Electromagnetic Fields and Energy

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Dedicated to

Professors R. B. Adler, L. J. Chu, and R. M. Fano in recognition and gratitude for their inspiration.

0.1 PREFACE

The text is aimed at an audience that has seen Maxwell's equations in integral or differential form (second-term Freshman Physics) and had some exposure to integral theorems and differential operators (second term Freshman Calculus). The first two chapters and supporting problems and appendices are a review of this material.

In Chap. 3, a simple and physically appealing argument is presented to show that Maxwell's equations predict the time evolution of a field, produced by free charges, given the initial charge densities and velocities, and electric and magnetic fields. This is a form of the uniqueness theorem that is established more rigorously later. As part of this development, it is shown that a field is completely specified by its divergence and its curl throughout all of space, a proof that explains the general form of Maxwell's equations.

With this background, Maxwell's equations are simplified into their electroquasistatic (EQS) and magnetoquasistatic (MQS) forms. The stage is set for taking a structured approach that gives a physical overview while developing the mathematical skills needed for the solution of engineering problems.

The text builds on and reinforces an understanding of analog circuits. The fields are never static. Their dynamics are often illustrated with step and sinusoidal steady state responses in systems where the spatial dependence has been encapsulated in time-dependent coefficients (of solutions to partial differential equations) satisfying ordinary differential equations. However, the connection with analog circuits goes well beyond the same approach to solving differential equations as used in circuit theory. The approximations inherent in the development of circuit theory from Maxwell's equations are brought out very explicitly, so that the student appreciates under what conditions the assumptions implicit in circuit theory cease to be applicable.

To appreciate the organization of material in this text, it may be helpful to make a more subtle connection with electrical analog circuits. We think of circuit theory as being analogous to field theory. In this analogy, our development begins with capacitors– charges and their associated fields, equipotentials used to represent perfect conductors. It continues with resistors– steady conduction to represent losses. Then these elements are combined to represent charge relaxation, i.e. "RC" systems dynamics (Chaps. 4-7). Because EQS fields are not necessarily static, the student can appreciate R-C type dynamics, where the distribution of free charge is determined by the continuum analog of R-C systems.

Using the same approach, we then take up the continuum generalization of L-R systems (Chaps. 8–10). As before, we first are given the source (the current density) and find the magnetic field. Then we consider perfectly conducting systems and once again take the boundary value point of view. With the addition of finite conductivity to this continuum analog of systems of inductors, we arrive at the dynamics of systems that are L-R-like in the circuit analogy.

Based on an appreciation of the connection between sources and fields afforded by these quasistatic developments, it is natural to use the study of electric and magnetic energy storage and dissipation as an entree into electrodynamics (Chap. 11).

Central to electrodynamics are electromagnetic waves in loss-free media (Chaps. 12–14). In this limit, the circuit analog is a system of distributed differential inductors and capacitors, an L-C system. Following the same pattern used for EQS and MQS systems, fields are first found for given sources– antennae and arrays. The boundary value point of view then brings in microwave and optical waveguides and transmission lines.

We conclude with the electrodynamics of lossy material, the generalization of L-R-C systems (Chaps. 14–15). Drawing on what has been learned for EQS, MQS, and electrodynamic systems, for example, on the physical significance of the dominant characteristic times, we form a perspective as to how electromagnetic fields are exploited in practical systems. In the circuit analogy, these characteristic times are RC, L/R, and $1/\sqrt{LC}$. One benefit of the field theory point of view is that it shows the influence of physical scale and configuration on the dynamics represented by these times. The circuit analogy gives a hint as to why it is so often possible to view the world as either EQS or MQS. The time $1/\sqrt{LC}$ is the geometric mean of RC and L/R. Either RC or L/R is smaller than $1/\sqrt{LC}$, but not both. For large R, RC dynamics comes first as the frequency is raised (EQS), followed by electrodynamics. For small R, L/R dynamics comes first (MQS), again followed by electrodynamics. Implicit is the enormous difference between what is meant by a "perfect conductor" in systems appropriately modeled as EQS and MQS.

This organization of the material is intended to bring the student to the realization that electric, magnetic, and electromagnetic devices and systems can be broken into parts, often described by one or another limiting form of Maxwell's equations. Recognition of these limits is part of the art and science of modeling, of making the simplifications necessary to make the device or system amenable to analytic treatment or computer analysis and of effectively using appropriate simplifications of the laws to guide in the process of invention.

With the EQS approximation comes the opportunity to treat such devices as transistors, electrostatic precipitators, and electrostatic sensors and actuators, while relays, motors, and magnetic recording media are examples of MQS systems. Transmission lines, antenna arrays, and dielectric waveguides (i.e., optical fibers) are examples where the full, dynamic Maxwell's equations must be used.

In connection with examples, about 40 demonstrations are described in this text. These are designed to make the mathematical results take on physical meaning. Based upon relatively simple configurations and arrangements of equipment, they incorporate no more complexity then required to make a direct connection between what has been derived and what is observed. Their purpose is to help the student observe physically what has been described symbolically. Often coming with a plot of the theoretical predictions that can be compared to data taken in the classroom, they give the opportunity to test the range of validity of the theory and to promulgate a quantitative approach to dealing with the physical world. More detailed consideration of the demonstrations can be the basis for special projects, often bringing in computer modeling. For the student having only the text as a resource, the descriptions of the experiments stand on their own as a connection between the abstractions and the physical reality. For those fortunate enough to have some of the demonstrations used in the classroom, they serve as documentation of what was done. All too often, students fail to profit from demonstrations because conventional note taking fails to do justice to the presentation.

The demonstrations included in the text are of physical phenomena more than of practical applications. To fill out the classroom experience, to provide the engineering motivation, applications should also be exemplified. In the subject as we teach it, and as a practical matter, these are more of the nature of "show and tell" than of working demonstrations, often reflecting the current experience and interests of the instructor and usually involving more complexity than appropriate for more than a qualitative treatment.

