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CK-12 Trigonometry

Second Edition

CK-12 Trigonometry - Second Edition

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CHAPTER 1 Right Triangles and an Introduction to Trigonometry

Chapter Outline

1.1 THE PYTHAGOREAN THEOREM
1.2 SPECIAL RIGHT TRIANGLES
1.3 BASIC TRIGONOMETRIC FUNCTIONS
1.4 SOLVING RIGHT TRIANGLES
1.5 MEASURING ROTATION
1.6 APPLYING TRIG FUNCTIONS TO ANGLES OF ROTATION
1.7 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE
1.8 RELATING TRIGONOMETRIC FUNCTIONS

1.1 The Pythagorean Theorem

Introduction

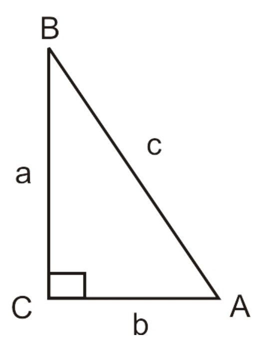
Right triangles play an integral part in the study of trigonometry. It is from right triangles that the basic definitions of the trigonometric functions are formed. In this chapter we will explore right triangles and their properties. Through this, we will introduce the six basic trig functions and the unit circle.

Learning Objectives

- Recognize and use the Pythagorean Theorem.
- Recognize basic Pythagorean Triples.
- Use the Distance Formula.

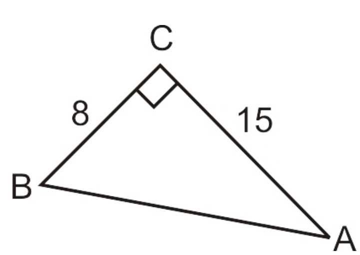
The Pythagorean Theorem

From Geometry, recall that the Pythagorean Theorem is $a^2+b^2=c^2$ where a and b are the legs of a right triangle and c is the hypotenuse. Also, the side opposite the angle is lower case and the angle is upper case. For example, angle A is opposite side a.



The Pythagorean Theorem is used to solve for the sides of a right triangle.

Example 1: Use the Pythagorean Theorem to find the missing side.

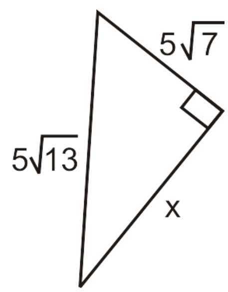


Solution: $a = 8,\ b = 15$, we need to find the hypotenuse.

$$\begin{aligned}8^2 + 15^2 &= c^2\\64 + 225 &= c^2\\289 &= c^2\\17 &= c\end{aligned}$$

Notice, we do not include -17 as a solution because a negative number cannot be a side of a triangle.

Example 2: Use the Pythagorean Theorem to find the missing side.



Solution: Use the Pythagorean Theorem to find the missing leg.

$$\begin{aligned}\left(5\sqrt{7}\right)^2 + x^2 &= \left(5\sqrt{13}\right)^2\\25 \cdot 7 + x^2 &= 25 \cdot 13\\175 + x^2 &= 325\\x^2 &= 150\\x &= 5\sqrt{6}\end{aligned}$$

Pythagorean Triples

Pythagorean Triples are sets of whole numbers for which the Pythagorean Theorem holds true. The most well-known triple is 3, 4, 5. This means, that 3 and 4 are the lengths of the legs and 5 is the hypotenuse. *The largest length is always the hypotenuse*. If we were to multiply any triple by a constant, this new triple would still represent sides of a right triangle. Therefore, 6, 8, 10 and 15, 20, 25, among countless others, would represent sides of a right triangle.

Example 3: Determine if the following lengths are Pythagorean Triples.

a. 7, 24, 25

b. 9, 40, 41

c. 11, 56, 57

Solution: Plug each set of numbers into the Pythagorean Theorem.

a.

$$\begin{aligned}7^2+24^2 &\stackrel{?}{=} 25^2\\ 49+576 &= 625\\ 625 &= 625\end{aligned}$$

Yes, 7, 24, 25 is a Pythagorean Triple and sides of a right triangle.

b.

$$\begin{aligned}9^2+40^2 &\stackrel{?}{=} 41^2\\ 81+1600 &= 1681\\ 1681 &= 1681\end{aligned}$$

Yes, 9, 40, 41 is a Pythagorean Triple and sides of a right triangle.

c.

$$\begin{aligned}11^2+56^2 &\stackrel{?}{=} 57^2\\ 121+3136 &= 3249\\ 3257 &\neq 3249\end{aligned}$$

No, 11, 56, 57 do not represent the sides of a right triangle.

Converse of the Pythagorean Theorem

Using the technique from Example 3, we can determine if sets of numbers are acute, right or obtuse triangles. Examples 3a and 3b were both right triangles because the two sides equaled each other and made the Pythagorean Theorem true. However in Example 3c, the two sides were not equal. Because $3257 > 3249$, we can say that 11, 56, and 57 are the sides of an acute triangle. To help you visualize this, think of an equilateral triangle with sides of length 5. We know that this is an acute triangle. If you plug in 5 for each number in the Pythagorean Theorem we get $5^2+5^2=5^2$ and $50>25$. Therefore, if $a^2+b^2>c^2$, then lengths a, b, and c make up an acute triangle. Conversely, if $a^2+b^2<c^2$, then lengths a, b, and c make up the sides of an obtuse triangle. *It is important to note that the length "c'' is always the longest.*

Example 4: Determine if the following lengths make an acute, right or obtuse triangle.

a. 5, 6, 7

b. 5, 10, 14

c. 12, 35, 37

Solution: Plug in each set of lengths into the Pythagorean Theorem.

a.

$$\begin{aligned}5^2+6^2 &\ ?\ 7^2\\ 25+36 &\ ?\ 49\\ 61 &> 49\end{aligned}$$

Because $61 > 49$, this is an acute triangle.

b.

$$5^2 + 10^2 \ ? \ 14^2$$
$$25 + 100 \ ? \ 196$$
$$125 < 196$$

Because $125 < 196$, this is an obtuse triangle.

c.

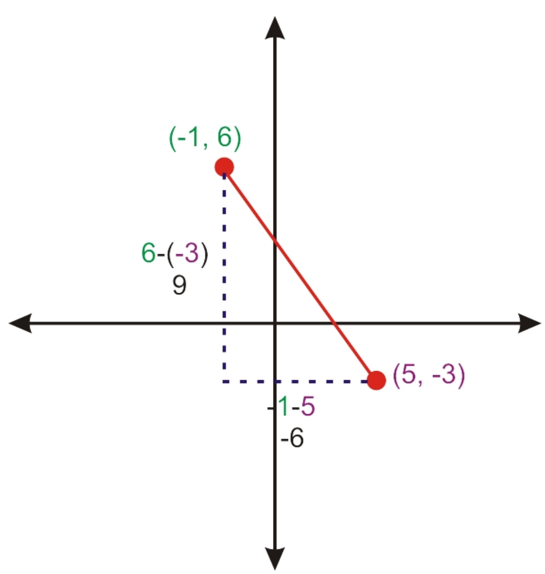
$$12^2 + 35^2 \ ? \ 37^2$$
$$144 + 1225 \ ? \ 1369$$
$$1369 = 1369$$

Because the two sides are equal, this is a right triangle.

NOTE: All of the lengths in Example 4 represent the lengths of the sides of a triangle. Recall the Triangle Inequality Theorem from geometry which states: The length of a side in a triangle is less than the sum of the other two sides. For example, 4, 7 and 13 cannot be the sides of a triangle because $4+7$ is not greater than 13.

The Distance Formula

An application of the Pythagorean Theorem is to find the distance between two points. Consider the points (-1, 6) and (5, -3). If we plot them on a grid, they make a diagonal line. Draw a vertical line down from (-1, 6) and a horizontal line to the left of (5, -3) to make a right triangle.

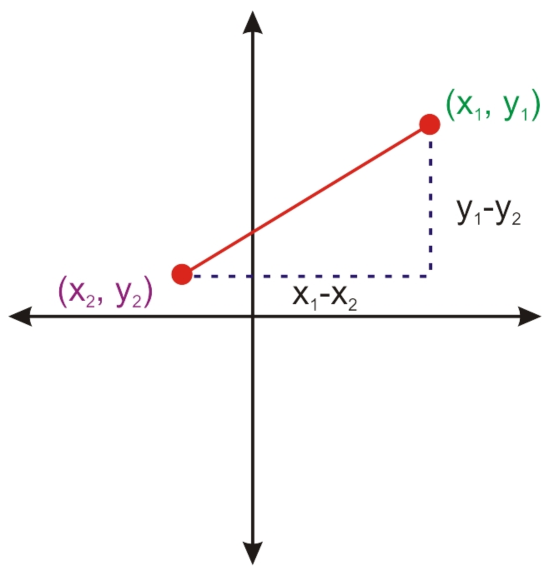


Now we can find the distance between these two points by using the vertical and horizontal distances that we determined from the graph.

$$\begin{aligned}9^2+(-6)^2&=d^2\\81+36&=d^2\\117&=d^2\\\sqrt{117}&=d\\3\sqrt{13}&=d\end{aligned}$$

Notice, that the $x-$values were subtracted from each other to find the horizontal distance and the $y-$values were subtracted from each other to find the vertical distance. If this process is generalized for two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, the Distance Formula is derived.

$$(x_1-x_2)^2+(y_1-y_2)^2=d^2$$



This is the Pythagorean Theorem with the vertical and horizontal differences between $(x_1,\ y_1)$ and $(x_2,\ y_2)$. Taking the square root of both sides will solve the right hand side for d, the distance.

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=d$$

This is the Distance Formula. The following example shows how to apply the distance formula.

Example 5: Find the difference between the two points.

a. (4, 2) and (-9, 5)

b. (-10, 3) and (0, -15)

Solution: Plug each pair of points into the distance formula.

a.

$$\begin{aligned} d &= \sqrt{(4-(-9))^2+(2-5)^2} \\ &= \sqrt{13^2+(-3)^2} \\ &= \sqrt{169+9} \\ &= \sqrt{178} \end{aligned}$$

b.

$$\begin{aligned} d &= \sqrt{(-10-0)^2+(3-(-15))^2} \\ &= \sqrt{(-10)^2+(18)^2} \\ &= \sqrt{100+324} \\ &= \sqrt{424} = 2\sqrt{106} \end{aligned}$$

Points to Consider

- Does the Pythagorean Theorem apply to all real numbers?
- Can a Pythagorean Triple have irrational numbers in the set?
- What is the difference between the Distance Formula and the Pythagorean Theorem?

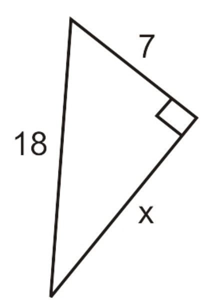
Review Questions

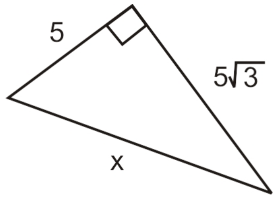
Determine if the lengths below represent the sides of a right triangle. If not, state if the triangle is acute or obtuse.

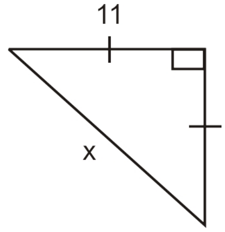
1. 6, 9, 13
2. 9, 10, 11
3. 16, 30, 34
4. 20, 23, 40
5. 11, 16, 29
6. $2\sqrt{6},\ 6\sqrt{3},\ 2\sqrt{33}$

Find the missing side of each right triangle below. Leave the answer in simplest radical form.

