multiply 3 (−4), you can face left (toward the negative side) and make three “jumps” forward (in a negative direction).



The product of a positive number and a negative number (or a negative and a positive) is negative. You can also see this by using patterns. In the following list of products, the first number is always 3. The second number decreases by 1 with each row (3, 2, 1, 0, −1, −2). Look for a pattern in the products of these numbers. What numbers would fit the pattern for the last two products?

$3(3) = 9$
$3(2) = 6$
$3(1) = 3$
$3(0) = 0$
$3(-1) = ?$
$3(-2) = ?$

Notice that the pattern is the same if the order of the **factors** is switched:

$3(3) = 9$
$2(3) = 6$
$1(3) = 3$
$0(3) = 0$
$-1(3) = ?$
$-2(3) = ?$

Take a moment to think about that pattern before you read on.

As the factor decreases by 1, the product decreases by 3. So $3(-1) = -3$ and $3(-2) = -6$.

If you continue the pattern further, you see that multiplying 3 by a negative integer gives a negative number. This is true in general.

The Product of a Positive Number and a Negative Number.

To multiply a **positive number** and a **negative number**, multiply their absolute values. The product is **negative.**

You can use the pattern idea to see how to multiply two negative numbers. Think about how you would complete this list of products.

$-3(3) = -9$
$-3(2) = -6$
$-3(1) = -3$
$-3(0) = 0$
$-3(-1) = ?$
$-3(-2) = ?$

As the factor decreases by 1, the product increases by 3. So $-3(-1) = 3$, $-3(-2) = 6$.

Multiplying −3 by a negative integer gives a *positive* number. This is true in general.

The Product of Two Numbers with the Same Sign (both positive or both negative).

To multiply two **positive numbers**, multiply their absolute values. The product is **positive**.

To multiply two **negative numbers**, multiply their absolute values. The product is **positive**.

Example		
Problem	**Find −3.8(0.6).**	
	$\begin{array}{r} 3.8 \\ \underline{\times\ 0.6} \\ 2.28 \end{array}$	Multiply the absolute values as you normally would. Place the decimal point by counting place values. 3.8 has 1 place after the decimal point, and 0.6 has 1 place after the decimal point, so the product has 1 + 1 or 2 places after the decimal point.
Answer	−3.8(0.6) = −2.28	The product of a negative and a positive is negative.

Example		
Problem	**Find** $\left(-\frac{3}{4}\right)\left(-\frac{2}{5}\right)$	
	$\left(\frac{3}{4}\right)\left(\frac{2}{5}\right)=\frac{6}{20}=\frac{3}{10}$	Multiply the absolute values of the numbers. First, multiply the numerators together to get the product's numerator. Then, multiply the denominators together to get the product's denominator. Rewrite in lowest terms, if needed.
Answer	$\left(-\frac{3}{4}\right)\left(-\frac{2}{5}\right)=\frac{3}{10}$	The product of two negative numbers is positive.

To summarize:

positive • *positive*: The product is **positive**.
negative • *negative*: The product is **positive**.

negative • *positive*: The product is **negative**.
positive • *negative*: The product is **negative**.

You can see that the product of two negative numbers is a positive number. So, if you are multiplying more than two numbers, you can count the number of negative factors.

Multiplying More Than Two Negative Numbers

If there are an **even** number (0, 2, 4, ...) of negative factors to multiply, the product is **positive**. If there are an **odd** number (1, 3, 5, ...) of negative factors, the product is **negative**.

Example

Problem **Find 3(−6)(2)(−3)(−1).**

	3(6)(2)(3)(1) 18(2)(3)(1) 36(3)(1) 108(1) 108	Multiply the absolute values of the numbers.
	3(−6)(2)(−3)(−1)	Count the number of negative factors. There are three (−6, −3, −1).
Answer	3(−6)(2)(−3)(−1) = −108	Since there are an odd number of negative factors, the product is negative.

Self Check A
Find (−30)(−0.5).

Multiplicative Inverses

You might recall that two numbers are additive inverses if their sum is 0, the additive identity.

3 and −3 are additive inverses because 3 + (−3) = 0.

Two numbers are **multiplicative inverses** if their product is 1, the multiplicative identity.

$\frac{2}{3}$ and $\frac{3}{2}$ are multiplicative inverses because $\frac{2}{3}\left(\frac{3}{2}\right)=\frac{6}{6}=1$.

You may remember that when you divided fractions, you multiplied by the **reciprocal**. *Reciprocal* is another name for the multiplicative inverse (just as *opposite* is another name for additive inverse).

An easy way to find the multiplicative inverse is to just "flip" the numerator and denominator as you did to find the reciprocal. Here are some examples:

The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$ because $\frac{4}{9}\left(\frac{9}{4}\right)=\frac{36}{36}=1$.

The reciprocal of 3 is $\frac{1}{3}$ because $\frac{3}{1}\left(\frac{1}{3}\right)=\frac{3}{3}=1$.

The reciprocal of $-\frac{5}{6}$ is $-\frac{6}{5}$ because $-\frac{5}{6}\left(-\frac{6}{5}\right)=\frac{30}{30}=1$.

The reciprocal of 1 is 1 as 1(1) = 1.

Self Check B

What is the reciprocal, or multiplicative inverse, of −12?

Dividing Real Numbers

Objective **3**

When you divided by positive fractions, you learned to multiply by the reciprocal. You also do this to divide **real numbers**.

Think about dividing a bag of 26 marbles into two smaller bags with the same number of marbles in each. You can also say each smaller bag has *one half* of the marbles.

$$26\div 2=26\left(\frac{1}{2}\right)=13$$

Notice that 2 and $\frac{1}{2}$ are reciprocals.