The text provides a natural frame of reference for developing numerical approaches to the details of geometry and nonlinearity, beginning with the method of moments as the superposition integral approach to boundary value problems and culminating in energy methods as a basis for the finite element approach. Professor J. L. Kirtley and Dr. S. D. Umans are currently spearheading our efforts to expose the student to the "muscle" provided by the computer for making practical use of field theory while helping the student gain physical insight. Work stations, finite element packages, and the like make it possible to take detailed account of geometric effects in routine engineering design. However, no matter how advanced the computer packages available to the student may become in the future, it will remain essential that a student comprehend the physical phenomena at work with the aid of special cases. This is the reason for the emphasis of the text on simple geometries to provide physical insight into the processes at work when fields interact with media.

The mathematics of Maxwell's equations leads the student to a good understanding of the gradient, divergence, and curl operators. This mathematical conversance will help the student enter other areas– such as fluid and solid mechanics, heat and mass transfer, and quantum mechanics– that also use the language of classical fields. So that the material serves this larger purpose, there is an emphasis on source-field relations, on scalar and vector potentials to represent the irrotational and solenoidal parts of fields, and on that understanding of boundary conditions that accounts for finite system size and finite time rates of change.

Maxwell's equations form an intellectual edifice that is unsurpassed by any other discipline of physics. Very few equations encompass such a gamut of physical phenomena. Conceived before the introduction of relativity Maxwell's equations not only survived the formulation of relativity, but were instrumental in shaping it. Because they are linear in the fields, the replacement of the field vectors by operators is all that is required to make them quantum theoretically correct; thus, they also survived the introduction of quantum theory.

The introduction of magnetizable materials deviates from the usual treatment in that we use paired magnetic charges, magnetic dipoles, as the source of magnetization. The often-used alternative is circulating Ampèrian currents. The magnetic charge approach is based on the Chu formulation of electrodynamics. Chu exploited the symmetry of the equations obtained in this way to facilitate the study of magnetism by analogy with polarization. As the years went by, it was unavoidable that this approach would be criticized, because the dipole moment of the electron, the main source of ferromagnetism, is associated with the spin of the electron, i.e., seems to be more appropriately pictured by circulating currents.

Tellegen in particular, of Tellegen-theorem fame, took issue with this approach. Whereas he conceded that a choice between two approaches that give identical answers is a matter of taste, he gave a derivation of the force on a current loop (the Ampèrian model of a magnetic dipole) and showed that it gave a different answer from that on a magnetic dipole. The difference was small, the correction term was relativistic in nature; thus, it would have been difficult to detect the effect in macroscopic measurements. It occurred only in the presence of a time-varying electric field. Yet this criticism, if valid, would have made the treatment of magnetization in terms of magnetic dipoles highly suspect.

The resolution of this issue followed a careful investigation of the force exerted on a current loop on one hand, and a magnetic dipole on the other. It turned out that Tellegen's analysis, in postulating a constant circulating current around the loop, was in error. A time-varying electric field causes changes in the circulating current that, when taken into account, causes an additional force that cancels the critical term. Both models of a magnetic dipole yield the same force expression. The difficulty in the analysis arose because the current loop contains "moving parts," i.e., a circulating current, and therefore requires the use of relativistic corrections in the rest-frame of the loop. Hence, the current loop model is inherently much harder to analyze than the magnetic charge–dipole model.

The resolution of the force paradox also helped clear up the question of the symmetry of the energy momentum tensor. At about the same time as this work was in progress, Shockley and James at Stanford independently raised related questions that led to a lively exchange between them and Coleman and Van Vleck at Harvard. Shockley used the term "hidden momentum" for contributions to the momentum of the electromagnetic field in the presence of magnetizable materials. Coleman and Van Vleck showed that a proper formulation based on the Dirac equation (i.e., a relativistic description) automatically includes such terms. With all this theoretical work behind us, we are comfortable with the use of the magnetic charge–dipole model for the source of magnetization. The student is not introduced to the intricacies of the issue, although brief mention is made of them in the text.

As part of curriculum development over a period about equal in time to the age of a typical student studying this material (the authors began their collaboration in 1968) this text fits into an evolution of field theory with its origins in the "Radiation Lab" days during and following World War II. Quasistatics, promulgated in texts by Professors Richard B. Adler, L.J. Chu, and Robert M. Fano, is a major theme in this text as well. However, the notion has been broadened and made more rigorous and useful by recognizing that electromagnetic phenomena that are "quasistatic," in the sense that electromagnetic wave phenomena can be ignored, can nevertheless be rate dependent. As used in this text, a quasistatic regime includes dynamical phenomena with characteristic times longer than those associated with electromagnetic waves. (A model in which no time-rate processes are included is termed "quasistationary" for distinction.)

In recognition of the lineage of our text, it is dedicated to Professors R. B. Adler, L. J. Chu and R. M. Fano. Professor Adler, as well as Professors J. Moses, G. L. Wilson, and L. D. Smullin, who headed the department during the period of development, have been a source of intellectual, moral, and financial support. Our inspiration has also come from colleagues in teaching– faculty and teaching assistants, and those students who provided insight concerning the many evolutions of the "notes." The teaching of Professor Alan J. Grodzinsky, whose latterday lectures have been a mainstay for the course, is reflected in the text itself. A partial list of others who contributed to the curriculum development includes Professors J. A. Kong, J. H. Lang, T. P. Orlando, R. E. Parker, D. H. Staelin, and M. Zahn (who helped with a final reading of the text). With "macros" written by Ms. Amy Hendrickson, the text was "Tex't" by Ms. Cindy Kopf, who managed to make the final publication process a pleasure for the authors.

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APPENDIX

1.1 VECTOR OPERATIONS

A vector is a quantity which possesses magnitude and direction. In order to describe a vector mathematically, a coordinate system having orthogonal axes is usually chosen. In this text, use is made of the Cartesian, circular cylindrical, and spherical coordinate systems. In these three-dimensional systems, any vector is completely described by three scalar quantities. For example, in Cartesian coordinates, a vector is described with reference to mutually orthogonal coordinate axes. Then the magnitude and orientation of the vector are described by specifying the three projections of the vector onto the three coordinate axes.

