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MATH S122
A Foundation In Applied Mathematics (Free Courseware)

香港公開大學
THE OPEN UNIVERSITY OF HONG KONG

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Chapter 1 Applied Maths - Defining Functions

1.1 About this module

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Welcome to this free courseware module 'Applied Maths - Defining Functions'!

This module is taken from the OUHK course *MATH S122 A Foundation in Applied Mathematics (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&lang=eng&TYPE=CI&CODE=M122&ncode=MATH%20S122&shopping=Y)*, a ten-credit, Foundation level course that is compulsory for a large number of Science and Technology programmes and an option in a number of other programmes, offered by the School of Science and Technology (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcSubWeb&l=C_ST&lid=191133000200&lang=eng) of the OUHK. MATH S122 focuses on the branch of mathematics called calculus and analytic geometry. Its intention is to broaden students' view of mathematics, equip them for advances in technology and prepare them for studying higher level courses.

MATH S122 is mainly presented in printed format and comprises 13 study units. Each unit contains study content, activities, examples, exercises, self-tests, assigned readings, etc for students' self-learning. This module (The materials for this module, taken from S122, have been specially adapted to make them more suitable for studying online, and multimedia elements have been added where appropriate. In addition to this topic on 'Functions', which is an extract from Unit 1 of the course, the original Unit 1 also includes the topic 'Conic sections', 'Parametrizations of plane curves' and 'Polar coordinates') retains most of these elements, so you can have a taste of what an OUHK course is like. Please note that no credits can be earned on completion of this module. If you would like to pursue it further, you are welcome to enrol in *MATH S122 A Foundation in Applied Mathematics (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&lang=eng&TYPE=CI&CODE=M122&ncode=MATH%20S122&shopping=Y)*.

This module will take you about **eight hours** to complete, including the time for completing the activities and exercises.

Good luck, and enjoy your study!

1.2 Introduction

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This module develops concepts and techniques for studying functions. You will learn about one of the foundations of applied mathematics, i.e. the algebraic and graphic methods for studying functions.

You'll be introduced to clear define and recognize functions, see how they constitute a special type of relation, and then move on to consider the simple, useful class of functions called linear functions.

Along the way you'll have plenty of chances to practise what you're learning in examples and exercises.

1.3 Introduction to functions

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This module prepares you to apply mathematical relations to solving practical problems. The first question is: what are relations? This can be a pure mathematical question that demands a lengthy discussion. For our purposes, however, all you need to know is that a relation is a mathematical statement that describes the relationship among different variables, so

$y = 2x - 3$, $r^2 + s^2 + t^2 = 1$ and $p - 2 > q + 4$

are all examples of relations.

1.3.1 Video activity 1

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You should now watch the following video introduction to functions. It introduces you to the basics of what functions are; shows you how to manually draw functions on a graph; considers the validity of functions; and provides an extensive set of examples.

Video (http://www.mathtutor.ac.uk/functions/inroductiontofunctions/video)

Source: This video clip is available at http://www.mathtutor.ac.uk/functions/inroductiontofunctions/video (http://www.mathtutor.ac.uk/functions/inroductiontofunctions/video).

1.4 Relations and functions

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A relation imposes restrictions on the values of its variables. For instance, when the values of all except one variable of a relation are specified, the possible value(s) of the remaining variable are defined. Note that there can be more than one possible value.

Example 1

1. For $y = 2x - 3$,

   $y = -1$ when $x = 1$, $y = 1$ when $x = 2$, ...

   that is, there is one possible y for a given x.

2. For $r^2 + s^2 + t^2 = 1$,

   $t^2 = 1$ or $t = \pm 1$ when $r = s = 0$, ...

   that is, there are two possible t's for a given pair of r and s.

3. For $p - 2 > q + 4$,

   $p > 5$ when $q = -1$, ...

   that is, there are infinitely many possible p's for a given q.

In general, a relation can involve two or more variables. This module, however, only considers relations of two variables. This section focuses on a special case of relations - functions. To begin, let's review some basic definitions of functions.

Reading

You should now work through the following interactive introduction to the domain and range of a function.

http://www.intmath.com/Functions-and-graphs/2a_Domain-and-range.php. (http://www.intmath.com/Functions-and-graphs/2a_Domain-and-range.php.%20http://www.intmath.com/functions-and-graphs/2a-domain-and-range.php)

Example 2

For each of the following equations, discuss whether y is a function of x.