Try again, dividing a bag of 36 marbles into smaller bags.

Number of bags	**Dividing by number of bags**	**Multiplying by reciprocal**
3	$\frac{36}{3}=12$	$36\left(\frac{1}{3}\right)=\frac{36}{3}=\frac{12(3)}{3}=12$
4	$\frac{36}{4}=9$	$36\left(\frac{1}{4}\right)=\frac{36}{4}=\frac{9(4)}{4}=9$
6	$\frac{36}{6}=6$	$36\left(\frac{1}{6}\right)=\frac{36}{6}=\frac{6(6)}{6}=6$

Dividing by a number is the same as multiplying by its reciprocal. (That is, you use the reciprocal of the **divisor**, the second number in the division problem.)

Example

Problem **Find $28 \div \frac{4}{3}$**

$28 \div \frac{4}{3} = 28\left(\frac{3}{4}\right)$ Rewrite the division as multiplication by the reciprocal. The reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

$\frac{28}{1}\left(\frac{3}{4}\right)=\frac{28(3)}{4}=\frac{4(7)(3)}{4}=7(3)=21$ Multiply.

Answer $28 \div \frac{4}{3} = 21$

Self Check C

Find $\frac{6}{7} \div \frac{3}{10}$. Write the answer in lowest terms.

Now let's see what this means when one or more of the numbers is negative. A number and its reciprocal have the same sign. Since division is rewritten as multiplication using the reciprocal of the divisor, and taking the reciprocal doesn't change any of the signs, division follows the same rules as multiplication.

Rules of Division

When dividing, rewrite the problem as multiplication using the reciprocal of the divisor as the second factor.

When one number is **positive** and the other is **negative,** the **quotient** is **negative**.

When **both** numbers are **negative,** the quotient is **positive**.

When **both** numbers are **positive,** the quotient is **positive**.

Example

Problem	**Find** $24 \div \left(-\frac{5}{6}\right)$	
	$24 \div \left(-\frac{5}{6}\right) = 24\left(-\frac{6}{5}\right)$	Rewrite the division as multiplication by the reciprocal.
	$\frac{24}{1}\left(-\frac{6}{5}\right) = -\frac{144}{5}$	Multiply. Since one number is positive and one is negative, the product is negative.
Answer	$24 \div \left(-\frac{5}{6}\right) = -\frac{144}{5}$	

Example

Problem	**Find** $4\left(-\frac{2}{3}\right) \div (-6)$	
	$\frac{4}{1}\left(-\frac{2}{3}\right)\left(-\frac{1}{6}\right)$	Rewrite the division as multiplication by the reciprocal.
	$\frac{4(2)(1)}{3(6)} = \frac{8}{18}$	Multiply. There is an even number of negative numbers, so the product is positive.
Answer	$4\left(-\frac{2}{3}\right) \div (-6) = \frac{4}{9}$	Write the fraction in lowest terms.

Remember that a fraction bar also indicates division, so, a negative sign in front of a fraction goes with the numerator, the denominator, or the whole fraction:

$$-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4}.$$

In each case, the overall fraction is negative because there's only one negative in the division.

Applications of Multiplication and Division

Objective **4**

Situations that require multiplication or division may use negative numbers and rational numbers.

Example		
Problem	**Carl didn't know his bank account was at exactly 0 when he wrote a series of $100 checks. As each check went through, $125 was charged against his account. (In addition to the $100 for the check, there was a $25 overdraft charge.) After 1 check, his account was −$125 dollars. After 6 of these checks, what was his account balance?**	
	−$125(6)	Each check reduces the account by $125; this is represented by −$125. To find the amount it reduces for multiple checks, multiply the number of checks by the amount charged.
	−$125(6) = −$750	Multiply. Since there is one negative number, the product is negative.
Answer	Carl's account balance is now −$750.	

Example		
Problem	**Brenda thought she was taking 3 chocolate bars to a picnic with 5 friends. When she brought out the chocolate, she discovered her brother had eaten half of one bar, so she only had $2\frac{1}{2}$ bars to divide among the 6 people (herself and her 5 friends). If each person gets the same amount, how much of a full bar does each person get?**	
	$2\frac{1}{2} \div \frac{1}{6}$	Since the candy is being shared among 6 people, divide the amount of chocolate by 6.
	$2\frac{1}{2}\left(\frac{1}{6}\right)$	Rewrite the problem as multiplication, using the reciprocal of the divisor.
	$\frac{5}{2}\left(\frac{1}{6}\right)$	Change the mixed number to an improper fraction. Multiply.
Answer	Each person gets $\frac{5}{12}$ of a full bar.	

Example

Problem	**A business was owned by three partners. When it dissolved, they had $50,000 in cash, $120,000 in assets they can sell, and $260,000 in debt. If the partners split everything equally, how much will each have or owe in debt?**	
	$50,000+120,000+(-260,000)=-90,000$	The cash and assets are positive quantities, but the debt is negative. Start by adding these
	$-90,000\div 3=-30,000$	Now, divide this between the three partners. Since you are dividing one negative and one positive number, the quotient is negative.
Answer	Each partner will still owe $30,000.	

Self Check D

Over the course of an 18-year research project, the height of an oceanside cliff actually dropped due to soil erosion. At the end of this period, its height was measured as −3 inches compared to what it had been at the beginning of the research project. What was the average amount that the cliff’s height changed each year?