In representing a vector[1] $\mathbf{A}$ mathematically, its direction along the three orthogonal coordinate axes must be given. The direction of each axis is represented by a unit vector $\mathbf{i}$, that is, a vector of unit magnitude directed along the axis. In Cartesian coordinates, the three unit vectors are denoted $\mathbf{i_x}$, $\mathbf{i_y}$, $\mathbf{i_z}$. In cylindrical coordinates, they are $\mathbf{i_r}$, $\mathbf{i}_\phi$, $\mathbf{i_z}$, and in spherical coordinates, $\mathbf{i_r}$, $\mathbf{i}_\theta$, $\mathbf{i}_\phi$. $\mathbf{A}$, then, has three vector components, each component corresponding to the projection of $\mathbf{A}$ onto the three axes. Expressed in Cartesian coordinates, a vector is defined in terms of its components by

$$\mathbf{A} = A_x\mathbf{i_x} + A_y\mathbf{i_y} + A_z\mathbf{i_z} \tag{1}$$

These components are shown in Fig. A.1.1.

[1] Vectors are usually indicated either with boldface characters, such as $\mathbf{A}$, or by drawing a line (or an arrow) above a character to indicate its vector nature, as in $\bar{A}$ or $\vec{A}$.

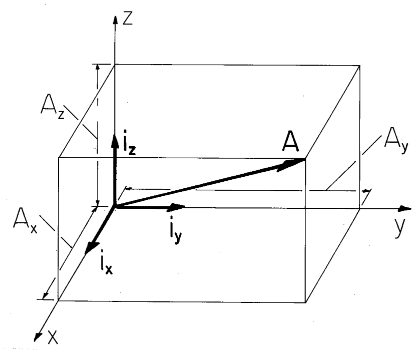

Fig. A.1.1 Vector **A** represented by its components in Cartesian coordinates and unit vectors **i**.

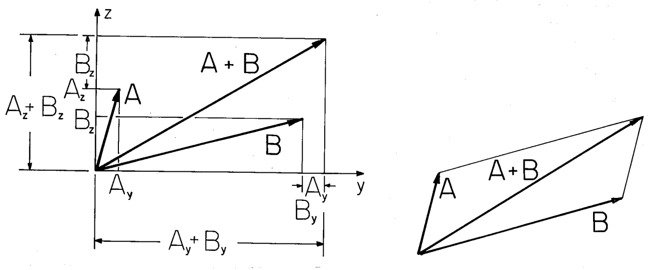

Fig. A.1.2 (a) Graphical representation of vector addition in terms of specific coordinates. (b) Representation of vector addition independent of specific coordinates.

Vector Addition. The sum of two vectors $\mathbf{A} = A_x\mathbf{i_x} + A_y\mathbf{i_y} + A_z\mathbf{i_z}$ and $\mathbf{B} = B_x\mathbf{i_x} + B_y\mathbf{i_y} + B_z\mathbf{i_z}$ is effected by adding the coefficients of each of the components, as shown in two dimensions in Fig. A.1.2a.

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i_x} + (A_y + B_y)\mathbf{i_y} + (A_z + B_z)\mathbf{i_z} \tag{2}$$

From (2), then, it should be clear that vector addition is both commutative, $\mathbf{A}+\mathbf{B} = \mathbf{B} + \mathbf{A}$, and associative, $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Graphically, vector summation can be performed without regard to the coordinate system, as shown in Fig. A.1.2b, by noticing that the sum $\mathbf{A} + \mathbf{B}$ is a vector directed along the diagonal of a parallelogram formed by **A** and **B**.

It should be noted that the representation of a vector in terms of its components is dependent on the coordinate system in which it is carried out. That is, changes of coordinate system will require an appropriate vector transformation. Further, the variables used must also be transformed. The transformation of variables and vectors from one coordinate system to another is illustrated by considering a transformation from Cartesian to spherical coordinates.

Example 1.1.1. Transformation of Variables and Vectors

We are given variables in terms of x, y, and z and vectors such as $\mathbf{A} = A_x\mathbf{i_x} + A_y\mathbf{i_y} + A_z\mathbf{i_z}$. We wish to obtain variables in terms of r, θ, and ϕ and vectors expressed as $\mathbf{A} = A_r\mathbf{i_r} + A_\theta\mathbf{i_\theta} + A_\phi\mathbf{i_\phi}$. In Fig. A.1.3a, we see that the point P has two

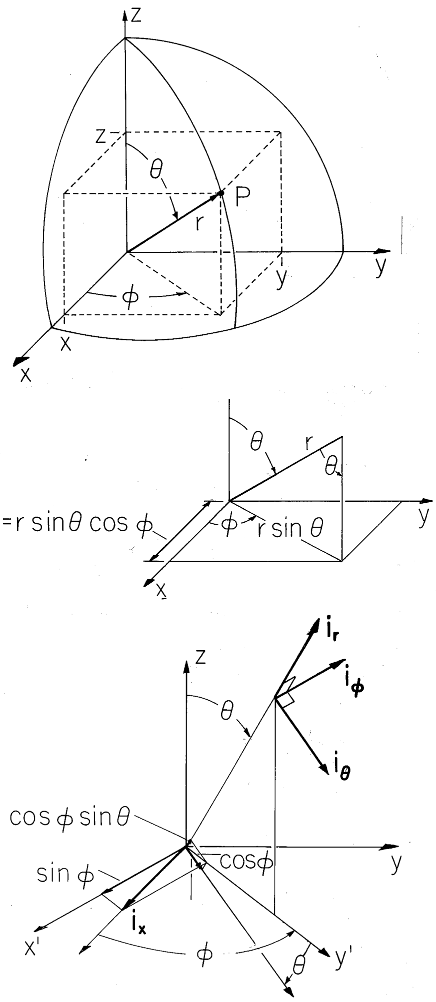

Fig. A.1.3 Specification of a point P in Cartesian and spherical coordinates. (b) Transformation from Cartesian coordinate x to spherical coordinates. (c) Transformation of unit vector in x direction into spherical coordinate coordinates.