7.



8. 

9. 

10. The general formula for a Pythagorean Triple is $n^2 - m^2, 2nm, n^2 + m^2$ where n and m are natural numbers. Use the Pythagorean Theorem to prove this is true.
11. Find the distance between the pair of points.
    a. (5, -6) and (18, 3)
    b. $\left(\sqrt{3},\ -\sqrt{2}\right)$ and $\left(-2\sqrt{3},\ 5\sqrt{2}\right)$

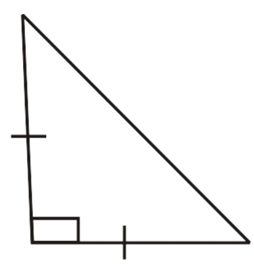
1.2 Special Right Triangles

Learning Objectives

- Recognize special right triangles.
- Use the special right triangle ratios to solve special right triangles.

Special Right Triangle #1: Isosceles Right Triangle

An isosceles right triangle is an isosceles triangle and a right triangle. This means that it has two congruent sides and one right angle. Therefore, the two congruent sides must be the legs.



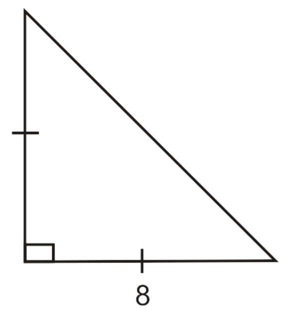
Because the two legs are congruent, we will call them both a and the hypotenuse c. Plugging both letters into the Pythagorean Theorem, we get:

$$\begin{aligned} a^2 + a^2 &= c^2 \\ 2a^2 &= c^2 \\ \sqrt{2a^2} &= \sqrt{c^2} \\ a\sqrt{2} &= c \end{aligned}$$

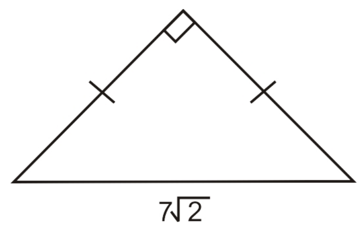
From this we can conclude that the hypotenuse length is the length of a leg multiplied by $\sqrt{2}$. Therefore, we only need one of the three lengths to determine the other two lengths of the sides of an isosceles right triangle. The ratio is usually written $x : x : x\sqrt{2}$, where x is the length of the legs and $x\sqrt{2}$ is the length of the hypotenuse.

Example 1: Find the lengths of the other two sides of the isosceles right triangles below.

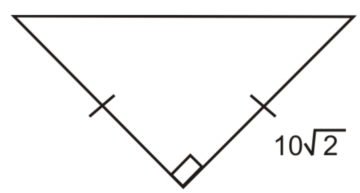
a.



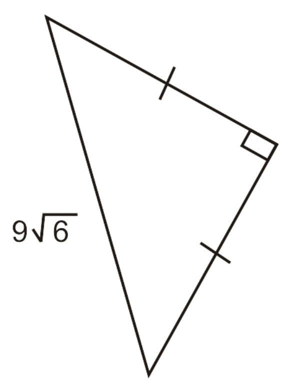
b.



c.



d.



Solution:

a. If a leg has length 8, by the ratio, the other leg is 8 and the hypotenuse is $8\sqrt{2}$.

b. If the hypotenuse has length $7\sqrt{2}$, then both legs are 7.

c. Because the leg is $10\sqrt{2}$, then so is the other leg. The hypotenuse will be $10\sqrt{2}$ multiplied by an additional $\sqrt{2}$.

$$10\sqrt{2}\cdot\sqrt{2} = 10\cdot 2 = 20$$

d. In this problem set $x\sqrt{2} = 9\sqrt{6}$ because $x\sqrt{2}$ is the hypotenuse portion of the ratio.

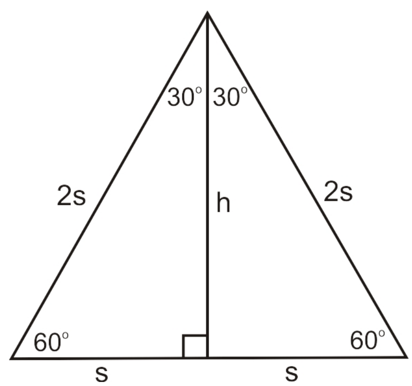
$$x\sqrt{2} = 9\sqrt{6}$$
$$x = \frac{9\sqrt{6}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{9\sqrt{12}}{2} = \frac{18\sqrt{3}}{2} = 9\sqrt{3}$$

So, the length of each leg is $9\sqrt{3}$.

What are the angle measures in an isosceles right triangle? Recall that the sum of the angles in a triangle is 180° and there is one 90° angle. Therefore, the other two angles add up to 90°. Because this is an isosceles triangle, these two angles are equal and 45° each. Sometimes an isosceles right triangle is also referred to as a $45-45-90$ triangle.

Special Right Triangle #2: 30-60-90 Triangle

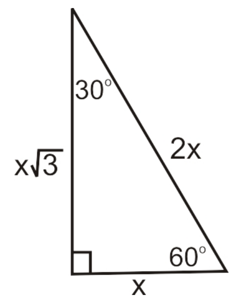
$30-60-90$ refers to each of the angles in this special right triangle. To understand the ratios of the sides, start with an equilateral triangle with an altitude drawn from one vertex.



Recall from geometry, that an altitude, h, cuts the opposite side directly in half. So, we know that one side, the hypotenuse, is $2s$ and the shortest leg is s. Also, recall that the altitude is a perpendicular and angle bisector, which is why the angle at the top is split in half. To find the length of the longer leg, use the Pythagorean Theorem:

$$s^2 + h^2 = (2s)^2$$
$$s^2 + h^2 = 4s^2$$
$$h^2 = 3s^2$$
$$h = s\sqrt{3}$$

From this we can conclude that the length of the longer leg is the length of the short leg multiplied by $\sqrt{3}$ or $s\sqrt{3}$. Just like the isosceles right triangle, we now only need one side in order to determine the other two in a $30-60-90$ triangle. The ratio of the three sides is written $x : x\sqrt{3} : 2x$, where x is the shortest leg, $x\sqrt{3}$ is the longer leg and $2x$ is the hypotenuse.

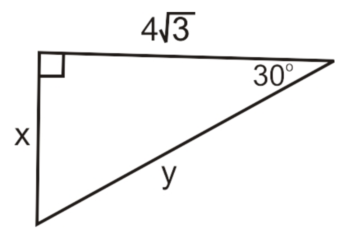


Notice, that the shortest side is *always* opposite the smallest angle and the longest side is *always* opposite 90°.

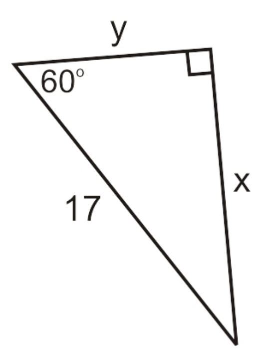
If you look back at the Review Questions from the last section we now recognize #8 as a $30-60-90$ triangle.

Example 2: Find the lengths of the two missing sides in the $30-60-90$ triangles.

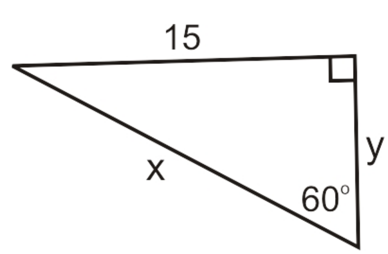
a.



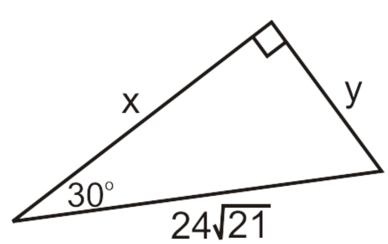
b.



c.



d.



Solution: Determine which side in the $30-60-90$ ratio is given and solve for the other two.

a. $4\sqrt{3}$ is the longer leg because it is opposite the 60°. So, in the $x : x\sqrt{3} : 2x$ ratio, $4\sqrt{3} = x\sqrt{3}$, therefore $x = 4$ and $2x = 8$. The short leg is 4 and the hypotenuse is 8.

b. 17 is the hypotenuse because it is opposite the right angle. In the $x : x\sqrt{3} : 2x$ ratio, $17 = 2x$ and so the short leg is $\frac{17}{2}$ and the long leg is $\frac{17\sqrt{3}}{2}$.

c. 15 is the long leg because it is opposite the 60°. Even though 15 does not have a radical after it, we can still set it equal to $x\sqrt{3}$.

$$x\sqrt{3} = 15$$
$$x = \frac{15}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3} \quad \text{So, the short leg is } 5\sqrt{3}.$$

Multiplying $5\sqrt{3}$ by 2, we get the hypotenuse length, which is $10\sqrt{3}$.

d. $24\sqrt{21}$ is the length of the hypotenuse because it is opposite the right angle. Set it equal to $2x$ and solve for x to get the length of the short leg.

$$2x = 24\sqrt{21}$$
$$x = 12\sqrt{21}$$

To find the length of the longer leg, we need to multiply $12\sqrt{21}$ by $\sqrt{3}$.

$$12\sqrt{21} \cdot \sqrt{3} = 12\sqrt{3 \cdot 3 \cdot 7} = 36\sqrt{7}$$

The length of the longer leg is $36\sqrt{7}$.

Be careful when doing these problems. You can always check your answers by finding the decimal approximations of each side. For example, in 2d, short leg $= 12\sqrt{21} \approx 54.99$, long leg $= 36\sqrt{7} \approx 95.25$ and the hypotenuse $= 24\sqrt{21} \approx 109.98$. This is an easy way to double-check your work and verify that the hypotenuse is the longest side.

Using Special Right Triangle Ratios

Special right triangles are the basis of trigonometry. The angles 30°, 45°, 60° and their multiples have special properties and significance in the unit circle (sections 1.5 and 1.6). Students are usually required to memorize these two ratios because of their importance.

First, let's compare the two ratios, so that we can better distinguish the difference between the two. For a $45-45-90$ triangle the ratio is $x : x : x\sqrt{2}$ and for a $30-60-90$ triangle the ratio is $x : x\sqrt{3} : 2x$. An easy way to tell the difference between these two ratios is the isosceles right triangle has two congruent sides, so its ratio has the $\sqrt{2}$, whereas the $30-60-90$ angles are all divisible by 3, so that ratio includes the $\sqrt{3}$. Also, if you are ever in doubt or forget the ratios, you can always use the Pythagorean Theorem. The ratios are considered a short cut.

Example 3: Determine if the sets of lengths below represent special right triangles. If so, which one?

a. $8\sqrt{3} : 24 : 16\sqrt{3}$

b. $\sqrt{5} : \sqrt{5} : \sqrt{10}$

c. $6\sqrt{7} : 6\sqrt{21} : 12$

Solution:

a. Yes, this is a $30-60-90$ triangle. If the short leg is $x = 8\sqrt{3}$, then the long leg is $8\sqrt{3} \cdot \sqrt{3} = 8 \cdot 3 = 24$ and the hypotenuse is $2 \cdot 8\sqrt{3} = 16\sqrt{3}$.

b. Yes, this is a $45-45-90$ triangle. The two legs are equal and $\sqrt{5} \cdot \sqrt{2} = \sqrt{10}$, which would be the length of the hypotenuse.

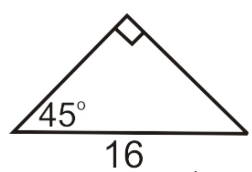
c. No, this is not a special right triangle, nor a right triangle. The hypotenuse should be $12\sqrt{7}$ in order to be a $30-60-90$ triangle.