Summary

With multiplication and division, you can find the sign of the final answer by counting how many negative numbers are used in the product or quotient. If there are an even number of negatives, the result is positive. If there are an odd number of negatives, the result is negative. Division can be rewritten as multiplication, by using the reciprocal or multiplicative inverse of the divisor.

5.3 Self Check Solutions

Self Check A

Find (−30)(−0.5).

15

First, multiply 30 and 0.5. Multiply 30 and 5 to get 150, then place the decimal point. Since 0.5 has one digit to the right of the decimal point, the decimal point in the product needs to be placed with one digit to the right of it, to get 15. The two original factors are both negative. Since there is an even number of negative factors, the product is positive.

Self Check B

What is the reciprocal, or multiplicative inverse, of −12?

$-\frac{1}{12}$

$-12 = -\frac{12}{1}$ and $-\frac{12}{1}\left(-\frac{1}{12}\right) = 1$.

Self Check C

Find $\frac{6}{7} \div \frac{3}{10}$. Write the answer in lowest terms.

Rewrite the problem as multiplication, using the reciprocal of the divisor. Then multiply and simplify. $\frac{6}{7} \div \frac{3}{10} = \frac{6}{7}\left(\frac{10}{3}\right) = \frac{60}{7(3)} = \frac{20(3)}{7(3)} = \frac{20}{7}$.

Self CheckD

Over the course of an 18-year research project, the height of an oceanside cliff actually dropped due to soil erosion. At the end of this period, its height was measured as −3 inches compared to what it had been at the beginning of the research project. What was the average amount that the cliff's height changed each year?

$\frac{-3}{18} = \frac{-1}{6} = -\frac{1}{6}$, or $-\frac{1}{6}$ in.

5.4 Order of Operations with Real Numbers

Learning Objective(s)

1 Use the order of operations to simplify expressions.

2 Simplify expressions containing absolute values.

Introduction

Recall the order of operations discussed previously.

The Order of Operations

- Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
- Evaluate exponents or square roots.
- Multiply or divide, from left to right.
- Add or subtract, from left to right.

This order of operations is true for all real numbers.

Example		
Problem		**Simplify 7 – 5 + 3 • 8.**
	7 – 5 + **3 • 8**	According to the order of operations, multiplication comes before addition and subtraction. Multiply 3 • 8.
	7 – 5 + 24	Now, add and subtract from left to right. 7 – 5 comes first.
	2 + 24 = 26	Finally, add 2 + 24.
Answer	7 – 5 + 3 • 8 = 26	

When you are applying the order of operations to expressions that contain fractions, decimals, and negative numbers, you will need to recall how to do these computations as well.

Example		
Problem	**Simplify** $3 \bullet \frac{1}{3} - 8 \div \frac{1}{4}$	
	$3 \bullet \frac{1}{3} - 8 \div \frac{1}{4}$	According to the order of operations, multiplication comes before addition and subtraction. Multiply $3 \bullet \frac{1}{3}$ first.
	$1 - 8 \div \frac{1}{4}$	Now, divide $8 \div \frac{1}{4}$.
	$8 \div \frac{1}{4} = \frac{8}{1} \bullet \frac{4}{1} = 32$	
	$1 - 32 = -31$	Subtract.
Answer	$3 \bullet \frac{1}{3} - 8 \div \frac{1}{4} = -31$	

Exponents

When you are evaluating expressions, you will sometimes see exponents used to represent repeated multiplication. Recall that an expression such as 7^2 is **exponential notation** for 7 • 7. (Exponential notation has two parts: the **base** and the **exponent** or the **power**. In 7^2, 7 is the base and 2 is the exponent; the exponent determines how many times the base is multiplied by itself.)

Exponents are a way to represent repeated multiplication; the order of operations places it *before* any other multiplication, division, subtraction, and addition is performed.

Example		
Problem	**Simplify** $3^2 \bullet 2^3$**.**	
	$3^2 \bullet 2^3$	This problem has exponents and multiplication in it. According to the order of operations, simplifying 3^2 and 2^3 comes before multiplication.
	$9 \bullet 2^3$	3^2 is 3 • 3, which equals 9.
	$9 \bullet 8$	2^3 is 2 • 2 • 2, which equals 8.
	$9 \bullet 8 = 72$	Multiply.
Answer	$3^2 \bullet 2^3 = 72$	

Example

Problem

Simplify $\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3\cdot 32$.

$\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3\cdot 32$	This problem has exponents, multiplication, and addition in it. According to the order of operations, simplify the terms with the exponents first, then multiply, then add.
$\frac{1}{4}+\left(\frac{1}{4}\right)^3\cdot 32$	Evaluate: $\left(\frac{1}{2}\right)^2=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
$\frac{1}{4}+\frac{1}{64}\cdot 32$	Evaluate: $\left(\frac{1}{4}\right)^3=\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{64}$
$\frac{1}{4}+\frac{32}{64}$	Multiply.
$\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$	Simplify. $\frac{32\div 32}{64\div 32}=\frac{1}{2}$, so you can add $\frac{1}{4}+\frac{1}{2}$.

Answer $\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3\cdot 32=\frac{3}{4}$

Self Check A

Simplify: $100-5^2\cdot 4$.

When there are grouping symbols within grouping symbols, calculate from the inside to the outside. That is, begin simplifying within the innermost grouping symbols first.

Remember that parentheses can also be used to show multiplication. In the example that follows, both uses of parentheses—as a way to represent a group, as well as a way to express multiplication—are shown.