representations, one involving the variables x, y and z and the other, r, θ and ϕ. In particular, from Fig. A.1.3b, x is related to the spherical coordinates by

$$x = r\sin\theta\cos\phi \tag{3}$$

In a similar way, the variables y and z evaluated in spherical coordinates can be shown to be

$$y = r\sin\theta\sin\phi \tag{4}$$

$$z = r\cos\theta \tag{5}$$

The vector $\mathbf{A}$ is transformed by resolving each of the unit vectors $\mathbf{i_x}$, $\mathbf{i_y}$, $\mathbf{i_z}$ in terms of the unit vectors in spherical coordinates. For example, $\mathbf{i_x}$ can first be

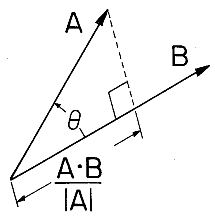

Fig. A.1.4 Illustration for definition of dot product.

resolved into components in the orthogonal coordinates (x', y', z) shown in Fig. A.1.3c. By definition, y' is along the intersection of the $\phi =$ constant and the $x - y$ planes. Also in the $x-y$ plane is x', which is perpendicular to the $y'-z$ plane. Thus, $\sin\phi$, $\cos\phi$, and 0 are the components of $\mathbf{i_x}$ along the x', y', and z axes respectively. These components are in turn resolved into components along the spherical coordinate directions by recognizing that the component $\sin\phi$ along the x' axis is in the $-\mathbf{i}_\phi$ direction while the component of $\cos\phi$ along the y' axis resolves into components $\cos\phi\cos\theta$ in the direction of $\mathbf{i}_\theta$, and $\cos\phi\sin\theta$ in the $\mathbf{i_r}$ direction. Thus,

$$\mathbf{i_x} = \sin\theta\cos\phi\mathbf{i_r} + \cos\theta\cos\phi\mathbf{i}_\theta - \sin\phi\mathbf{i}_\phi \tag{6}$$

Similarly,

$$\mathbf{i_y} = \sin\theta\sin\phi\mathbf{i_r} + \cos\theta\sin\phi\mathbf{i}_\theta + \cos\phi\mathbf{i}_\phi \tag{7}$$

$$\mathbf{i_z} = \cos\theta\mathbf{i_r} - \sin\theta\mathbf{i}_\theta \tag{8}$$

It must be emphasized that the concept of a vector is independent of the coordinate system. (In the same sense, in Chaps. 2 and 4, vector operations are defined independently of the coordinate system in which they are expressed.) A vector can be visualized as having the direction and magnitude of an arrow-tipped line element. This picture makes it possible to deal with vectors in a geometrical language that is independent of the choice of a particular coordinate system, one that will now be used to define the most important vector operations.

For analytical or numerical purposes, the operations are usually carried out in coordinate notation. Then, as illustrated, either in the text that follows or in the problems, each operation will be evaluated in a Cartesian coordinate system.

Definition of Scalar Product. Given vectors $\mathbf{A}$ and $\mathbf{B}$ as illustrated in Fig. A.1.4, the scalar, or dot product, between the two vectors is defined as

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta \tag{9}$$

where θ is the angle between the two vectors.

It follows directly from its definition that the scalar product is commutative.

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \tag{10}$$

The scalar product is also distributive.

$$(\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C} \tag{11}$$

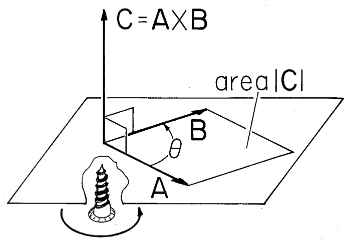

Fig. A.1.5 Illustration for definition of vector-product.

To see this, note that $\mathbf{A}\cdot\mathbf{C}$ is the projection of $\mathbf{A}$ onto $\mathbf{C}$ times the magnitude of $\mathbf{C}$, $|\mathbf{C}|$, and $\mathbf{B}\cdot\mathbf{C}$ is the projection of $\mathbf{B}$ onto $\mathbf{C}$ times $|\mathbf{C}|$. Because projections are additive, (11) follows.

These two properties can be used to define the scalar product in terms of the vector components in Cartesian coordinates. According to the definition of the unit vectors,

$$\mathbf{i_x}\cdot\mathbf{i_x}=\mathbf{i_y}\cdot\mathbf{i_y}=\mathbf{i_z}\cdot\mathbf{i_z}=1$$

$$\mathbf{i_x}\cdot\mathbf{i_y}=\mathbf{i_x}\cdot\mathbf{i_z}=\mathbf{i_y}\cdot\mathbf{i_z}=0 \tag{12}$$

With $\mathbf{A}$ and $\mathbf{B}$ expressed in terms of these components, it follows from the distributive and commutative properties that

$$\mathbf{A}\cdot\mathbf{B}=A_xB_x+A_yB_y+A_zB_z \tag{13}$$

Thus, in agreement with (9), the square of the magnitude of a vector is

$$\mathbf{A}\cdot\mathbf{A}=|\mathbf{A}|^2=A_x^2+A_y^2+A_z^2 \tag{14}$$

Definition of Vector Product. The cross-product of vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector $\mathbf{C}$ having a magnitude

$$|\mathbf{C}|=|\mathbf{A}||\mathbf{B}|\sin\theta \tag{15}$$

and having a direction perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Geometrically, the magnitude of $\mathbf{C}$ is the area of the parallelogram formed by the vectors $\mathbf{A}$ and $\mathbf{B}$. The vector $\mathbf{C}$ has the direction of advance of a right-hand screw, as though driven by rotating $\mathbf{A}$ into $\mathbf{B}$. Put another way, a right-handed coordinate system is formed by $\mathbf{A}-\mathbf{B}-\mathbf{C}$, as is shown in Fig. A.1.5. The commonly accepted notation for the cross-product is

$$\mathbf{C}=\mathbf{A}\times\mathbf{B} \tag{16}$$

It is useful to note that if the vector $\mathbf{A}$ is resolved into two mutually perpendicular vectors, $\mathbf{A}=\mathbf{A}_\perp+\mathbf{A}_\|$, where $\mathbf{A}_\perp$ lies in the plane of $\mathbf{A}$ and $\mathbf{B}$ and is perpendicular to $\mathbf{B}$ and $\mathbf{A}_\|$ is parallel to $\mathbf{B}$, then

$$\mathbf{A}\times\mathbf{B}=\mathbf{A}_\perp\times\mathbf{B} \tag{17}$$

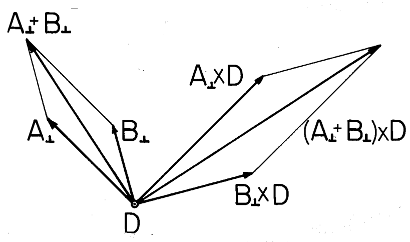

Fig. A.1.6 Graphical representation showing that the vector-product is distributive.