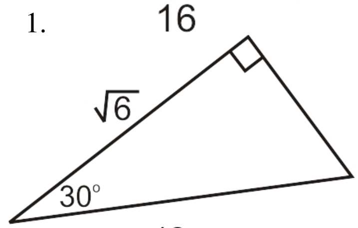
Points to Consider

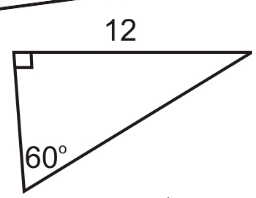
- What is the difference between Pythagorean triples and special right triangle ratios?
- Why are these two ratios considered "special"?

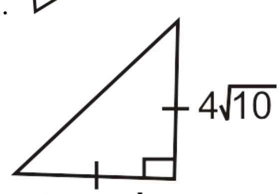
Review Questions

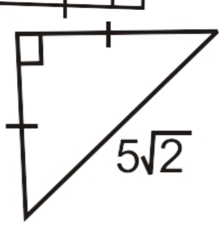
Solve each triangle using the special right triangle ratios.

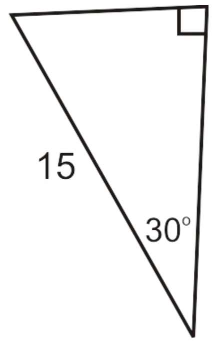
1. 

2. 

3. 

4. 

5. 

6. 

7. A square window has a diagonal of 6 ft. To the nearest hundredth, what is the height of the window?
8. Pablo has a rectangular yard with dimensions 10 ft by 20 ft. He is decorating the yard for a party and wants to hang lights along both diagonals of his yard. How many feet of lights does he need? Round your answer to the nearest foot.
9. Can $2 : 2 : 2\sqrt{3}$ be the sides of a right triangle? If so, is it a special right triangle?
10. Can $\sqrt{5} : \sqrt{15} : 2\sqrt{5}$ be the sides of a right triangle? If so, is it a special right triangle?

1.3 Basic Trigonometric Functions

Learning Objectives

- Find the values of the six trigonometric functions for angles in right triangles.

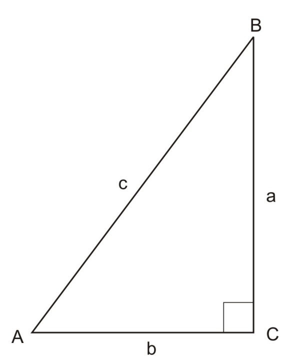
Introduction

Consider a situation in which you are building a ramp for wheelchair access to a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

Solving this kind of problem requires trigonometry. The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define six trigonometric functions. For each of these functions, the elements of the domain are angles. We will define these functions in two ways: first, using right triangles, and second, using angles of rotation. Once we have defined these functions, we will be able to solve problems like the one above.

The Sine, Cosine, and Tangent Functions

The first three trigonometric functions we will work with are the sine, cosine, and tangent functions. As noted above, the elements of the domains of these functions are angles. We can define these functions in terms of a right triangle: The elements of the range of the functions are particular ratios of sides of triangles.



We define the sine function as follows: For an acute angle x in a right triangle, the $\sin x$ is equal to the ratio of the side opposite of the angle over the hypotenuse of the triangle. For example, using this triangle, we have: $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Since all right triangles with the same acute angles are similar, this function will produce the same ratio, no matter which triangle is used. Thus, it is a well-defined function.

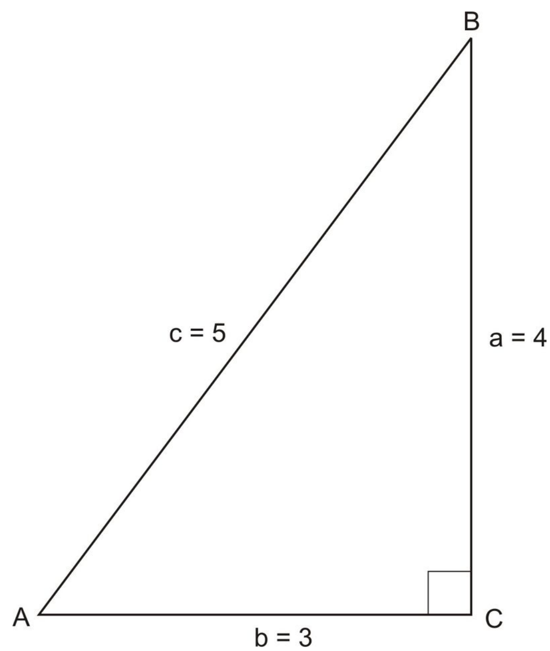
Similarly, the cosine of an angle is defined as the ratio of the side adjacent (next to) the angle over the hypotenuse of the triangle. Using this triangle, we have: $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Finally, the tangent of an angle is defined as the ratio of the side opposite the angle to the side adjacent to the angle. In the triangle above, we have: $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these functions. First, keep in mind that the abbreviations $sinx, cosx$, and $tanx$ are just like $f(x)$. They simply stand for specific kinds of functions. Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write $sinx$ it is still pronounced *sine*, with a long "i." When we write $cosx$, we still say co-sine. And when we write $tanx$, we still say tangent.

We can use these definitions to find the sine, cosine, and tangent values for angles in a right triangle.

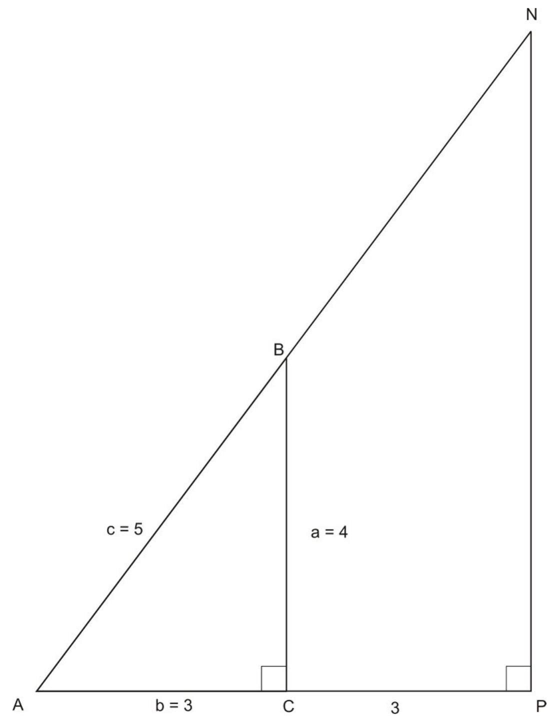
Example 1: Find the sine, cosine, and tangent of $\angle A$:



Solution:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{3}$$

One of the reasons that these functions will help us solve problems is that these ratios will always be the same, as long as the angles are the same. Consider for example, a triangle similar to triangle ABC.



If CP has length 3, then side AP of triangle NAP is 6. Because NAP is similar to ABC, side NP has length 8. This means the hypotenuse AN has length 10. (This can be shown either by using Pythagorean Triples or the Pythagorean Theorem.)

If we use triangle NAP to find the sine, cosine, and tangent of angle A, we get:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{8}{6} = \frac{4}{3}$$

Also notice that the tangent function is the same as the slope of the hypotenuse. $\tan A = \frac{4}{3}$, which is the same as $\frac{rise}{run}$ or $\frac{\text{change in } y}{\text{change in } x}$. The $\tan B$ does not equal the slope because it is the reciprocal of $tanA$.

Example 2: Find $\sin B$ using triangle ABC and triangle NAP.

Solution:

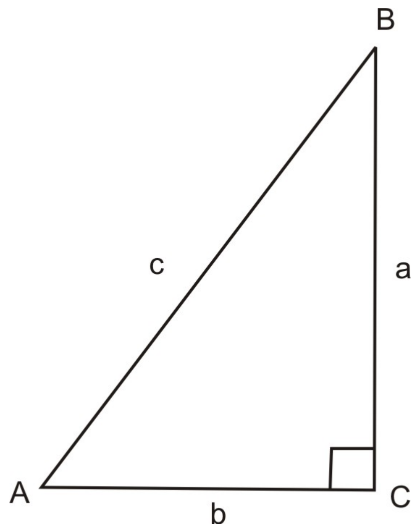
Using triangle ABC : $\sin B = \frac{3}{5}$

Using triangle NAP : $\sin B = \frac{6}{10} = \frac{3}{5}$

An easy way to remember the ratios of the sine, cosine, and tangent functions is SOH-CAH-TOA. $\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}}, \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$.

Secant, Cosecant, and Cotangent Functions

We can define three more functions also based on a right triangle. They are the reciprocals of sine, cosine and tangent.

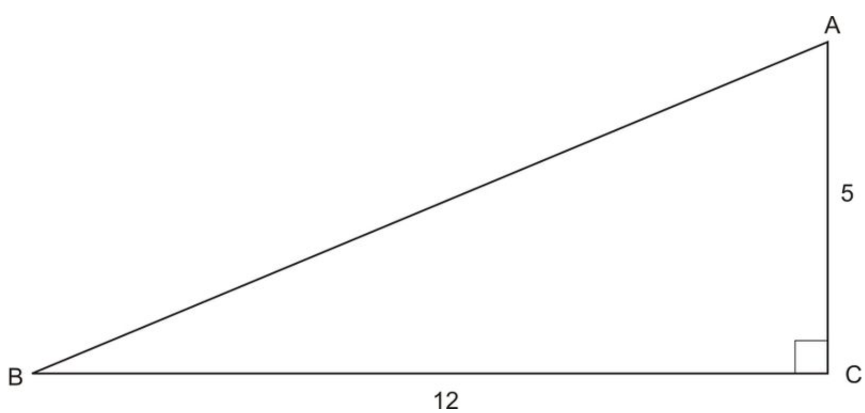


If $\sin A = \frac{a}{c}$, then the definition of cosecant, or csc, is $\csc A = \frac{c}{a}$.

If $\cos A = \frac{b}{c}$, then the definition of secant, or sec, is $\sec A = \frac{c}{b}$.

If $\tan A = \frac{a}{b}$, then the definition of cotangent, or cot, is $\cot A = \frac{b}{a}$.

Example 3: Find the secant, cosecant, and cotangent of angle B.



Solution:

First, we must find the length of the hypotenuse. We can do this using the Pythagorean Theorem:

$$\begin{aligned} 5^2 + 12^2 &= H^2 \\ 25 + 144 &= H^2 \\ 169 &= H^2 \\ H &= 13 \end{aligned}$$

Now we can find the secant, cosecant, and cotangent of angle B:

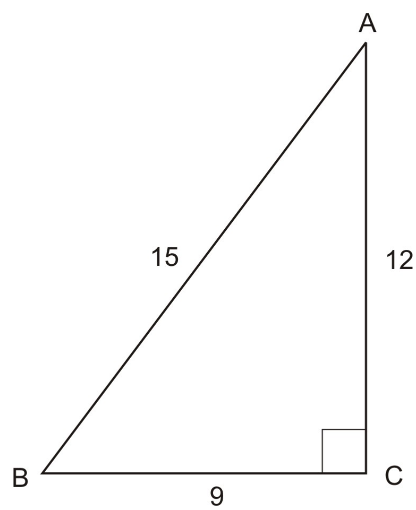
$$\sec B = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{13}{12}$$
$$\csc B = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{13}{5}$$
$$\cot B = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{12}{5}$$

Points to Consider

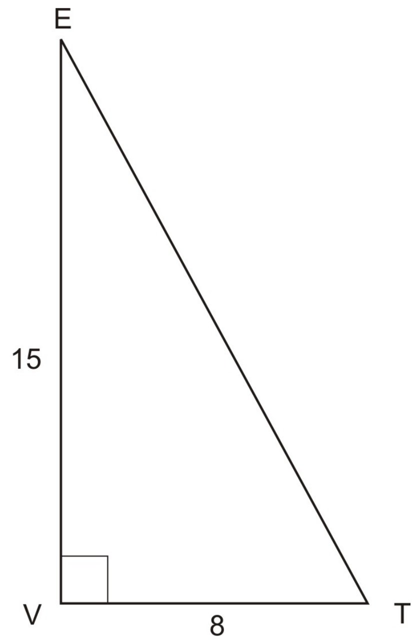
- Do you notice any similarities between the sine of one angle and the cosine of the other, in the same triangle?