Example

Problem	**Simplify $(3 + 4)^2 + (8)(4)$.**	
	$(3 + 4)^2 + (8)(4)$	This problem has parentheses, exponents, multiplication, and addition in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication.
	$\mathbf{(3 + 4)}^2 + (8)(4)$	Grouping symbols are handled first. Add numbers in parentheses.
	$\mathbf{7^2} + (8)(4)$	Simplify 7^2.
	$49 + \mathbf{(8)(4)}$	Perform multiplication.
	$\mathbf{49 + 32} = 81$	Perform addition.
Answer	$(3 + 4)^2 + (8)(4) = 81$	

Example

Problem	**Simplify $(1.5 + 3.5) – 2(0.5 \bullet 6)^2$.**	
	$(1.5 + 3.5) – 2(0.5 \bullet 6)^2$	This problem has parentheses, exponents, multiplication, subtraction, and addition in it. Grouping symbols are handled first. Add numbers in the first set of parentheses.
	$5 – 2(0.5 \bullet 6)^2$	Multiply numbers in the second set of parentheses.
	$5 – 2(3)^2$	Evaluate exponents.
	$5 – 2 \bullet 9$	Multiply.
	$5 – 18 = −13$	Subtract.
Answer	$(1.5 + 3.5) – 2(0.5 \bullet 6)^2 = −13$	

Example

Problem

Simplify $\frac{5-[3+(2\cdot(-6))]}{3^2+2}$

$\frac{5-[3+(2\cdot(-6))]}{3^2+2}$	This problem has brackets, parentheses, fractions, exponents, multiplication, subtraction, and addition in it. Grouping symbols are handled first. The parentheses around the -6 aren’t a grouping symbol, they are simply making it clear that the negative sign belongs to the 6. Start with the innermost set of parentheses that are a grouping symbol , here it is in the numerator of the fraction, (2 • −6), and begin working out. (The fraction line acts as a type of grouping symbol, too; you simplify the numerator and denominator independently, and then divide the numerator by the denominator at the end.)
$\frac{5-[3+(-12)]}{3^2+2}$	Add the values in the brackets.
$\frac{5-[-9]}{3^2+2}$	Subtract 5 – [−9] = 5 + 9 = 14.
$\frac{14}{3^2+2}$	The top of the fraction is all set, but the bottom (denominator) has remained untouched. Apply the order of operations to that as well. Begin by evaluating $3^2 = 9$.
$\frac{14}{9+2}$	Now add. 9 + 2 = 11.
$\frac{14}{11}$	
Answer $\frac{5-[3+(2\cdot(-6))]}{3^2+2}=\frac{14}{11}$	

Self Check B

Simplify $\left[\frac{3^3+3}{(-2)(-3)}\right]^2+1$.

Absolute Value Expressions

Objective 2

Absolute value expressions are one final method of grouping that you may see. Recall that the absolute value of a quantity is always positive or 0.

When you see an absolute value expression included within a larger expression, follow the regular order of operations and evaluate the expression within the absolute value sign. Then take the absolute value of that expression. The example below shows how this is done.

Example

Problem	**Simplify** $\dfrac{3+\lvert 2-6\rvert}{2\lvert 3\bullet 1.5\rvert-(-3)}$	
	$\dfrac{3+\lvert 2-6\rvert}{2\lvert 3\bullet 1.5\rvert-(-3)}$	This problem has absolute values, decimals, multiplication, subtraction, and addition in it. Grouping symbols, including absolute value, are handled first. Simplify the numerator, then the denominator. Evaluate |2 – 6|.
	$\dfrac{3+\lvert -4\rvert}{2\lvert 3\bullet 1.5\rvert-(-3)}$	Take the absolute value of |−4|.
	$\dfrac{3+4}{2\lvert 3\bullet 1.5\rvert-(-3)}$	Add the numbers in the numerator.
	$\dfrac{7}{2\lvert 3\bullet 1.5\rvert-(-3)}$	Now that the numerator is simplified, turn to the denominator. Evaluate the absolute value expression first.
	$\dfrac{7}{2\lvert 4.5\rvert-(-3)}$	The expression "2|4.5|" reads "2 times the absolute value of 4.5." Multiply 2 times 4.5.
	$\dfrac{7}{2\bullet 4.5-(-3)}$	
	$\dfrac{7}{9-(-3)}$	Subtract.
	$\dfrac{7}{12}$	
Answer	$\dfrac{3+\lvert 2-6\rvert}{2\lvert 3\bullet 1.5\rvert-(-3)}=\dfrac{7}{12}$	

Self Check C

Simplify: $(5|3 - 4|)^3$.

5.4 Self Check Solutions

Self Check A

Simplify: $100 - 5^2 \bullet 4$.

To simplify this expression, simplify the term with the exponent first, then multiply, then subtract. $5^2 = 25$, and $25 \bullet 4 = 100$, and $100 - 100 = 0$.

Self Check B

Simplify $\left[\dfrac{3^3 + 3}{(-2)(-3)}\right]^2 + 1$.

The entire quantity within the brackets is 5. 5^2 is 25, and $25 + 1 = 26$.

Self Check C

Simplify: $(5|3 - 4|)^3$.

$|3 - 4| = |-1| = 1$, and 5 times 1 is 5. 5 cubed is 125.

5.5.1 Variables and Expressions

Learning Objective(s)

1 Evaluate expressions with one variable for given values for the variable.

2 Evaluate expressions with two variables for given values for the variables.

Introduction

Algebra involves the solution of problems using variables, expressions, and equations. This topic focuses on variables and expressions and you will learn about the types of expressions used in algebra.

Variables and Expressions

One thing that separates algebra from arithmetic is the **variable**. A variable is a letter or symbol used to represent a quantity that can change. Any letter can be used, but x and y are common. You may have seen variables used in formulas, like the area of a rectangle. To find the area of a rectangle, you multiply *length* times *width*, written using the two variables l and w.

$$l \bullet w$$

Here, the variable l represents the length of the rectangle. The variable w represents the width of the rectangle.