This equality follows from the fact that both cross-products have equal magnitude (since $|\mathbf{A}_\perp \times \mathbf{B}| = |\mathbf{A}_\perp||\mathbf{B}|$ and $|\mathbf{A}|_\perp| = |\mathbf{A}|\sin\theta$) and direction (perpendicular to both $\mathbf{A}$ and $\mathbf{B}$).

The distributive property for the cross-product,

$$(\mathbf{A}+\mathbf{B})\times\mathbf{D} = \mathbf{A}\times\mathbf{D}+\mathbf{B}\times\mathbf{D} \tag{18}$$

can be shown using (17) and the geometrical construction in Fig. A.1.6 as follows. First, note that $(\mathbf{A}+\mathbf{B})_\perp = (\mathbf{A}_\perp + \mathbf{B}_\perp)$, where $\perp$ denotes a component in the planes of $\mathbf{A}$ and $\mathbf{D}$ or $\mathbf{B}$ and $\mathbf{D}$, respectively, and perpendicular to $\mathbf{D}$. Thus,

$$(\mathbf{A}+\mathbf{B})\times\mathbf{D} = (\mathbf{A}+\mathbf{B})_\perp\times\mathbf{D} = (\mathbf{A}_\perp+\mathbf{B}_\perp)\times\mathbf{D} \tag{19}$$

Now, we need only show that

$$(\mathbf{A}_\perp+\mathbf{B}_\perp)\times\mathbf{D} = \mathbf{A}_\perp\times\mathbf{D}+\mathbf{B}_\perp\times\mathbf{D} \tag{20}$$

This equation is given graphical expression in Fig. A.1.6 by the vectors $\mathbf{A}_\perp$, $\mathbf{B}_\perp$, and their sum. To within a factor of $|\mathbf{D}|$, the three vectors $\mathbf{A}_\perp\times\mathbf{D}$, $\mathbf{B}_\perp\times\mathbf{D}$, and their sum, are, respectively, the vectors $\mathbf{A}_\perp$, $\mathbf{B}_\perp$, and their sum, rotated by 90 degrees. Thus, the vector addition property already shown for $\mathbf{A}_\perp+\mathbf{B}_\perp$ also applies to $\mathbf{A}_\perp\times\mathbf{D}+\mathbf{B}_\perp\times\mathbf{D}$.

Because interchanging the order of two vectors calls for a reassignment of the direction of the product vector (the direction of $\mathbf{C}$ in Fig. A.1.5), the commutative property does not hold. Rather,

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{21}$$

Using the distributive law, the vector product of two vectors can be constructed in terms of their Cartesian coordinates by using the following properties of the vector products of the unit vectors.

$$\begin{aligned}
&\mathbf{i_x}\times\mathbf{i_x} = 0 \qquad \mathbf{i_x}\times\mathbf{i_y} = \mathbf{i_z}\\
&\mathbf{i_y}\times\mathbf{i_y} = 0 \qquad \mathbf{i_y}\times\mathbf{i_z} = -\mathbf{i_z}\times\mathbf{i_y} = \mathbf{i_x}\\
&\mathbf{i_z}\times\mathbf{i_z} = 0 \qquad \mathbf{i_x}\times\mathbf{i_z} = -\mathbf{i_z}\times\mathbf{i_x} = -\mathbf{i_y}
\end{aligned} \tag{22}$$

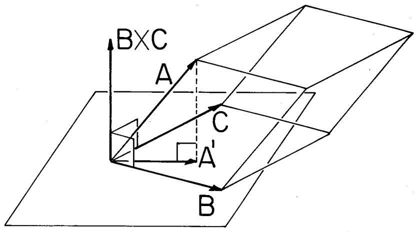

Fig. A.1.7 Graphical representation of scalar triple product.

Thus,

$$\mathbf{A}\times\mathbf{B} = \mathbf{i_x}(A_yB_z - A_zB_y) + \mathbf{i_y}(A_zB_x - A_xB_z) + \mathbf{i_z}(A_xB_y - A_yB_x) \tag{23}$$

A useful mnemonic for finding the cross-product in Cartesian coordinates is realized by noting that the right-hand side of (23) is the determinant of a matrix:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{24}$$

The Scalar Triple Product. The definition of the scalar triple product of vectors **A**, **B**, and **C** follows from Fig. A.1.7, and the definition of the scalar and vector products.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = [|\mathbf{A}|\cos(\mathbf{A},\mathbf{B}\times\mathbf{C})][|\mathbf{B}||\mathbf{C}|\sin(\mathbf{B},\mathbf{C})] \tag{25}$$

The scalar triple product is equal to the volume of the parallelepiped having the three vectors for its three bases. That is, in (25) the second term in square brackets is the area of the base parallelogram in Fig. A.1.7 while the first is the height of the parallelopiped. The scalar triple product is positive if the three vectors form a right-handed coordinate system in the order in which they are written; otherwise it is negative. Hence, a cyclic rearrangement in the order of the vectors leaves the value of the product unchanged.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{26}$$

It follows that the placing of the cross and the dot in a scalar triple product is arbitrary. The cross and dot can be interchanged without affecting the product.

Using the rules for evaluating the dot product and the cross-product in Cartesian coordinates, we have

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x) \tag{27}$$

The Double Cross-Product. Consider the vector product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. Is there another, sometimes more useful, way of expressing this double cross-product?