Review Questions

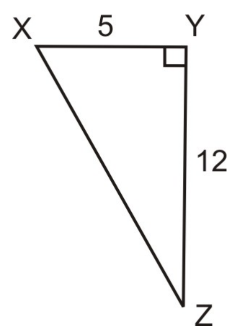
1. Find the values of the six trig functions of angle A.



2. Consider triangle VET below. Find the length of the hypotenuse and values of the six trig functions of angle T.



3. Consider the right triangle below.



   a. Find the hypotenuse.
   b. Find the six trigonometric functions of $\angle X$.
   c. Find the six trigonometric functions of $\angle Z$.

4. Looking back at #3, are any functions of $\angle X$ equal to any of the functions of $\angle Z$? If so, which ones? Do you think this could be generalized for ANY pair of acute angles in the same right triangle (also called complements)?
5. Consider an isosceles right triangle with legs of length 2. Find the sine, cosine and tangent of both acute angles.
6. Consider an isosceles right triangle with legs of length x. Find the sine, cosine and tangent of both acute angles. Write down any similarities or patterns you notice with #5.
7. Consider a $30-60-90$ triangle with hypotenuse of length 10. Find the sine, cosine and tangent of both acute angles.
8. Consider a $30-60-90$ triangle with short leg of length x. Find the sine, cosine and tangent of both acute angles. Write down any similarities or patterns you notice with #7.
9. Consider a right triangle, ABC. If $\sin A = \frac{9}{41}$, find the length of the third side.

1.4 Solving Right Triangles

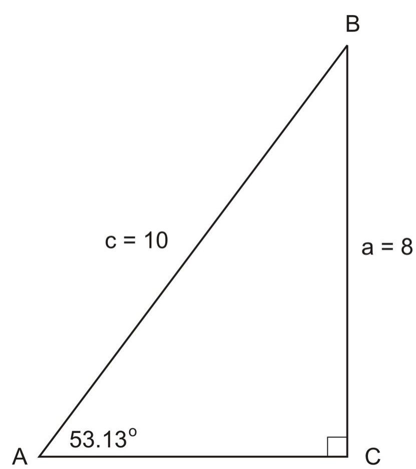
Learning Objectives

- Solve right triangles.
- Find the area of any triangle using trigonometry.
- Solve real-world problems that require you to solve a right triangle.
- Find angle measures using inverse trigonometric functions.

Solving Right Triangles

You can use your knowledge of the Pythagorean Theorem and the six trigonometric functions to solve a right triangle. Because a right triangle is a triangle with a 90 degree angle, solving a right triangle requires that you find the measures of one or both of the other angles. How you solve will depend on how much information is given. The following examples show two situations: a triangle missing one side, and a triangle missing two sides.

Example 1: Solve the triangle shown below.



Solution:

We need to find the lengths of all sides and the measures of all angles. In this triangle, two of the three sides are given. We can find the length of the third side using the Pythagorean Theorem:

$$\begin{aligned}8^2+b^2&=10^2\\64+b^2&=100\\b^2&=36\\b&=\pm 6\Rightarrow b=6\end{aligned}$$

(You may have also recognized the "Pythagorean Triple," 6, 8, 10, instead of carrying out the Pythagorean Theorem.)

You can also find the third side using a trigonometric ratio. Notice that the missing side, b, is adjacent to $\angle A$, and the hypotenuse is given. Therefore we can use the cosine function to find the length of b:

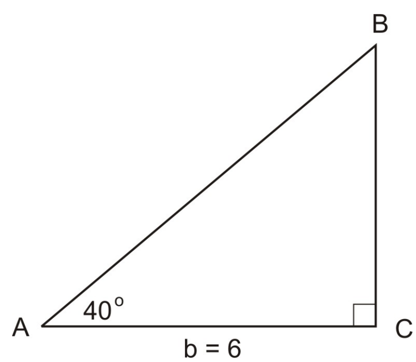
$$\begin{aligned}\cos 53.13^\circ &= \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{10}\\ 0.6 &= \frac{b}{10}\\ b &= 0.6(10) = 6\end{aligned}$$

We could also use the tangent function, as the opposite side was given. It may seem confusing that you can find the missing side in more than one way. The point is, however, not to create confusion, but to show that you must look at what information is missing, and choose a strategy. Overall, when you need to identify one side of the triangle, you can either use the Pythagorean Theorem, or you can use a trig ratio.

To solve the above triangle, we also have to identify the measures of all three angles. Two angles are given: 90 degrees and 53.13 degrees. We can find the third angle using the Triangle Sum Theorem, $180-90-53.13=36.87^\circ$.

Now let's consider a triangle that has two missing sides.

Example 2: Solve the triangle shown below.



Solution:

In this triangle, we need to find the lengths of two sides. We can find the length of one side using a trig ratio. Then we can find the length of the third side by using a trig ratio with the same given information, not the side we solved for. This is because the side we found is an *approximation* and would not yield the most accurate answer for the other missing side. *Only use the given information when solving right triangles.*

We are given the measure of angle A, and the length of the side adjacent to angle A. If we want to find the length of the hypotenuse, c, we can use the cosine ratio:

$$\begin{aligned}\cos 40^\circ &= \frac{adjacent}{hypotenuse} = \frac{6}{c}\\ \cos 40^\circ &= \frac{6}{c}\\ c\cos 40^\circ &= 6\\ c &= \frac{6}{\cos 40^\circ} \approx 7.83\end{aligned}$$

If we want to find the length of the other leg of the triangle, we can use the tangent ratio. This will give us the most accurate answer because we are not using approximations.

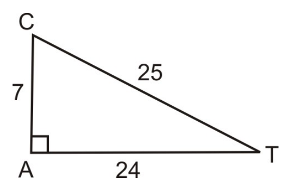
$$\tan 40^\circ = \frac{opposite}{adjacent} = \frac{a}{6}$$
$$a = 6\tan 40^\circ \approx 5.03$$

Now we know the lengths of all three sides of this triangle. In the review questions, you will verify the values of c and a using the Pythagorean Theorem. Here, to finish solving the triangle, we only need to find the measure of $\angle B : 180 - 90 - 40 = 50^\circ$

Notice that in both examples, one of the two non-right angles was given. If neither of the two non-right angles is given, you will need a new strategy to find the angles.

Inverse Trigonometric Functions

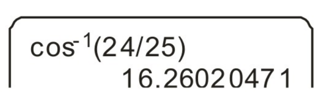
Consider the right triangle below.



From this triangle, we know how to determine all six trigonometric functions for both $\angle C$ and $\angle T$. From any of these functions we can also find the value of the angle, using our graphing calculators. If you look back at #7 from 1.3, we saw that $\sin 30^\circ = \frac{1}{2}$. If you type 30 into your graphing calculator and then hit the SIN button, the calculator yields 0.5. (Make sure your calculator's mode is set to degrees.)

Conversely, with the triangle above, we know the trig ratios, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. These functions are located above the SIN, COS, and TAN buttons on the calculator. To access this function, press 2^{nd} and the appropriate button and the measure of the angle appears on the screen.

$\cos T = \frac{24}{25} \rightarrow \cos^{-1}\frac{24}{25} = T$ from the calculator we get



Example 3: Find the angle measure for the trig functions below.

a. $\sin x = 0.687$

b. $\tan x = \frac{4}{3}$

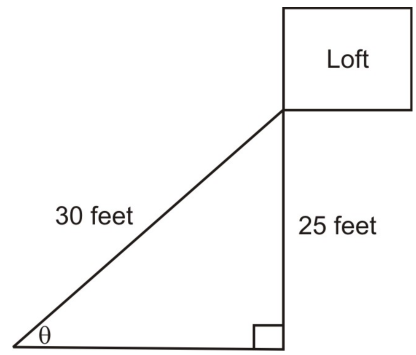
Solution: Plug into calculator.

a. $\sin^{-1} 0.687 = 43.4^\circ$

b. $\tan^{-1}\frac{4}{3} = 53.13^\circ$

Example 4: You live on a farm and your chore is to move hay from the loft of the barn down to the stalls for the horses. The hay is very heavy and to move it manually down a ladder would take too much time and effort. You decide to devise a make shift conveyor belt made of bed sheets that you will attach to the door of the loft and anchor securely in the ground. If the door of the loft is 25 feet above the ground and you have 30 feet of sheeting, at what angle do you need to anchor the sheets to the ground?

Solution:



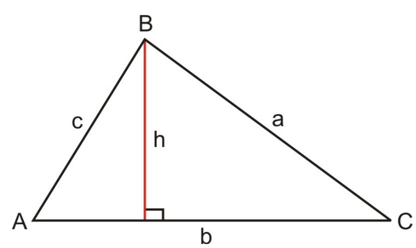
From the picture, we need to use the inverse sine function.

$$\sin\theta = \frac{25\ feet}{30\ feet}$$
$$\sin\theta = 0.8333$$
$$\sin^{-1}(\sin\theta) = \sin^{-1} 0.8333$$
$$\theta = 56.4^\circ$$

The sheets should be anchored at an angle of 56.4°.

Finding the Area of a Triangle

In Geometry, you learned that the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height, or altitude. Now that you know the trig ratios, this formula can be changed around, using sine.



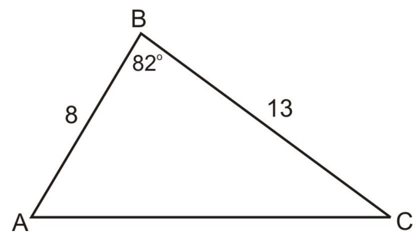
Looking at the triangle above, you can use sine to determine $h, \sin C = \frac{h}{a}$. So, solving this equation for h, we have $a \sin C = h$. Substituting this for h, we now have a new formula for area.

$$A = \frac{1}{2}ab\sin C$$

What this means is you do not need the height to find the area anymore. All you now need is two sides and the angle between the two sides, called the included angle.

Example 5: Find the area of the triangle.

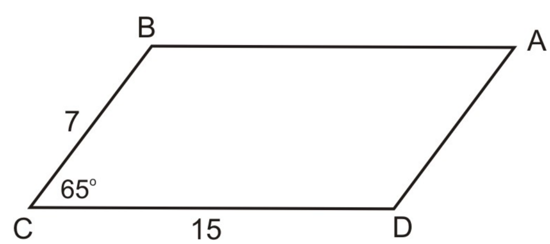
a.



Solution: Using the formula, $A = \frac{1}{2}\, ab \sin C$, we have

$$\begin{aligned}A &= \frac{1}{2} \cdot 8 \cdot 13 \cdot \sin 82^\circ \\ &= 4 \cdot 13 \cdot 0.990 \\ &= 51.494\end{aligned}$$

Example 6: Find the area of the parallelogram.