You may be familiar with the formula for the area of a triangle. It is $\frac{1}{2}b \bullet h$.

Here, the variable b represents the base of the triangle, and the variable h represents the height of the triangle. The $\frac{1}{2}$ in this formula is a **constant**. A constant, unlike a variable, is a quantity that does **not** change. A constant is often a number.

An **expression** is a mathematical phrase made up of a sequence of mathematical symbols. Those symbols can be numbers, variables, or operations (+, –, •, ÷). Examples of expressions are $l \bullet w$ and $\frac{1}{2}b \bullet h$.

Example		
Problem	**Identify the constant and variable in the expression 24 – *x*.**	
Answer	24 is the constant. x is the variable.	Since 24 cannot change its value, it is a constant. The variable is x, because it could be 0, or 2, or many other numbers.

Substitution and Evaluation

Objective **1, 2**

In arithmetic, you often evaluated, or simplified, expressions involving numbers.

$3 \bullet 25 + 4$ $\quad$ $25 \div 5$ $\quad$ $142 - \frac{12}{4}$ $\quad$ $\frac{3}{4} - \frac{1}{4}$ $\quad$ $2.45 + 13$

In algebra, you will evaluate many expressions that contain variables.

$a + 10$ $\quad$ $48 \bullet c$ $\quad$ $100 - x$ $\quad$ $l \bullet w$ $\quad$ $\frac{1}{2}b \bullet h$

To **evaluate** an expression means to find its value. If there are variables in the expression, you will be asked to evaluate the expression for a specified value for the variable.

The first step in evaluating an expression is to **substitute** the given value of a variable into the expression. Then you can finish evaluating the expression using arithmetic.

Example		
Problem	**Evaluate $24 - x$ when $x = 3$.**	
	$24 - x$ $24 - 3$	Substitute 3 for the x in the expression.
	$24 - 3 = 21$	Subtract to complete the evaluation.
Answer	21	

When you have two variables, you substitute each given value for each variable.

Example		
Problem	**Evaluate $l \bullet w$ when $l = 3$ and $w = 8$.**	
	$l \bullet w$ $3 \bullet 8$	Substitute 3 for l in the expression and 8 for w.
	$3 \bullet 8 = 24$	Multiply.
Answer	24	

When you multiply a variable by a constant number, you don't need to write the multiplication sign or use parentheses. For example, $3a$ is the same as $3 \bullet a$.

Notice that the sign • is used to represent multiplication. This is especially common with algebraic expressions because the multiplication sign × looks a lot like the letter *x*, especially when hand written. Because of this, it's best to use parentheses or the • sign to indicate multiplication of numbers.

	Example	
Problem	**Evaluate $4x - 4$ when $x = 10$.**	
	$4x - 4$ $4(10) - 4$	Substitute 10 for *x* in the expression.
	$40 - 4$ 36	Remember that you must multiply before you do the subtraction.
Answer	36	

Since the variables are allowed to vary, there are times when you want to evaluate the same expression for different values for the variable.

	Example	
Problem	**John is planning a rectangular garden that is 2 feet wide. He hasn't decided how long to make it, but he's considering 4 feet, 5 feet, and 6 feet. He wants to put a short fence around the garden. Using x to represent the length of the rectangular garden, he will need $x + x + 2 + 2$, or $2x + 4$, feet of fencing.** **How much fencing will he need for each possible garden length? Evaluate the expression when $x = 4$, $x = 5$, and $x = 6$ to find out.**	
	$2x + 4$ $2(4) + 4$ $8 + 4$ 12	For $x = 4$, substitute 4 for *x* in the expression. Evaluate by multiplying and adding.
	$2x + 4$ $2(5) + 4$ $10 + 4$ 14	For $x = 5$, substitute 5 for *x*. Evaluate by multiplying and adding.
	$2x + 4$ $2(6) + 4$ $12 + 4$ 16	For $x = 6$, substitute 6 for *x* and evaluate.
Answer	*John needs 12 feet of fencing when* $x = 4$*, 14 feet when* $x = 5$*, and 16 feet when* $x = 6$*.*	

Evaluating expressions for many different values for the variable is one of the powers of algebra. Computer programs are written to evaluate the same expression (usually a very complicated expression) for millions of different values for the variable(s).

Self Check A

Evaluate $8x - 1$ when $x = 2$.

Summary

Variables are an important part of algebra. Expressions made from variables, constants, and operations can represent a numerical value. You can evaluate an expression when you are provided with one or more values for the variables: substitute each variable's value for the variable, then perform any necessary arithmetic.

5.5.1 Self Check Solutions

Self Check A

Evaluate $8x - 1$ when $x = 2$.

Substituting 2 for x gives $8(2) - 1$. First multiply to get $16 - 1$, then subtract to get 15.

5.5.2 Associative, Commutative, and Distributive Properties

Learning Objective(s)
1 Identify and use the commutative properties for addition and multiplication.
2 Identify and use the associative properties for addition and multiplication.
3 Identify and use the distributive property.

Introduction

There are many times in algebra when you need to simplify an expression. The properties of real numbers provide tools to help you take a complicated expression and simplify it.

The associative, commutative, and distributive properties of algebra are the properties most often used to simplify algebraic expressions. You will want to have a good understanding of these properties to make the problems in algebra easier to work.