1. $y = x^2$.
2. $y^2 = x$

Solution

1. y is a function of x since for any given x, y is uniquely defined.
2. y is not a function of x since for any positive x, y assumes two possible values.

Remark

1. By convention, the notation f(x) refers to a function of x. Writing f(x) for y assumes that y is a function, so we can write

   $f(x) = x^2$ for $y = x^2$

   but not

   $f(x)]^2 = x$ for $y^2 = x$

2. Although f is commonly used as the name of a function and x and y as variables, you can use other names for identification, including symbols. For example:

$$\theta\,(?) = \frac{1-?}{1+?}$$

$$Volume\,(radius) = \frac{4}{3}\pi\,(radius)^3$$

3. Don't use the same name for different objects (functions or variables) or different names for the same object in the same problem, as this can cause confusion.

You have learned what a function is and how it is defined algebraically. In general, for

f(x) = an expression of x,

the value of the function f at the point anything is just

f(anything) = the expression with x replaced by anything.

Example 3

Consider

$F(x) = 2x - 3$

The function F at the point anything is

F(anything) = 2(anything) − 3

1. If *anything* = 1, then $F(1) = 2(1) - 3$.
2. If *anything* = t, then $F(t) = 2(t) - 3$.
3. If *anything* = x^2, then $F(x^2) = 2(x^2) - 3$.
4. If *anything* = $2x - 3$, then $F(2x - 3) = 2(2x - 3) - 3$

The next example describes how the domain and range of a function are defined.

Example 4

1. $[-2,3]$ represents the closed interval $-2 \leq x \leq 3$.
2. $(1,\infty)$ represents the infinite interval $x > 1$.
3. $(-16,-9)$ represents the open interval $-16 < x < -9$.
4. $[-7,45)$ represents the half-open interval $-7 \leq x < 45$.

Note that some books use]a,b[to denote an open interval.

Note that some books use]a,b[to denote an open interval.

T&F uses the symbols '$\cup$' and '$\cap$' to combine two intervals. If you are not familiar with the symbols, just consider that '$\cup$' gives the collection of values in at least one of the two intervals, and '$\cap$' gives the collection of values in both intervals.

Example 5

1. $(-\infty,1) \cup [2,\infty)$ is equivalent to $x < 1$ or $x \geq 2$.
2. $(-\infty,2) \cap [1,3)$ is equivalent to $1 \leq x < 2$.

Once the domain is defined, the range is implied by the function. Finding the range can be difficult and is sometimes omitted. For the natural domain of a function involving more than one term, we discount the impossible values of individual terms from all real numbers, rather than combine their possible values. To illustrate this point, let's consider the following example.

Example 6

Find the natural domain of the function

$$f(x) = \frac{1}{x} + \frac{1}{x-1}$$

Solution

Denominators cannot be zero, so we have the following for individual terms:

Term	Impossible values	Possible values
$\frac{1}{x}$	0	$(-\infty,0) \cup (0,\infty)$
$\frac{1}{x-1}$	1	$(-\infty,1) \cup (1,\infty)$

By removing the impossible values of individual terms, the natural domain consists of all real numbers except 0 and 1, that is,

$(-\infty,0) \cup (0,1) \cup (1,\infty)$

If we combine the possible values of the two terms, we get all real numbers and that is incorrect.

What you've learned so far in this module has concentrated on algebraic representations of relations. Algebraic representations are important for analysis. However, for the purpose of general understanding or communication, graphical representations can be more efficient.

There are two basic approaches to graph a relation: plotting and sketching. Plotting does not require any pre-knowledge of the graph, you just plot the relation point-by-point over the region of concern. Sketching, on the other hand, uses only a few points, which you then join together using your understanding of the graph. Generally, plotting can be very accurate, but sketching reveals only the general behaviour of a graph and is less time-consuming.

We did not explain how the basic points should be chosen and joined together. Unless we have some idea about what the graph is like, it may not be easy to answer these questions.

Once you understand the basic techniques, graphing is just a mechanical and time-consuming task. This is a typical situation in which algebraic software can really help us save time. The following video activity briefly introduces you to plotting functions using a specialized mathematics software application called Scientific Notebook.