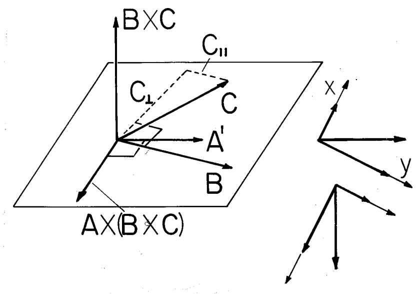

Fig. A.1.8 Graphical representation of double cross-product.

Since the product $\mathbf{B} \times \mathbf{C}$ is perpendicular to the plane defined by $\mathbf{B}$ and $\mathbf{C}$, then the final product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ must lie in the plane of $\mathbf{B}$ and $\mathbf{C}$. Hence, the vector product must be expressible as a linear combination of the vectors $\mathbf{B}$ and $\mathbf{C}$. One way to find the coefficients of this linear combination is to evaluate the product in Cartesian coordinates. Here we prefer to use a geometric derivation.

Because the vector $\mathbf{B} \times \mathbf{C}$ is perpendicular to the plane defined by the vectors $\mathbf{B}$ and $\mathbf{C}$, it follows from Fig. A.1.7 that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A}' \times (\mathbf{B} \times \mathbf{C}) \tag{28}$$

where $\mathbf{A}'$ is the projection of $\mathbf{A}$ onto the plane defined by $\mathbf{B}$ and $\mathbf{C}$. Next, we separate the vector $\mathbf{C}$ into a component parallel to $\mathbf{B}$, $\mathbf{C}_{\|}$, and a component perpendicular to $\mathbf{B}$, $\mathbf{C}_{\perp}$, as shown by Fig. A.1.8, so that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A}' \times (\mathbf{B} \times \mathbf{C}_{\perp}) \tag{29}$$

Then, according to the properties of the cross-product, the magnitude of the vector product is given by

$$|\mathbf{A} \times (\mathbf{B} \times \mathbf{C})| = |\mathbf{A}'||\mathbf{B}||\mathbf{C}_{\perp}| \tag{30}$$

and the direction of the vector product is orthogonal to $\mathbf{A}'$ and lies in the plane defined by the vectors $\mathbf{B}$ and $\mathbf{C}$, as shown in Fig. A.1.8

A rule for constructing a vector perpendicular to a given vector, $\mathbf{A}'$, in an $x-y$ plane is as follows. First, the two components of $\mathbf{A}'$ with respect to any two orthogonal axes (x, y) are determined. Here these are the directions of $\mathbf{C}_{\perp}$ and $\mathbf{B}$ with components $\mathbf{A}' \cdot \mathbf{C}_{\perp}$, and $\mathbf{A}' \cdot \mathbf{B}$, respectively. Then, a new vector is constructed by interchanging the x and y components and changing the sign of one of them. According to this rule, Fig. A.1.8 shows that the vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is given by

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}' \cdot \mathbf{C}_{\perp})\mathbf{B} - (\mathbf{A}' \cdot \mathbf{B})\mathbf{C}_{\perp} \tag{31}$$

Now, because $\mathbf{C}_{\|}$ has the same direction as $\mathbf{B}$,

$$(\mathbf{A}' \cdot \mathbf{B})\mathbf{C}_{\|} = (\mathbf{A}' \cdot \mathbf{C}_{\|})\mathbf{B}, \tag{32}$$

and addition of (31) gives

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A}' \cdot (\mathbf{C}_{\perp} + \mathbf{C}_{\|})\mathbf{B} - (\mathbf{A}' \cdot \mathbf{B})(\mathbf{C}_{\perp} + \mathbf{C}_{\|}) \tag{33}$$

Now observe that $\mathbf{A}' \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A}' \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B}$ (which follow from the definition of $\mathbf{A}'$ as the projection of $\mathbf{A}$ into the $\mathbf{B} - \mathbf{C}$ plane), and the double cross-product becomes

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{34}$$

This result is particularly convenient because it does not contain any special notation or projections.

The vector identities found in this Appendix are summarized in Table III at the end of the text.

2

APPENDIX

2.1 LINE AND SURFACE INTEGRALS

Consider a path connecting points (a) and (b) as shown in Fig. A.2.1. Assume that a vector field $\mathbf{A}(\mathbf{r})$ exists in the space in which the path is situated. Then the line integral of $\mathbf{A}(\mathbf{r})$ is defined by

$$\int_{(a)}^{(b)} \mathbf{A} \cdot d\mathbf{s} \tag{1}$$

To interpret (1), think of the path between (a) and (b) as subdivided into differential vector segments $d\mathbf{s}$. At every vector segment, the vector $\mathbf{A}(\mathbf{r})$ is evaluated and the dot product is formed. The line integral is then defined as the sum of these dot products in the limit as $d\mathbf{s}$ approaches zero. A line integral over a path that closes on itself is denoted by the symbol $\oint \mathbf{A} \cdot d\mathbf{s}$.

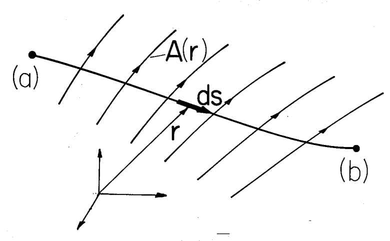

Fig. A.2.1 Configuration for integration of vector field **A** along line having differential length $d\mathbf{s}$ between points (a) and (b).

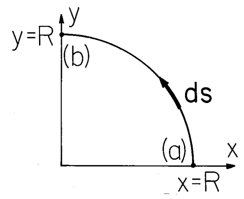

Fig. A.2.2 Integration line having shape of quarter segment of a circle with radius R and differential element $d\mathbf{s}$.

To perform a line integration, the integral must first be reduced to a form that can be evaluated using the rules of integral calculus. This is done with the aid of a coordinate system. The following example illustrates this process.

Example 2.1.1. Line Integral

Given the two-dimensional vector field

$$\mathbf{A} = x\mathbf{i_x} + axy\,\mathbf{i_y} \tag{2}$$

find the line integral along a quarter circle of radius R as shown in Fig. A.2.2.