The Commutative Properties of Addition and Multiplication

Objective **1**

You may encounter daily routines in which the order of tasks can be switched without changing the outcome. For example, think of pouring a cup of coffee in the morning. You would end up with the same tasty cup of coffee whether you added the ingredients in either of the following ways:

- Pour 12 ounces of coffee into mug, then add splash of milk.
- Add a splash of milk to mug, then add 12 ounces of coffee.

The order that you add ingredients does not matter. In the same way, it does not matter whether you put on your left shoe or right shoe first before heading out to work. As long as you are wearing both shoes when you leave your house, you are on the right track!

In mathematics, we say that these situations are commutative—the outcome will be the same (the coffee is prepared to your liking; you leave the house with both shoes on) no matter the order in which the tasks are done.

Likewise, the **commutative property of addition** states that when two numbers are being added, their order can be changed without affecting the sum. For example, 30 + 25 has the same sum as 25 + 30.

$$30 + 25 = 55$$
$$25 + 30 = 55$$

Multiplication behaves in a similar way. The **commutative property of multiplication** states that when two numbers are being multiplied, their order can be changed without affecting the product. For example, $7 \bullet 12$ has the same product as $12 \bullet 7$.

$$7 \bullet 12 = 84$$
$$12 \bullet 7 = 84$$

These properties apply to all real numbers. Let’s take a look at a few addition examples.

Original Equation	*Rewritten Equation*
1.2 + 3.8 = 5	3.8 + 1.2 = 5
$\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$	$\frac{1}{8}+\frac{1}{2}=\frac{5}{8}$
14 + (−10) = 4	(−10) + 14 = 4
$\frac{1}{3}+\left(-1\frac{2}{3}\right)=-1\frac{1}{3}$	$\left(-1\frac{2}{3}\right)+\frac{1}{3}=-1\frac{1}{3}$
(−5.2) + (−3.6) = −8.8	(−3.6) + (−5.2) = −8.8
x + 3 = 4	3 + x = 4

Commutative Property of Addition

For any real numbers *a* and *b*, *a* + *b* = *b* + *a*.

Subtraction is not commutative. For example, 4 − 7 does not have the same difference as 7 − 4. The − sign here means subtraction.

However, recall that 4 − 7 can be rewritten as 4 + (−7), since subtracting a number is the same as adding its opposite. Applying the commutative property for addition here, you can say that 4 + (−7) is the same as (−7) + 4. Notice how this expression is very different than 7 – 4.

Now look at some multiplication examples.

Original Equation	**Rewritten Equation**
4.5 • 2 = 9	2 • 4.5 = 9
(−5) • 3 = -15	3 • (−5) = -15
$\frac{1}{5}\bullet 5=1$	$5\bullet\frac{1}{5}=1$
$\left(-\frac{1}{4}\right)\bullet\left(-\frac{8}{10}\right)=\frac{1}{5}$	$\left(-\frac{8}{10}\right)\bullet\left(-\frac{1}{4}\right)=\frac{1}{5}$
y • 4	4 • y

Commutative Property of Multiplication

For any real numbers *a* and *b*, *a* • *b* = *b* • *a*.

Order does not matter as long as the two quantities are being multiplied together. This property works for real numbers and for variables that represent real numbers.

Just as subtraction is not commutative, neither is division commutative. 4 ÷ 2 does not have the same quotient as 2 ÷ 4.

Example

Problem **Write the expression (−15.5) + 35.5 in a different way, using the commutative property of addition, and show that both expressions result in the same answer.**

(−15.5) + 35.5 = 20	Adding.
35.5 + (−15.5)	Using the commutative property, you can switch the −15.5 and the 35.5 so that they are in a different order.
35.5 + (−15.5) 35.5 – 15.5 = 20	Adding 35.5 and −15.5 is the same as subtracting 15.5 from 35.5. The sum is 20.

Answer (−15.5) + 35.5 = 20 and 35.5 + (−15.5) = 20

Self Check A

Rewrite 52 • *y* in a different way, using the commutative property of multiplication. Note that *y* represents a real number.

The Associative Properties of Addition and Multiplication

Objective **2**

The **associative property of addition** states that numbers in an addition expression can be grouped in different ways without changing the sum. You can remember the meaning of the associative property by remembering that when you *associate* with family members, friends, and co-workers, you end up forming groups with them.

Below, are two ways of simplifying the same addition problem. In the first example, 4 is grouped with 5, and 4 + 5 = 9.

$$4 + 5 + 6 = 9 + 6 = 15$$

Here, the same problem is worked by grouping 5 and 6 first, 5 + 6 = 11.

$$4 + 5 + 6 = 4 + 11 = 15$$

In both cases, the sum is the same. This illustrates that changing the grouping of numbers when adding yields the same sum.

Mathematicians often use parentheses to indicate which operation should be done first in an algebraic equation. The addition problems from above are rewritten here, this time using parentheses to indicate the associative grouping.

$$(4 + 5) + 6 = 9 + 6 = 15$$

$$4 + (5 + 6) = 4 + 11 = 15$$

It is clear that the parentheses do not affect the sum; the sum is the same regardless of where the parentheses are placed.

Associative Property of Addition

For any real numbers a, b, and c, $(a + b) + c = a + (b + c)$.

The example below shows how the associative property can be used to simplify expressions with real numbers.