1.4.1 Exercise 1

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For each of the following equations, determine whether y is a function of x.

	Is it a function?	
	Yes	No
$y = x^3$		
$y^3 = x$		

1.4.1.1 Exercise 1 feedback

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	Is it a function?	
	Yes	No
$y = x^3$	o	
$y^3 = x$	o	

1.4.2 Exercise 2

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Consider

$$f(x) = 6x^2 - \frac{5}{x} + 1$$

Find

a. $f(\frac{1}{2})$

b. $f(-0.3)$

c. $f(a)$

d. $f(x + 1)$

e. *the difference* $f(x + 1) - f(x)$

1.4.2.1 Exercise 2 feedback

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a. $$f(\frac{1}{2}) = 6(\frac{1}{6})^2 - \frac{5}{1/2} + 1 = -\frac{15}{2}$$

b. $$f(-0.3) = 6(-0.3)^2 - \frac{5}{(-0.3)} + 1 = 18.207$$

c. $$f(a) = 6a^2 - \frac{5}{a} + 1$$

d. $$f(x+1) = 6(x+1)^2 - \frac{5}{(x+1)} + 1$$

e. $$f(x+1) - f(x) = 12x + 6 + \frac{5}{(x+1)x}$$

1.4.3 Exercise 3

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Complete the following table.

Notation	Interval	Type
(−1,1)		
(−∞,63)		
	2.5 < x < 4.9	Open
	0 £ x £ 100	
(−∞,∞)		

1.4.3.1 Exercise 3 feedback

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Notation	Interval	Type
(−1,1)	−1 < x £ 1	half-open
(−∞,63)	x < 63	infinite
(2.5,4.9)	2.5 < x < 4.9	Open
[0,100]	0 £ x £ 100	closed
(−∞,∞)	∞ < x < ∞	infinite

1.4.4 Exercise 4

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Find the natural domain of each of the following functions:

a.

$$f(x) = 1$$

b.

$$g(x) = \sqrt{(2x)} - \frac{1}{x}$$

c.

$$h(x) = \frac{x-1}{x^2-1}$$

1.4.4.1 Exercise 4 feedback

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a. Domain: (-∞,∞).

b. Domain: x > 0.

c. Domain: all real numbers except ±1.

1.4.5 Video Activity 2

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You should now watch the following two brief video clips. The point here is not for you to learn how to use Scientific Notebook in detail, but simply to see how computer software can make plotting functions much easier and more accurate.

Video 1: Scientific notebook

Click this link to watch the video:
http://www.opentextbooks.org.hk/system/files/resource/10/10561/10575/media/sci_notebook1.mp4

Video 2: Computer software

Click this link to watch the video:
http://www.opentextbooks.org.hk/system/files/resource/10/10561/10575/media/sci_notebook2.mp4

1.5 Linear functions

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Linear functions are any functions that can be expressed in the form of

$$y = mx + b$$

where m and b are constants.

Example 7

1. $2x - 3y = 1$ is a linear function.
2. $y = 2$ is a special linear function called a constant function.
3. $x = -1$ is linear but not a function.
4. $y = x^2 - 3$ is a function but not linear.

To sketch the graph of a linear function, you just need to locate any two points on the graph and then draw a straight line passing through them. This, of course, requires an understanding that the graph of a linear function is a straight line.

A line is defined when we have sufficient information to derive its equation. The next reading discusses various ways to define a line. Note that in the discussion (x,y) represents a general point, and (x1,y1) represents a specific point on the line.

Reading

Read the following sections from an introduction to linear functions from MIT.

http://dedekind.mit.edu/~djk/calculus_beginners/chapter03/section02.html (http://dedekind.mit.edu/~djk/calculus_beginners/chapter03/section02.html%20http://dedekind.mit.edu/~djk/calculus_beginners/chapter03/section02.html)

http://dedekind.mit.edu/~djk/calculus_beginners/chapter03/section03.html (http://dedekind.mit.edu/~djk/calculus_beginners/chapter03/section03.html)

Be sure to try out the applets built into these pages that allow you to experiment with graphing linear functions.