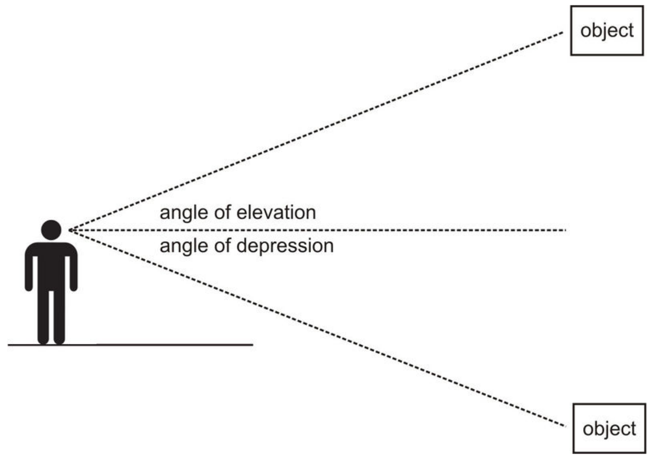


Solution: Recall that a parallelogram can be split into two triangles. So the formula for a parallelogram, using the new formula, would be: $A = 2 \cdot \frac{1}{2}\, ab \sin C$ or $A = ab \sin C$.

$$\begin{aligned}A &= 7 \cdot 15 \cdot \sin 65^\circ \\ &= 95.162\end{aligned}$$

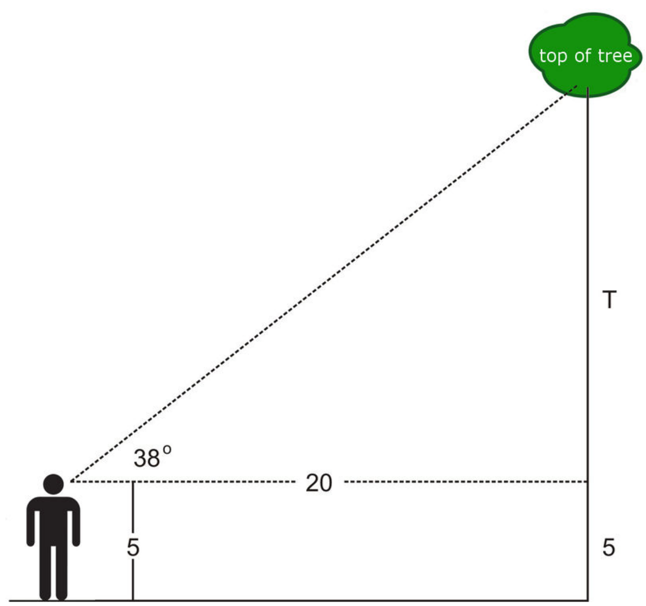
Angles of Elevation and Depression

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.



The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is the angle between the horizontal line of sight and the line of sight *down to* an object. For example, if you were standing on top of a hill or a building, looking down at an object, you could measure the angle of depression. You can measure these angles using a clinometer or a theodolite. People tend to use clinometers or theodolites to measure the height of trees and other tall objects. Here we will solve several problems involving these angles and distances.

Example 7: You are standing 20 feet away from a tree, and you measure the angle of elevation to be 38°. How tall is the tree?



Solution:

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall.

The figure shows us that once we find the value of T, we have to add 5 feet to this value to find the total height of the triangle. To find T, we should use the tangent value:

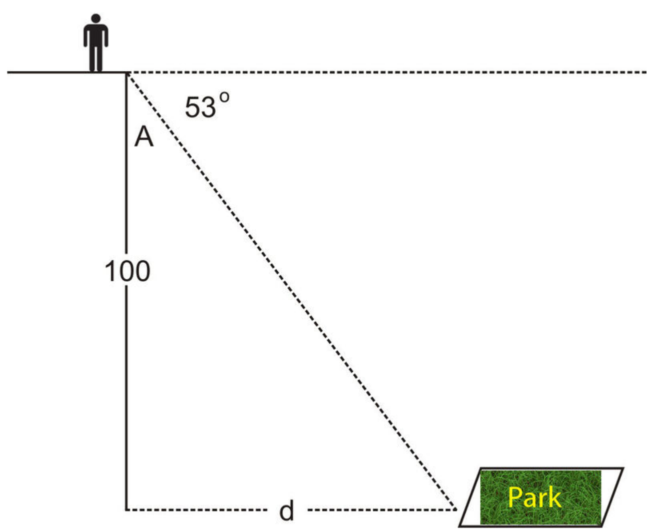
$$\tan 38^\circ = \frac{opposite}{adjacent} = \frac{T}{20}$$
$$\tan 38^\circ = \frac{T}{20}$$
$$T = 20 \tan 38^\circ \approx 15.63$$
$$\text{Height of tree} \approx 20.63\ ft$$

The next example shows an angle of depression.

Example 8: You are standing on top of a building, looking at a park in the distance. The angle of depression is 53°. If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Solution:

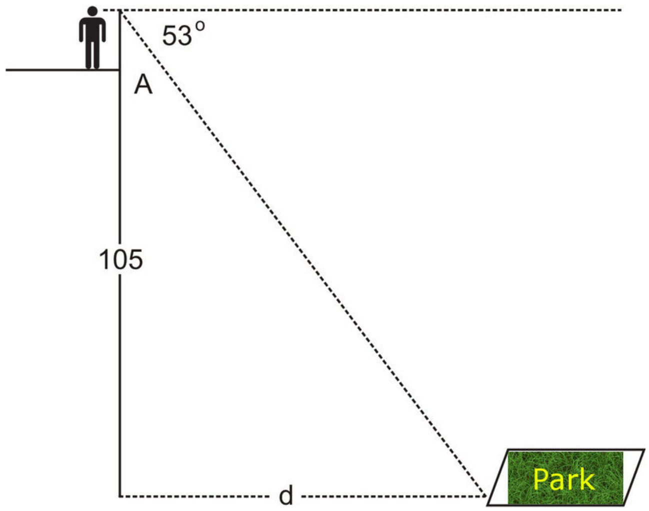
If we ignore the height of the person, we solve the following triangle:



Given the angle of depression is 53°, $\angle A$ in the figure above is 37°. We can use the tangent function to find the distance from the building to the park:

$$\tan 37^\circ = \frac{opposite}{adjacent} = \frac{d}{100}$$
$$\tan 37^\circ = \frac{d}{100}$$
$$d = 100 \tan 37^\circ \approx 75.36\ ft.$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:

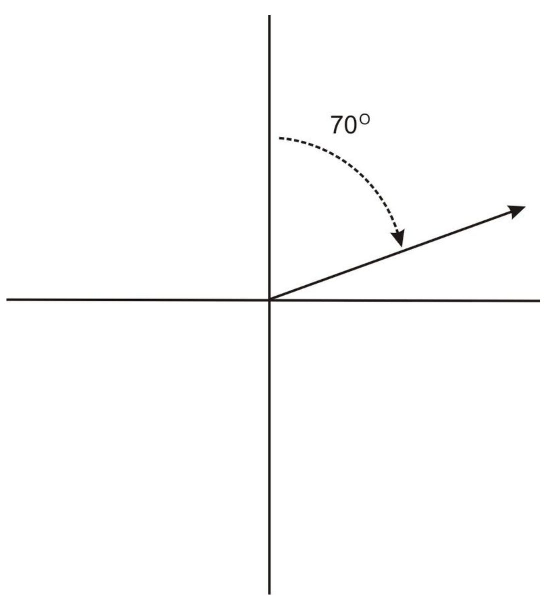


$$\tan 37^\circ = \frac{opposite}{adjacent} = \frac{d}{105}$$
$$\tan 37^\circ = \frac{d}{105}$$
$$d = 105 \tan 37^\circ \approx 79.12\ ft.$$

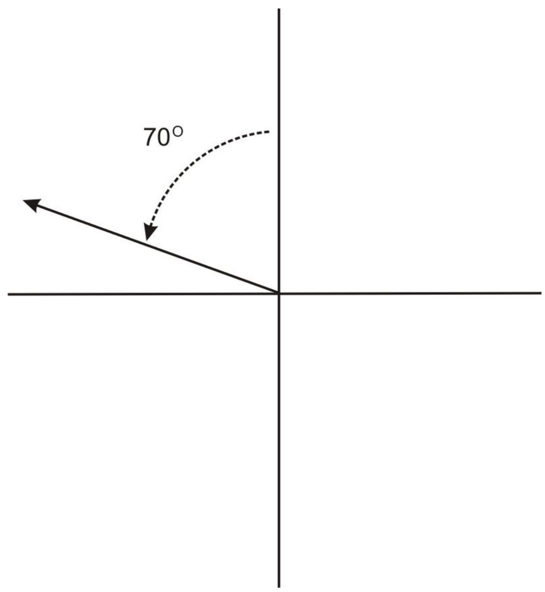
If you are only looking to estimate a distance, then you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Right Triangles and Bearings

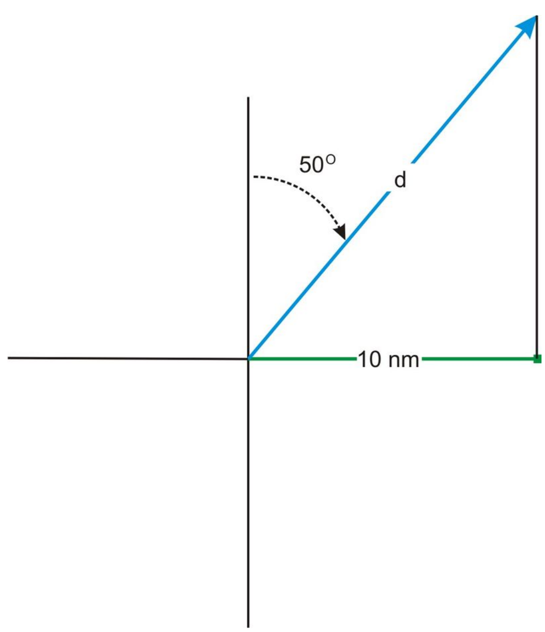
We can also use right triangles to find distances using angles given as bearings. In navigation, a bearing is the direction from one object to another. In air navigation, bearings are given as angles rotated clockwise from the north. The graph below shows an angle of 70 degrees:



It is important to keep in mind that angles in navigation problems are measured this way, and not the same way angles are measured in trigonometry. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, $N70^\circ E$ refers to an angle from the north, towards the east, while $N70^\circ W$ refers to an angle from the north, towards the west. $N70^\circ E$ is the same as the angle shown in the graph above. $N70^\circ W$ would result in an angle in the second quadrant.



Example 9: A ship travels on a $N50^\circ E$ course. The ship travels until it is due north of a port which is 10 nautical miles due east of the port from which the ship originated. How far did the ship travel?



Solution: The angle between d and 10 nm is the complement of 50°, which is 40°. Therefore we can find d using the cosine function:

$$\cos 40^\circ = \frac{adjacent}{hypotenuse} = \frac{10}{d}$$
$$\cos 40^\circ = \frac{10}{d}$$
$$d\cos 40^\circ = 10$$
$$d = \frac{10}{\cos 40^\circ} \approx 13.05\ nm$$

Other Applications of Right Triangles

In general, you can use trigonometry to solve any problem that involves right triangles. The next few examples show different situations in which a right triangle can be used to find a length or a distance.