Example

Problem **Rewrite 7 + 2 + 8.5 – 3.5 in two different ways using the associative property of addition. Show that the expressions yield the same answer.**

7 + 2 + 8.5 – 3.5 7 + 2 + 8.5 + (−3.5)	The associative property does not apply to expressions involving subtraction. So, re-write the expression as addition of a negative number.
(7 + 2) + 8.5 + (−3.5) 9 + 8.5 + (−3.5) 17.5 + (−3.5)	Group 7 and 2, and add them together. Then, add 8.5 to that sum. Finally, add −3.5, which is the same as subtracting 3.5.
17.5 – 3.5 = 14	Subtract 3.5. The sum is 14.
7 + 2 + (8.5 + (−3.5)) 7 + 2 + 5 9 + 5	Group 8.5 and –3.5, and add them together to get 5. Then add 7 and 2, and add that sum to the 5.
14	The sum is 14.

Answer (7 + 2) + 8.5 – 3.5 = 14 **and** 7 + 2 + (8.5 + (−3.5)) = 14

Multiplication has an associative property that works exactly the same as the one for addition. The **associative property of multiplication** states that numbers in a multiplication expression can be regrouped using parentheses. For example, the expression below can be rewritten in two different ways using the associative property.

Original expression: $-\frac{5}{2}\cdot 6\cdot 4$

Expression 1: $\left(-\frac{5}{2}\cdot 6\right)\cdot 4=\left(-\frac{30}{2}\right)\cdot 4=-15\cdot 4=-60$

Expression 2: $-\frac{5}{2}\cdot(6\cdot 4)=-\frac{5}{2}\cdot 24=-\frac{120}{2}=-60$

The parentheses do not affect the product, the product is the same regardless of where the parentheses are.

Associative Property of Multiplication

For any real numbers *a*, *b*, and *c*, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Self Check B

Rewrite $\frac{1}{2}\cdot\left(\frac{5}{6}\cdot 6\right)$ using only the associative property.

Using the Associative and Commutative Properties

You will find that the associative and commutative properties are helpful tools in algebra, especially when you evaluate expressions. Using the commutative and associative properties, you can reorder terms in an expression so that compatible numbers are next to each other and grouped together. Compatible numbers are numbers that are easy for you to compute, such as 5 + 5, or 3 • 10, or 12 – 2, or 100 ÷ 20. (The main criteria for compatible numbers is that they “work well” together.) The two examples below show how this is done.

Example

Problem **Evaluate the expression 4 • (x • 27) when $x = -\frac{3}{4}$.**

$4 \cdot (x \cdot 27)$	Original expression.
$4 \cdot \left(\left(-\frac{3}{4}\right) \cdot 27\right)$	Substitute $-\frac{3}{4}$ for x.
$\left(4 \cdot \left(-\frac{3}{4}\right)\right) \cdot 27$	Use the associative property of multiplication to regroup the factors so that 4 and $-\frac{3}{4}$ are next to each other.
$\left(-\frac{12}{4}\right) \cdot 27$	Multiplying 4 by $-\frac{3}{4}$ first makes the expression a bit easier to evaluate than multiplying $-\frac{3}{4}$ by 27.
$-3 \cdot 27 = -81$	Multiply. 4 times $-\frac{3}{4}$ = −3, and −3 times 27 is −81.

Answer $4 \cdot (x \cdot 27) = -81$ when $x = \left(-\frac{3}{4}\right)$.

Example

Problem **Simplify: 4 + 12 + 3 + 4 – 8.**

4 + 12 + 3 + 4 – 8	Original expression.
12 + 3 + 4 + 4 + (−8)	Identify compatible numbers. 4 + 4 is 8, and there is a −8 present. Recall that you can think of – 8 as + (−8). Use the commutative property of addition to group them together.
12 + 3 + (4 + 4 + (−8))	Use the associative property to group 4 + 4 + (−8).
12 + 3 + 0	Add 4 + 4 + (−8).
12 + 3 + 0 = 15	Add the rest of the terms.

Answer 4 + 12 + 3 + 4 – 8 = 15

Self Check C

Simplify the expression: −5 + 25 – 15 + 2 + 8

The Distributive Property

Objective **3**

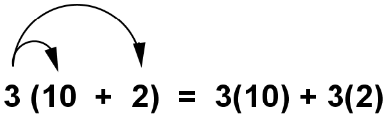
The **distributive property of multiplication** is a very useful property that lets you rewrite expressions in which you are multiplying a number by a sum or difference. The property states that the product of a sum or difference, such as 6(5 – 2), is equal to the sum or difference of products, in this case, 6(5) – 6(2).

$$6(5 - 2) = 6(3) = 18$$
$$6(5) - 6(2) = 30 - 12 = 18$$

The distributive property of multiplication can be used when you multiply a number by a sum. For example, suppose you want to multiply 3 by the sum of 10 + 2.

$$3(10 + 2) = ?$$

According to this property, you can add the numbers 10 and 2 first and then multiply by 3, as shown here: 3(10 + 2) = 3(12) = 36. Alternatively, you can first multiply *each* addend by the 3 (this is called **distributing** the 3), and then you can add the products. This process is shown here.

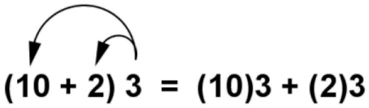


$$\mathbf{3\,(10 + 2) = 3(10) + 3(2)}$$

$$3\,(10 + 2) = 3(12) = 36$$
$$3(10) + 3(2) = 30 + 6 = 36$$

The products are the same.

Since multiplication is commutative, you can use the distributive property regardless of the order of the factors.



$$\mathbf{(10 + 2)\,3 = (10)3 + (2)3}$$

The Distributive Properties

For any real numbers *a*, *b*, and *c*:

Multiplication distributes over addition: $a(b + c) = ab + ac$
Multiplication distributes over subtraction: $a(b - c) = ab - ac$

Self Check D

Rewrite the expression 10(9 – 6) using the distributive property.