Using a Cartesian coordinate system, the differential line segment $d\mathbf{s}$ has the components dx and dy.

$$d\mathbf{s} = \mathbf{i_x}dx + \mathbf{i_y}dy \tag{3}$$

Now x and y are not independent but are constrained by the fact that the integration path follows a circle defined by the equation

$$x^2 + y^2 = R^2 \tag{4}$$

Differentiation of (4) gives

$$2xdx + 2ydy = 0 \tag{5}$$

and therefore

$$dy = -\frac{x}{y}dx \tag{6}$$

Thus, the dot product $\mathbf{A} \cdot d\mathbf{s}$ can be written as a function of the variable x alone.

$$\mathbf{A} \cdot d\mathbf{s} = xdx + a\,xydy = (x - ax^2)dx \tag{7}$$

When the path is described in the sense shown in Fig. A.2.4, x decreases from R to zero. Therefore,

$$\int \mathbf{A} \cdot d\mathbf{s} = \int_R^0 (x - ax^2)dx = \left(\frac{x^2}{2} - \frac{ax^3}{3}\right)\Big|_R^0 = \frac{aR^3}{3} - \frac{R^2}{2} \tag{8}$$

If the path is not expressible in terms of an analytic function, the evaluation of the line integral becomes difficult. If everything else fails, numerical methods can be employed.

Surface Integrals. Given a vector field $\mathbf{A}(\mathbf{r})$ in a region of space containing a specified (open or closed) surface S, an important form of the surface integral of $\mathbf{A}$ over S is

$$\int_S \mathbf{A} \cdot d\mathbf{a} \tag{9}$$

The vector $d\mathbf{a}$ has a magnitude that represents the differential area of a surface element and a direction that is normal to that area. To interpret (9), think of the surface S as subdivided into these differential area elements $d\mathbf{a}$. At each area element, the differential scalar $\mathbf{A} \cdot d\mathbf{a}$ is evaluated and the surface integral is defined as the sum of these dot products over S in the limit as $d\mathbf{a}$ approaches zero. The surface integral $\int_S \mathbf{A} \cdot d\mathbf{a}$ is also called the "flux" of the vector $\mathbf{A}$ through the surface S.

To evaluate a surface integral, a coordinate system is introduced in which the integration can be performed according to the methods of integral calculus. Then the surface integral is transformed into a double integral in two independent variables. This is best illustrated with the aid of a specific example.

Example 2.1.2. Surface Integral

Given the vector field

$$\mathbf{A} = \mathbf{i_x} x \tag{10}$$

find the surface integral $\int_S \mathbf{A} \cdot d\mathbf{a}$, where S is one eighth of a spherical surface of radius R in the first octant of a sphere $(0 \le \phi \le \pi/2,\ 0 \le \theta \le \pi/2)$.

Because the surface lies on a sphere, it is best to carry out the integration in spherical coordinates. To transform coordinates from Cartesian to spherical, recall from (A.1.3) that the x coordinate is related to r, θ, and ϕ by

$$x = r \sin\theta \cos\phi \tag{11}$$

and from (A.1.6), the unit vector $\mathbf{i_x}$ is

$$\mathbf{i_x} = \sin\theta\cos\phi\,\mathbf{i_r} + \cos\theta\cos\phi\,\mathbf{i}_\theta - \sin\phi\,\mathbf{i}_\phi \tag{12}$$

Therefore, because the area element $d\mathbf{a}$ is

$$d\mathbf{a} = \mathbf{i_r} R^2 \sin\theta d\theta d\phi \tag{13}$$

the surface integral becomes

$$\begin{aligned} \int_S \mathbf{A} \cdot d\mathbf{a} &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi R^3 \sin^3\theta \cos^2\phi \\ &= \frac{\pi R^3}{4} \int_0^{\pi/2} d\theta \sin^3\theta = \frac{\pi R^3}{6} \end{aligned} \tag{14}$$

A surface integral of a vector $\mathbf{A}$ over a closed surface is indicated by

$$\oint_S \mathbf{A} \cdot d\mathbf{a} \tag{15}$$

Note also that we use a single integral sign for a surface integral, even though, in fact, two integrations are involved when the integral is actually evaluated in terms of a coordinate system.

2.2 PROOF THAT THE CURL OPERATION RESULTS IN A VECTOR

The definition

$$[curl\,\mathbf{A}]_n = \lim_{a\to 0}\frac{1}{a}\oint \mathbf{A}\cdot d\mathbf{s} \tag{1}$$

assigns a scalar, $[curl\,\mathbf{A}]_n$, to each direction $\mathbf{n}$ at the point P under consideration. The limit must be independent of the shape of the contour C (as long as all its points approach the point P in the limit as the area a of the contour goes to zero). The identification of $curl\,\mathbf{A}$ as a vector also implies a proper dependence of this limit upon the orientation of the normal $\mathbf{n}$ of a. The purpose of this appendix is to show that these two requirements are indeed satisfied by (1). We shall prove the following facts:

1. At a particular point (x, y, z) lying in the plane specified by its normal vector $\mathbf{n}$, the quantity on the right in (1) is independent of the shape of the contour. (The notation $[curl\,\mathbf{A}]_n$, is introduced at this stage only as a convenient abbreviation for the expression on the right.)
2. If $[curl\,\mathbf{A}]_n$ is indeed the component of a vector $[curl\,\mathbf{A}]$ in the $\mathbf{n}$ direction and $\mathbf{n}$ is a unit normal in the $\mathbf{n}$ direction, then

$$[curl\,\mathbf{A}]_n = [\mathrm{curl}\,\mathbf{A}]\cdot\mathbf{n} \tag{2}$$

where $[curl\,\mathbf{A}]$ is a vector defined at the point (x, y, z).