Example 10: The wheelchair ramp

In lesson 3 we introduced the following situation: You are building a ramp so that people in wheelchairs can access a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

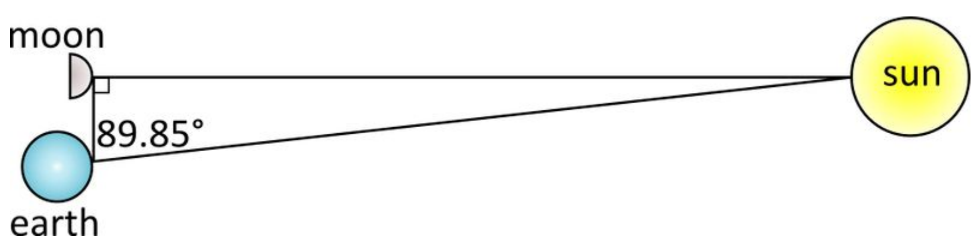
Given that we know the angle of the ramp and the length of the side opposite the angle, we can use the sine ratio to find the length of the ramp, which is the hypotenuse of the triangle:

$$\sin 5^\circ = \frac{8}{L}$$
$$L\sin 5^\circ = 8$$
$$L = \frac{8}{\sin 5^\circ} \approx 91.8\ ft$$

This may seem like a long ramp, but in fact a 5° ramp angle is what is required by the Americans with Disabilities Act (ADA). This explains why many ramps are comprised of several sections, or have turns. The additional distance is needed to make up for the small slope.

Right triangle trigonometry is also used for measuring distances that could not actually be measured. The next example shows a calculation of the distance between the moon and the sun. This calculation requires that we know the distance from the earth to the moon. In chapter 5 you will learn the Law of Sines, an equation that is necessary for the calculation of the distance from the earth to the moon. In the following example, we assume this distance, and use a right triangle to find the distance between the moon and the sun.

Example 11: The earth, moon, and sun create a right triangle during the first quarter moon. The distance from the earth to the moon is about 240,002.5 miles. What is the distance between the sun and the moon?



Solution:

Let $d =$ the distance between the sun and the moon. We can use the tangent function to find the value of d:

$$\tan 89.85^\circ = \frac{d}{240,002.5}$$
$$d = 240,002.5 \tan 89.85^\circ = 91,673,992.71 \textit{ miles}$$

Therefore the distance between the sun and the moon is much larger than the distance between the earth and the moon.

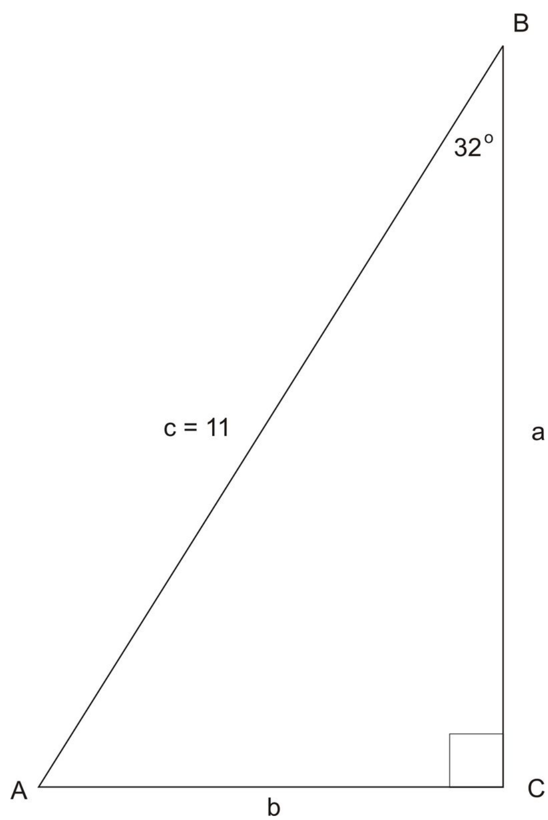
(Source: www.scribd.com, Trigonometry from the Earth to the Stars.)

Points to Consider

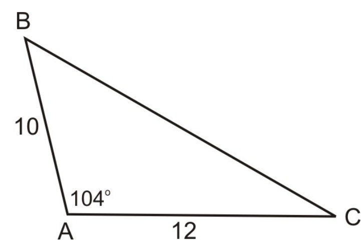
- In what kinds of situations do right triangles naturally arise?
- Are there right triangles that cannot be solved?
- Trigonometry can solve problems at an astronomical scale as well as problems at a molecular or atomic scale. Why is this true?

Review Questions

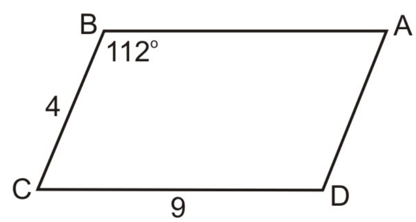
1. Solve the triangle.



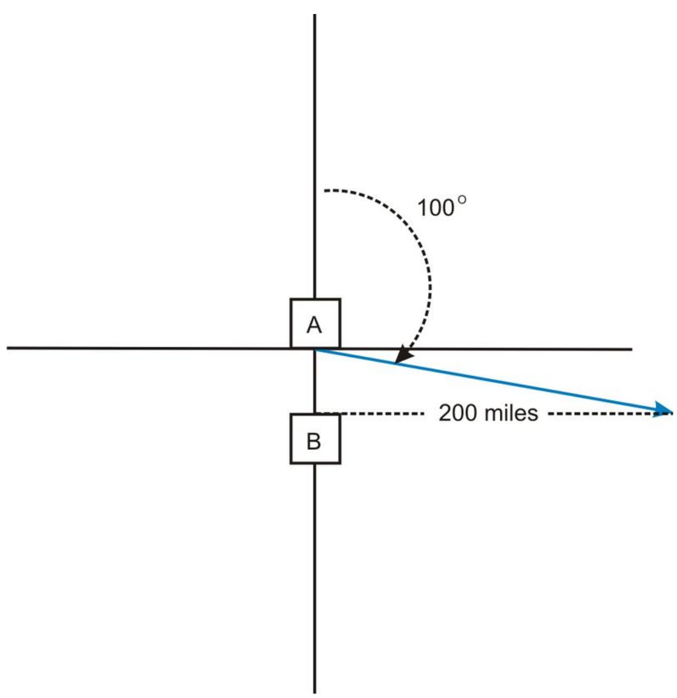
2. Two friends are writing practice problems to study for a trigonometry test. Sam writes the following problem for his friend Anna to solve: In right triangle ABC, the measure of angle C is 90 degrees, and the length of side c is 8 inches. Solve the triangle. Anna tells Sam that the triangle cannot be solved. Sam says that she is wrong. Who is right? Explain your thinking.
3. Use the Pythagorean Theorem to verify the sides of the triangle in example 2.
4. Estimate the measure of angle B in the triangle below using the fact that $\sin B = \frac{3}{5}$ and $\sin 30^\circ = \frac{1}{2}$. Use a calculator to find sine values. Estimate B to the nearest degree.
5. Find the area of the triangle.



6. Find the area of the parallelogram below.

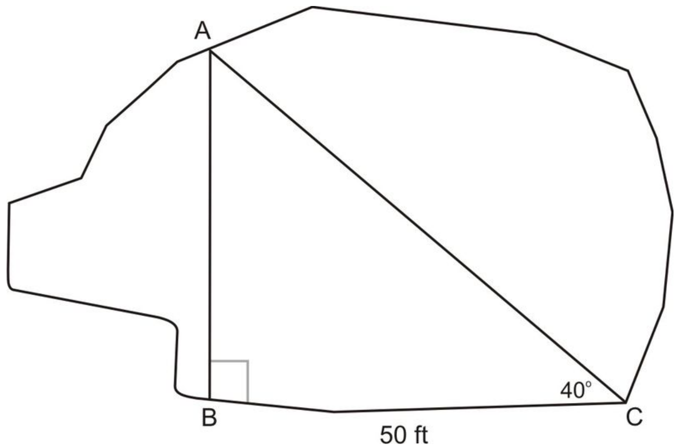


7. The angle of elevation from the ground to the top of a flagpole is measured to be 53°. If the measurement was taken from 15 feet away, how tall is the flagpole?
8. From the top of a hill, the angle of depression to a house is measured to be 14°. If the hill is 30 feet tall, how far away is the house?
9. An airplane departs city A and travels at a bearing of 100°. City B is directly south of city A. When the plane is 200 miles east of city B, how far has the plan traveled? How far apart are City A and City B?

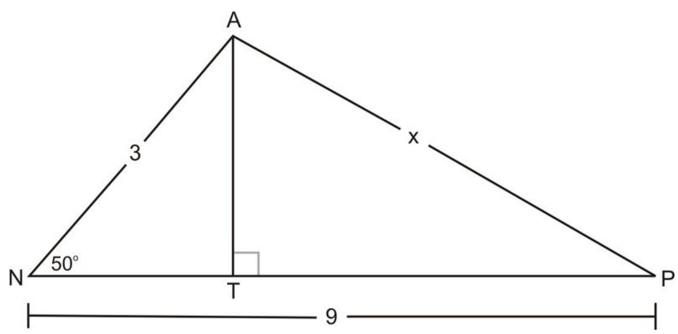


What is the length of the slanted outer wall, w? What is the length of the main floor, f?

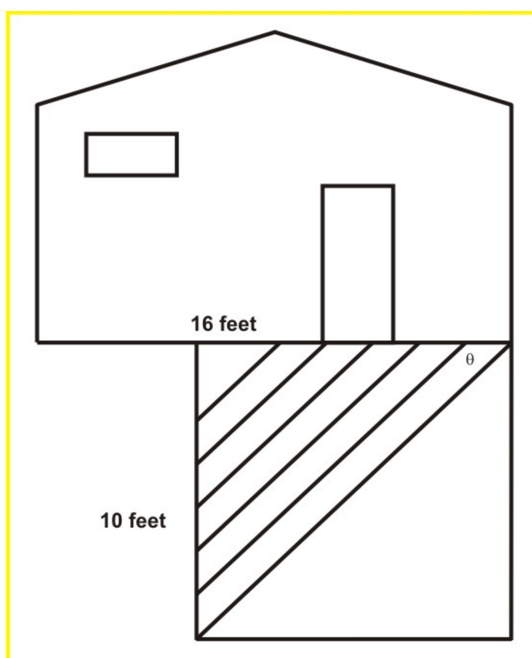
10. A surveyor is measuring the width of a pond. She chooses a landmark on the opposite side of the pond, and measures the angle to this landmark from a point 50 feet away from the original point. How wide is the pond?



11. Find the length of side x:



12. A deck measuring 10 feet by 16 feet will require laying boards with one board running along the diagonal and the remaining boards running parallel to that board. The boards meeting the side of the house must be cut prior to being nailed down. At what angle should the boards be cut?



1.5 Measuring Rotation

Learning Objectives

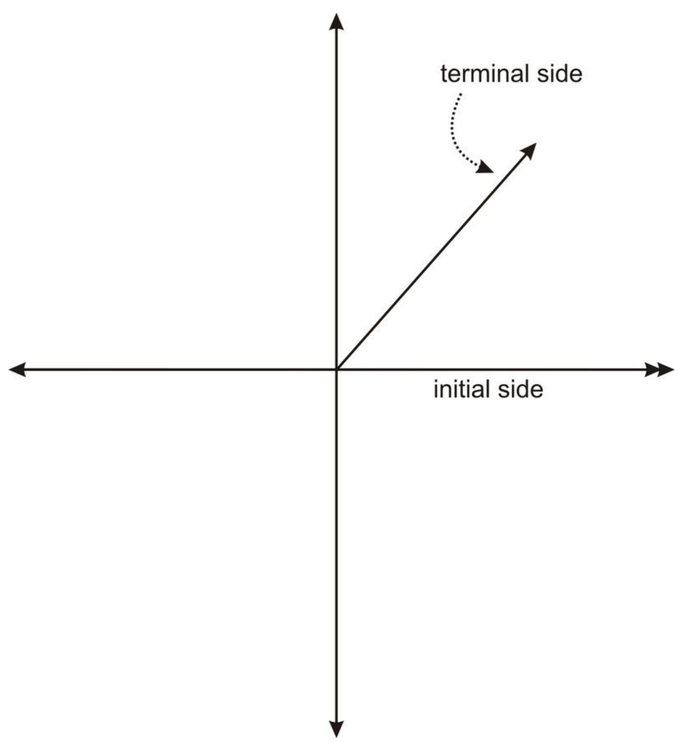
- Identify and draw angles of rotation in standard position.
- Identify quadrantal angles.
- Identify co-terminal angles.