Distributing with Variables

As long as variables represent real numbers, the distributive property can be used with variables. The distributive property is important in algebra, and you will often see expressions like this: $3(x-5)$. If you are asked to expand this expression, you can apply the distributive property just as you would if you were working with integers.

$$\mathbf{3\,(x-5\,)=3(x)-3(5)=3x-15}$$

Remember, when you multiply a number and a variable, you can just write them side by side to express the multiplied quantity. So, the expression “three times the variable *x*” can be written in a number of ways: $3x$, $3(x)$, or $3 \bullet x$.

Example

Problem **Use the distributive property to expand the expression $9(4+x)$.**

$9(4+x)$	Original expression.
$9(4)+9(x)$	Distribute the 9 and multiply.
$36+9x$	Multiply.

Answer $9(4+x)=36+9x$

Example

Problem **Use the distributive property to evaluate the expression $5(2x-3)$ when $x=2$.**

$5(2x-3)$	Original expression.
$5(2x)-5(3)$	Distribute the 5.
$10x-15$	Multiply.
$10(2)-15$	Substitute 2 for *x*, and evaluate.
$20-15=5$	

Answer When $x=2$, $5(2x-3)=5$.

In the example above, what do you think would happen if you substituted $x=2$ before distributing the 5? Would you get the same answer of 5? The example below shows what would happen.

Example	
Problem	**Use the distributive property to evaluate the expression $5(2x – 3)$ when $x = 2$.**
$5(2x – 3)$	Original expression.
$5(2(2) – 3)$	Substitute 2 for x.
$5(4 – 3)$	Multiply.
$5(4) – 5(3)$	Subtract and evaluate.
$20 – 15 = 5$	
Answer	When $x = 2$, $5(2x – 3) = 5$.

Combining Like Terms

The distributive property can also help you understand a fundamental idea in algebra: that quantities such as $3x$ and $12x$ can be added and subtracted in the same way as the numbers 3 and 12. Let's look at one example and see how it can be done.

Example	
Problem	**Add: $3x + 12x$.**
$3(x) + 12(x)$	$3x$ is 3 times x, and $12x$ is 12 times x.
$x(3 + 12)$	From studying the distributive property (and also using the commutative property), you know that $x(3 + 12)$ is the same as $3(x) + 12(x)$.
$x(15)$ or $15x$	Combine the terms within the parentheses: $3 + 12 = 15$.
Answer	$3x + 12x = 15x$

Do you see what happened? By thinking of the x as a distributed quantity, you can see that $3x + 12x = 15x$. (If you're not sure about this, try substituting any number for x in this expression…you will find that it holds true!)

Groups of terms that consist of a coefficient multiplied by the same variable are called "like terms". The table below shows some different groups of like terms:

Groups of Like Terms
$3x$, $7x$, $-8x$, $-0.5x$
$-1.1y$, $-4y$, $-8y$
$12t$, $25t$, $100t$, $1t$
$4ab$, $-8ab$, $2ab$

Whenever you see like terms in an algebraic expression or equation, you can add or subtract them just like you would add or subtract real numbers. So, for example, $10y + 12y = 22y$, and $8x – 3x – 2x = 3x$.

Be careful not to combine terms that do not have the same variable: $4x + 2y$ is not $6xy$!

Example		
Problem	**Simplify: $10y + 5y + 9x – 6x – x$.**	
	$10y + 5y + 9x – 6x – x$	There are like terms in this expression, since they all consist of a coefficient multiplied by the variable x or y. Note that $– x$ is the same as $(−1)x$.
	$15y \quad + \quad 2x$	Add like terms. $10y + 5y = 15y$, and $9x – 6x – x = 2x$.
Answer	$10y + 5y + 9x – 6x – x = 15y + 2x$	

Self Check E

Simplify: $12x – x + 2x – 8x$.

Summary

The commutative, associative, and distributive properties help you rewrite a complicated algebraic expression into one that is easier to deal with. When you rewrite an expression by a commutative property, you change the order of the numbers being added or multiplied. When you rewrite an expression using an associative property, you group a different pair of numbers together using parentheses. You can use the commutative and associative properties to regroup and reorder any number in an expression as long as the expression is made up entirely of addends or factors (and not a combination of them). The distributive property can be used to rewrite expressions for a variety of purposes. When you are multiplying a number by a sum, you can add and then multiply. You can also multiply each addend first and then add the products together. The same principle applies if you are multiplying a number by a difference.

5.5.2 Self Check Solutions

Self Check A

Rewrite 52 • y in a different way, using the commutative property of multiplication. Note that y represents a real number.

$y \bullet 52$
The order of factors is reversed.

Self Check B

Rewrite $\frac{1}{2} \bullet \left(\frac{5}{6} \bullet 6\right)$ using only the associative property.

$\left(\frac{1}{2} \bullet \frac{5}{6}\right) \bullet 6$

Here, the numbers are regrouped. Now $\frac{1}{2}$ and $\frac{5}{6}$ are grouped in parentheses instead of $\frac{5}{6}$ and 6.

Self Check C

Simplify the expression: −5 + 25 – 15 + 2 + 8

15
Correct. Use the commutative property to rearrange the expression so that compatible numbers are next to each other, and then use the associative property to group them.

Self Check D

Rewrite the expression 10(9 – 6) using the distributive property.

10(9) – 10(6)
Correct. The 10 is correctly distributed so that it is used to multiply the 9 and the 6 separately.

Self Check E

Simplify: $12x - x + 2x - 8x$.

$5x$
Correct. When you combine these like terms, you end up with a sum of $5x$.