The proof of (1) follows from the fact that any closed contour integral can be built up from a superposition of contour integrals around a large number of rectangular contours C_i, as shown in Fig. A.2.3. All rectangles have sides $\Delta\xi$, $\Delta\eta$. If the entire contour containing the rectangles is small ($a \to 0$), then the contour integral around each rectangle differs from that for the contour C_o at the origin only by a term on the order of the linear dimension of the contour, $a^{1/2}$, times the area $\Delta\xi\Delta\eta$. This is true provided that the distance from the origin to any point on the contour does not exceed $a^{1/2}$ by an order of magnitude and that $\mathbf{A}$ is once differentiable in the neighborhood of the origin. We have

$$\frac{1}{\Delta\xi\Delta\eta}\oint_{C_i}\mathbf{A}\cdot d\mathbf{s} = \frac{1}{\Delta\xi\Delta\eta}\oint_{C_o}\mathbf{A}\cdot d\mathbf{s} + O(a^{1/2}) \tag{3}$$

Therefore,

$$\begin{aligned}\frac{1}{a}\oint_C \mathbf{A}\cdot d\mathbf{s} &= \sum_i \frac{1}{a}\oint_{C_i}\mathbf{A}\cdot d\mathbf{s} = \frac{\Delta\xi\Delta\eta}{a}\sum_i \frac{1}{\Delta\xi\Delta\eta}\oint_{C_i}\mathbf{A}\cdot d\mathbf{s}\\ &= N\frac{\Delta\xi\Delta\eta}{a}\left[\frac{1}{\Delta\xi\Delta\eta}\oint_{C_o}\mathbf{A}\cdot d\mathbf{s} + O(a^{1/2})\right]\end{aligned} \tag{4}$$

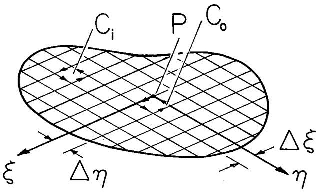

Fig. A.2.3 Separation of closed contour integral into large number of integrals over rectangular contours.

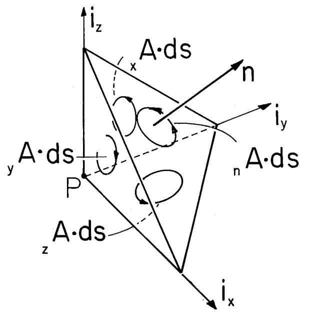

Fig. A.2.4 Arbitrary incremental contour integral having normal **n** analyzed into integration contours enclosing surface, having normals in the directions of the Cartesian coordinates.

where N is the number of rectangles into which the contour C has been subdivided. However, $N = a/(\Delta\xi\Delta\eta)$, and therefore we find

$$\lim_{a\to 0} \sum_i \frac{1}{a} \oint_{C_i} \mathbf{A}\cdot d\mathbf{s} = \lim_{a\to 0} \frac{1}{\Delta\xi\Delta\eta} \oint_{C_o} \mathbf{A}\cdot d\mathbf{s} \tag{5}$$

The expression on the left refers to the original contour, while the expression on the right refers to the rectangular contour at the origin. Since a contour of arbitrary shape can be constructed by a proper arrangement of rectangular contours, we have proven that the expression $\lim_{a\to 0} \int \mathbf{A}\cdot d\mathbf{s}/a$ is independent of the shape of the contour as long as (3) holds.

Turning to the proof that (1) defines the component of a vector, we recognize that the shape of the contour is arbitrary when evaluating $\int \mathbf{A}\cdot d\mathbf{s}/a$. We displace the plane in which the contour lies by a *differential* amount away from the point $P(x, y, z)$, as shown in Fig. A.2.4 which does not affect the value of $[curl\,\mathbf{A}]_n$ as defined in (1). The intersection of the plane with the three coordinate planes through P is a triangle. We pick the triangle for the contour C in (1).

It follows from Fig. A.2.4 that the contour integral around the triangular contour in the plane perpendicular to **n** can also be written as the sum of three integrals around the three triangular contours in the respective coordinate planes. Indeed, each of the added sections of line are traversed in one contour integration in the opposite direction, so that the integrals over the added sections of the line cancel upon summation and we have

$$\oint_n \mathbf{A}\cdot d\mathbf{s} = \oint_x \mathbf{A}\cdot d\mathbf{s} + \oint_y \mathbf{A}\cdot d\mathbf{s} + \oint_z \mathbf{A}\cdot d\mathbf{s} \tag{6}$$

where each contour integral is denoted by the subscript taken from the unit vector normal to the plane of the contour.

We further note that the areas a_x, a_y, a_z of the three triangles in the respective coordinate planes are the projections of the area a onto the corresponding coordinate plane.

$$a_x = a\mathbf{i_x} \cdot \mathbf{n} \tag{7}$$

$$a_y = a\mathbf{i_y} \cdot \mathbf{n} \tag{8}$$

$$a_z = a\mathbf{i_z} \cdot \mathbf{n} \tag{9}$$

Thus, by dividing (6) by a and making use of (7), (8), and (9), we have:

$$\begin{aligned}\frac{1}{a}\oint_n \mathbf{A} \cdot d\mathbf{s} =& \frac{1}{a_x}\oint_x \mathbf{A} \cdot d\mathbf{s}\mathbf{i_x} \cdot \mathbf{n} + \frac{1}{a_y}\oint_y \mathbf{A} \cdot d\mathbf{s}\mathbf{i_y} \cdot \mathbf{n} \\ &+ \frac{1}{a_z}\oint_z \mathbf{A} \cdot d\mathbf{s}\mathbf{i_z} \cdot \mathbf{n}\end{aligned} \tag{10}$$

Now, since the contours are already taken around differential area elements, the limit $a \to 0$ is already implied in (10). Thus, we have the quantities

$$[curl\,\mathbf{A}]_x = \lim_{a_x \to 0} \oint_x \mathbf{A} \cdot d\mathbf{s}/a_x \ldots \tag{11}$$

But (10) is the definition of the component in the $\mathbf{n}$ direction of a vector:

$$curl\,\mathbf{A} = [curl\,\mathbf{A}]_x\mathbf{i_x} + [curl\,\mathbf{A}]_y\mathbf{i_y} + [curl\,\mathbf{A}]_z\mathbf{i_z} \tag{12}$$

It is therefore legitimate to define at every point x, y, z in space a vector quantity, $curl\,\mathbf{A}$, whose x-, y-, and z-components are evaluated as the limiting expressions of (1).