Angles of Rotation in Standard Position

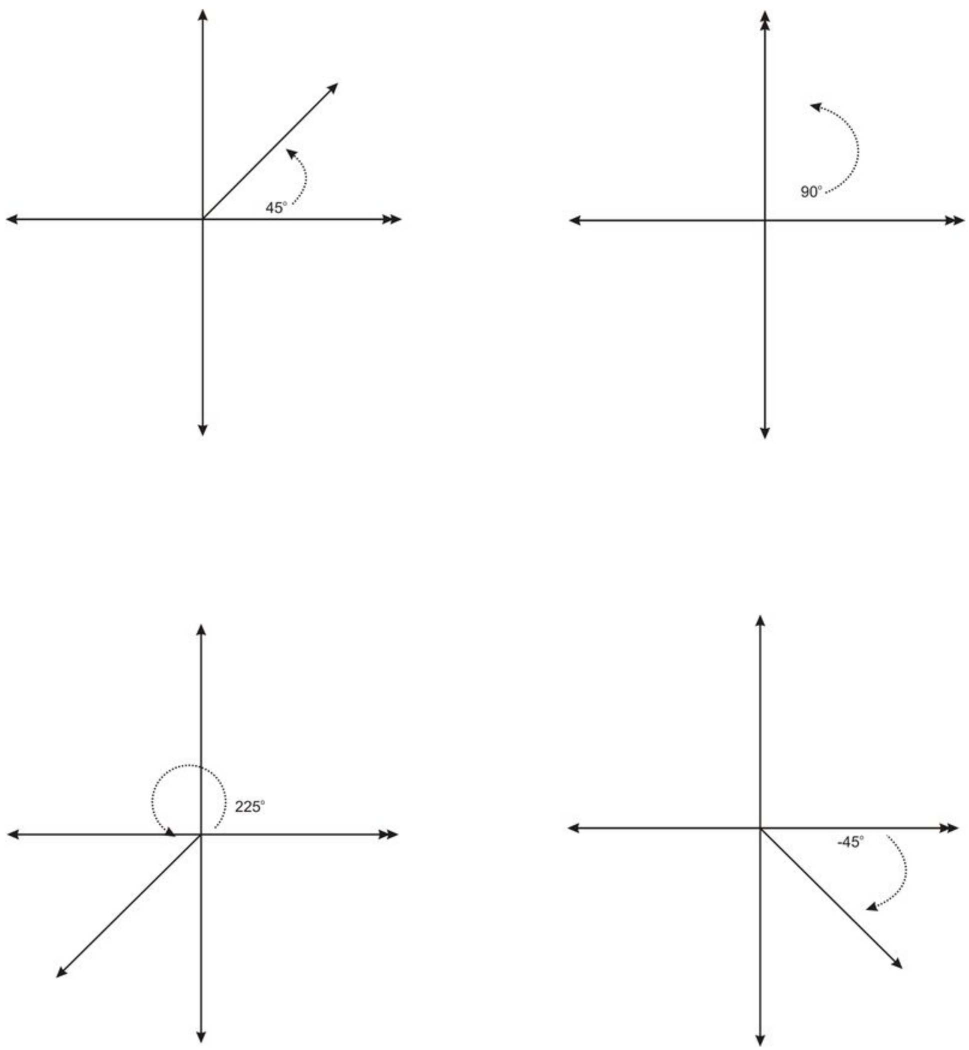
Consider, for example, a game that is played with a spinner. When you spin the spinner, how far has it gone? You can answer this question in several ways. You could say something like "the spinner spun around 3 times." This means that the spinner made 3 complete rotations, and then landed back where it started.

We can also measure the rotation in degrees. In the previous lesson we worked with angles in triangles, measured in degrees. You may recall from geometry that a full rotation is 360 degrees, usually written as 360°. Half a rotation is then 180° and a quarter rotation is 90°. Each of these measurements will be important in this lesson, as well as in the remainder of the chapter.

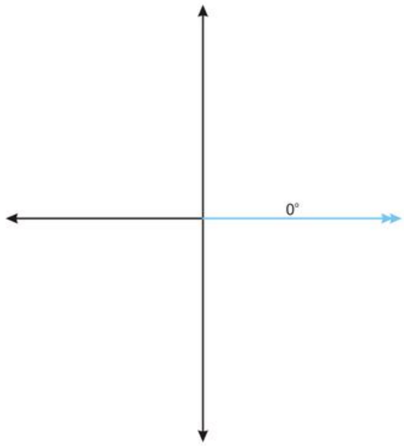
We can use our knowledge of graphing to represent any angle. The figure below shows an angle in what is called **standard position**.

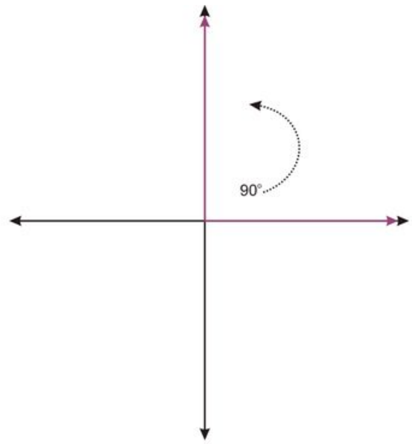


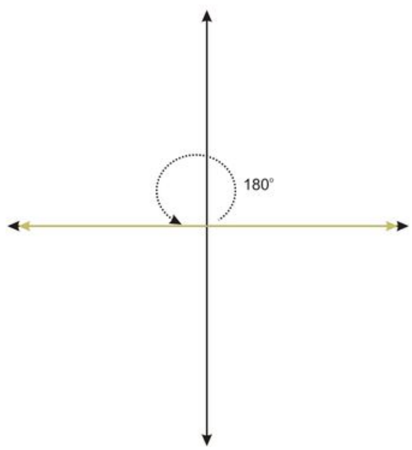
The initial side of an angle in standard position is always on the positive $x-$axis. The terminal side always meets the initial side at the origin. Notice that the rotation goes in a **counterclockwise** direction. This means that if we rotate **clockwise**, we will generate a negative angle. Below are several examples of angles in standard position.

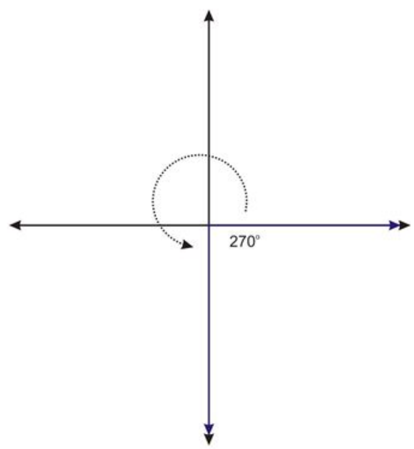


The 90 degree angle is one of four **quadrantal** angles. A quadrantal angle is one whose terminal side lies on an axis. Along with 90°, 0°, 180° and 270° are quadrantal angles.





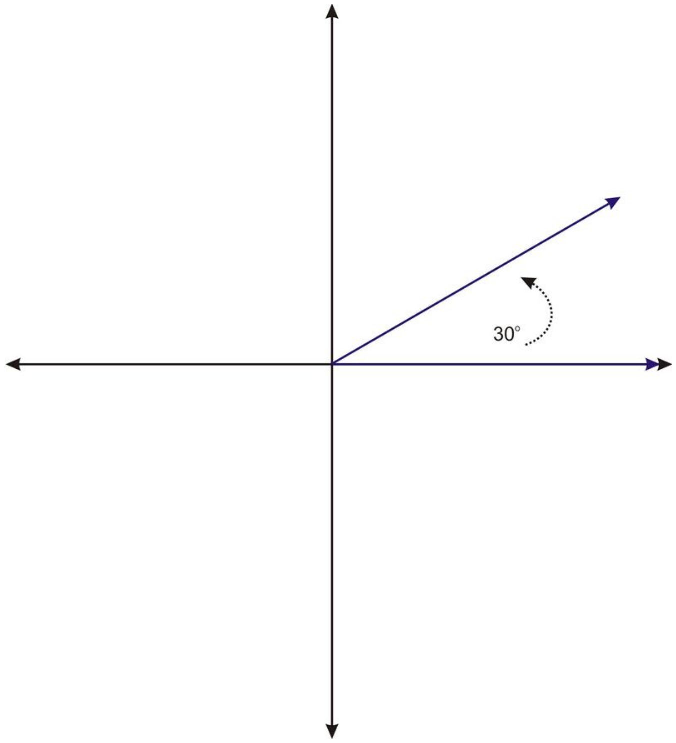




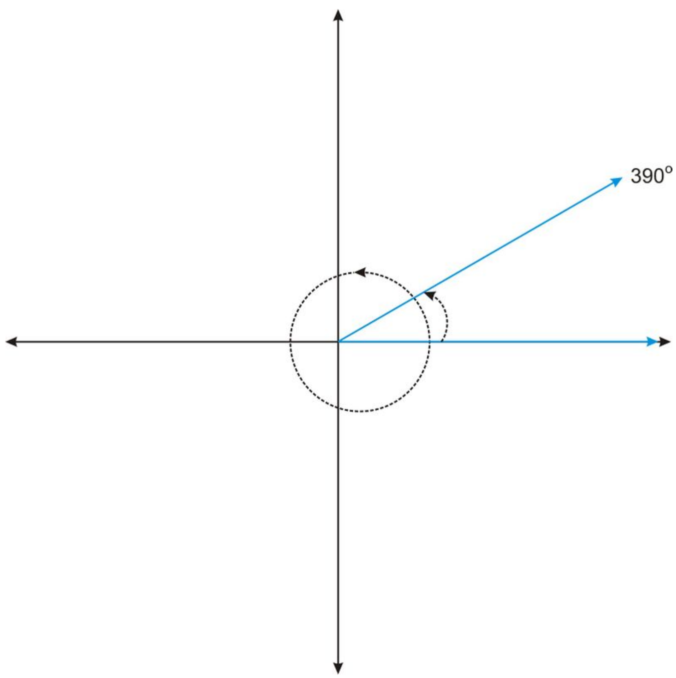
These angles are referred to as quadrantal because each angle defines a quadrant. Notice that without the arrow indicating the rotation, 270° looks as if it is a $-90°$, defining the fourth quadrant. Notice also that 360° would look just like 0°. The difference is in the action of rotation. This idea of two angles actually being the same angle is discussed next.

Coterminal Angles

Consider the angle 30°, in standard position.



Now consider the angle 390°. We can think of this angle as a full rotation (360°), plus an additional 30 degrees.



Notice that 390° looks the same as 30°. Formally, we say that the angles share the same terminal side. Therefore we call the angles **co-terminal**. Not only are these two angles co-terminal, but there are infinitely many angles that are co-terminal with these two angles. For example, if we rotate another 360°, we get the angle 750°. Or, if we create the angle in the negative direction (clockwise), we get the angle -330°. Because we can rotate in either direction, and we can rotate as many times as we want, we can continuously generate angles that are co-terminal with 30°.