Note that the slope of a vertical line is undefined - not infinity. Slope is an important measure, but is not sufficient to define a line by itself. There are lines of which the slopes are the same.

Example 8

Given that:

$$L_1 : y = x$$
$$L_2 : y = 2x - 1$$
$$L_3 : y = \frac{1}{2}x - 16$$
$$L_4 : y = -x + 3$$

Determine which of these lines are:

1. parallel; or
2. perpendicular.

Solution

Comparing the above equations with the point-slope equation we obtain

Line	L_1	L_2	L_3	L_4
Slope	1	2	$\frac{1}{2}$	-1

1. Since the slopes are all different, none of the lines are parallel.
2. Since the product of Slope(L1) and Slope(L4) is −1, L1 and L4are perpendicular.

(Note that Slope(L_2) × Slope(L_3) = 1, not −1, and therefore L_2 and L_3 are not perpendicular.)

Definition

The equation

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

is the two-point equation of the line that passes through the points (x_1,y_1) and (x_2,y_2).

Example 9

Find the equation of the line with x-intercept 2 and y-intercept -1. (Recall that the x-intercept is the x-coordinate where the graph and the x-axis intersect. The y-intercept is the y-coordinate where the graph and the y-axis intersect.)

Solution

The two intercepts imply that (2,0) and (0,-1) are two points on the line. Applying the two-point equation we have

$$\frac{y-0}{x-2}=\frac{-1-0}{0-2}$$

Simplifying the equation gives

$$y=\frac{1}{2}x-1$$

1.5.1 Video activity 3

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You should now watch the following video introduction to linear functions. It discusses the special characteristics that define linear functions, then introduces linear functions that are not written in the standard form.

Video (http://www.mathtutor.ac.uk/functions/linearfunctions/video)

Source: This video clip is available at http://www.mathtutor.ac.uk/functions/linearfunctions/video (http://www.mathtutor.ac.uk/functions/linearfunctions/video).

1.5.2 Exercise 5

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Let L be a straight line passing through (2,3) and (-1,5). Find

1. the slope of *L*
2. the angle of inclination of *L*
3. the slope of any line parallel to *L*
4. the slope of any line perpendicular to *L*.

1.5.2.1 Exercise 5 Feedback

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1. $$\text{Slope of } L = \frac{5-3}{-1-2} = -\frac{2}{3}$$
2. $$\text{Angle of inclination} : \tan\theta = \frac{2}{3} \Rightarrow \theta = 146.31^\circ$$
3. $$\text{Slope of parallel line} = -\frac{2}{3}$$
4. $$\text{Slope of perpendicular line} = -\frac{3}{2}$$

1.5.3 Exercise 6

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Given that

$$L_1 : y = 2x + 3$$
$$L_2 : y + 2x = 4$$
$$L_3 : 2y + x - 6 = 0$$
$$L_4 : y - 1 = x + 2$$

Determine which of these lines are

a. parallel
b. perpendicular.

1.5.3.1 Exercise 6 feedback

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$$L_1 : y = 2x + 3$$
$$L_2 : y = -2x + 4$$
$$L_3 : y = -\frac{1}{2} + 3$$
$$L_4 : y = x + 3$$

Line	L_1	L_2	L_3	L_4
Slope	2	-2	$-\frac{1}{2}$	1

1. Since the slopes are all different, none of the lines are parallel.
2. Since the product of Slope(L_1) and Slope(L_3) is -1, $L_1$and L_3 are perpendicular.

1.5.4 Exercise 7

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Find the equation of the line that passes through the point (-4,3) and has *y*-intercept 2.

1.5.4.1 Exercise 7 feedback

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Applying the two-point equation we have

$$\frac{y-2}{x-0} = \frac{3-2}{-4-0}$$

Simplifying the equation gives

$$y = -\frac{1}{4}x + 2$$

1.6 Conclusion

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This module has taken you through some of the basics of relations, functions and linear functions. You should now understand the algebraic and graphic methods for studying functions.

If you would like to learn more on this subject, you are welcome to enrol in *MATH S122 A Foundation in Applied Mathematics (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&lang=eng&ncode=MATH%20S122&shopping=Y&TYPE=CI&CODE=M122)* offered by the School of Science and Technology (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcSubWeb&l=C_ST&lid=191133000200&lang=eng) of the OUHK.