Example 1: Which angles are co-terminal with 45°?

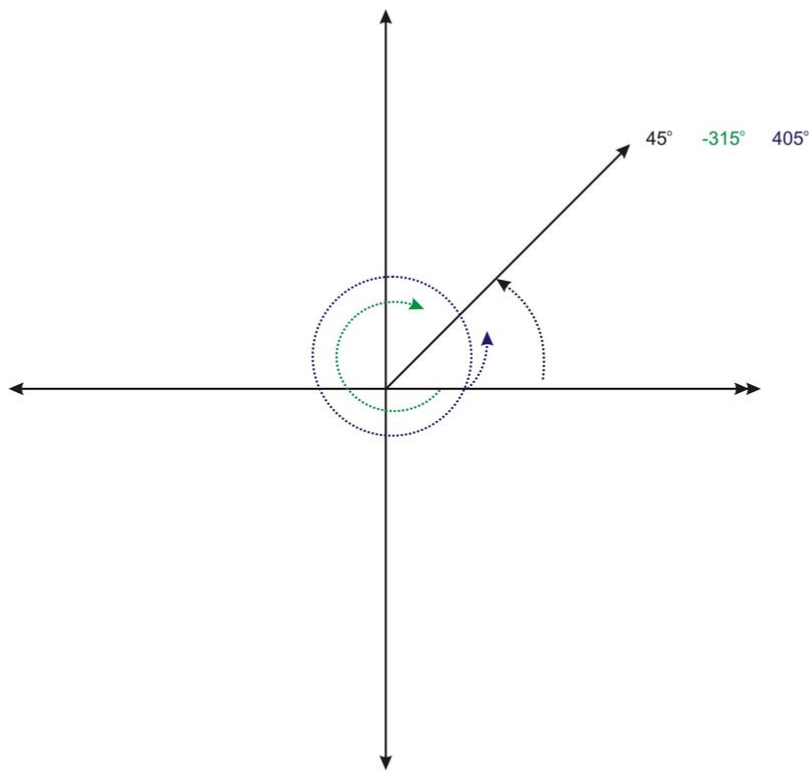
a. -45°

b. 405°

c. -315°

d. 135°

Solution: b. 405° and c. -315° are co-terminal with 45°.



Notice that terminal side of the first angle, -45°, is in the 4^{th} quadrant. The last angle, 135° is in the 2^{nd} quadrant. Therefore neither angle is co-terminal with 45°.

Now consider 405°. This is a full rotation, plus an additional 45 degrees. So this angle is co-terminal with 45°. The angle -315° can be generated by rotating clockwise. To determine where the terminal side is, it can be helpful to use quadrantal angles as markers. For example, if you rotate clockwise 90 degrees 3 times (for a total of 270 degrees), the terminal side of the angle is on the positive $y-$axis. For a total clockwise rotation of 315 degrees, we have $315 - 270 = 45$ degrees more to rotate. This puts the terminal side of the angle at the same position as 45°.

Points to Consider

- How can one angle look exactly the same as another angle?
- Where might you see angles of rotation in real life?

Review Questions

1. Plot the following angles in standard position.
   a. 60°
   b. -170°

c. 365°
d. 325°
e. 240°

2. State the measure of an angle that is co-terminal with 90°.
3. Name a positive and negative angle that are co-terminal with:

a. 120°
b. 315°
c. -150°

4. A drag racer goes around a 180 degree circular curve in a racetrack in a path of radius 120 m. Its front and back wheels have different diameters. The front wheels are 0.6 m in diameter. The rear wheels are much larger; they have a diameter of 1.8 m. The axles of both wheels are 2 m long. Which wheel has more rotations going around the curve? How many more degrees does the front wheel rotate compared to the back wheel?

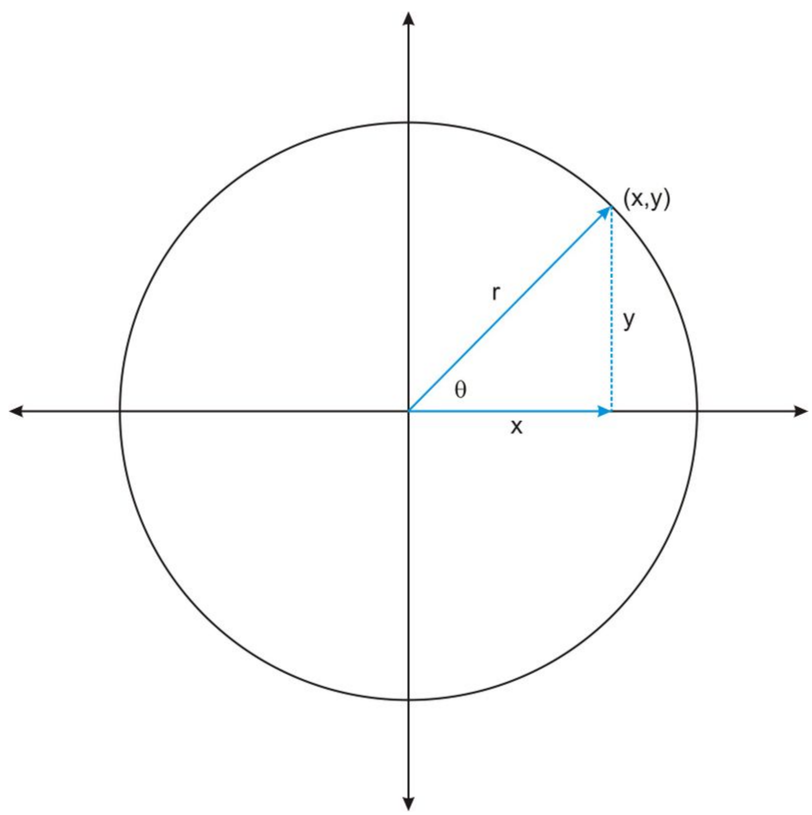
1.6 Applying Trig Functions to Angles of Rotation

Learning Objectives

- Find the values of the six trigonometric functions for angles of rotation.
- Recognize angles of the unit circle.

Trigonometric Functions of Angles in Standard Position

In section 1.3, we defined the six trigonometric functions for angles in right triangles. We can also define the same functions in terms of angles of rotation. Consider an angle in standard position, whose terminal side intersects a circle of radius r. We can think of the radius as the hypotenuse of a right triangle:



The point (x, y) where the terminal side of the angle intersects the circle tells us the lengths of the two legs of the triangle. Now, we can define the trigonometric functions in terms of x, y, and r:

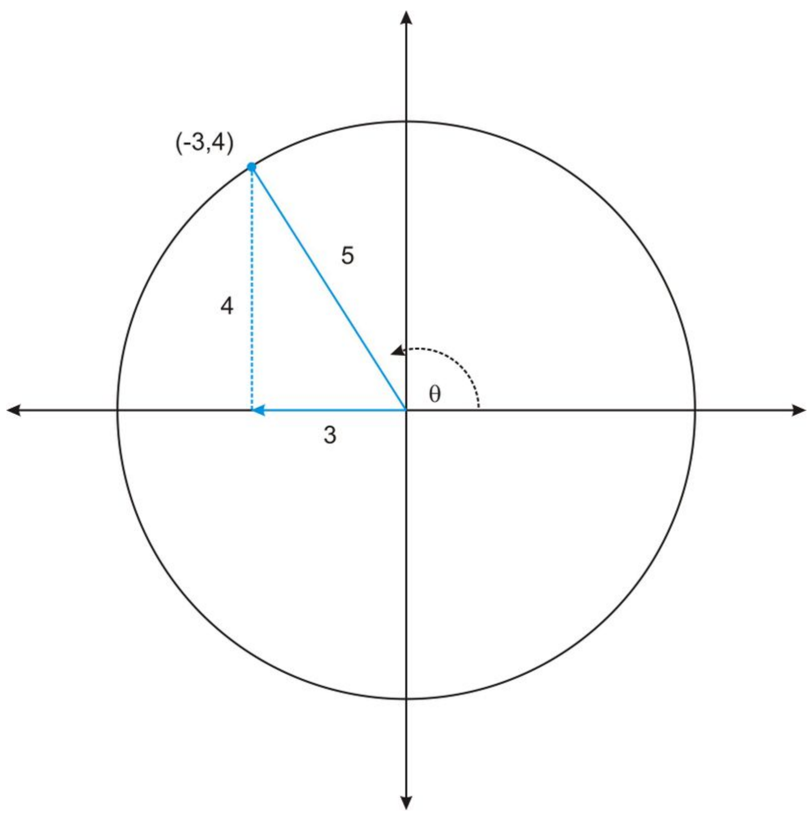
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

And, we can extend these functions to include non-acute angles.

Example 1: The point (-3, 4) is a point on the terminal side of an angle in standard position. Determine the values of the six trigonometric functions of the angle.

Solution:

Notice that the angle is more than 90 degrees, and that the terminal side of the angle lies in the second quadrant. This will influence the signs of the trigonometric functions.

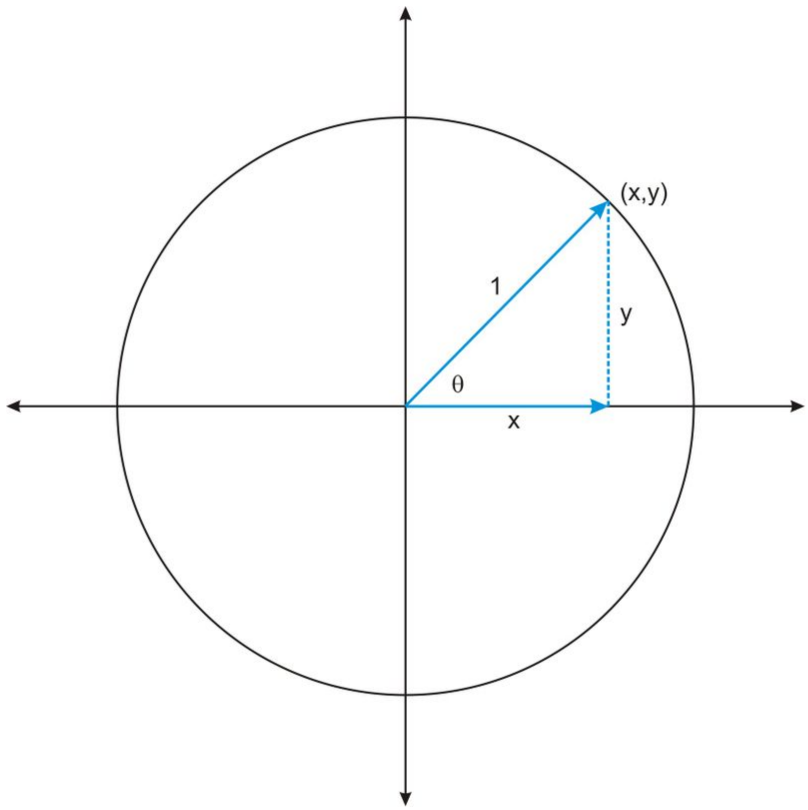


$$\cos\theta = \frac{-3}{5} \qquad \sec\theta = \frac{5}{-3}$$
$$\sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$
$$\tan\theta = \frac{4}{-3} \qquad \cot\theta = \frac{-3}{4}$$

Notice that the value of r depends on the coordinates of the given point. You can always find the value of r using the Pythagorean Theorem. However, often we look at angles in a circle with radius 1. As you will see next, doing this allows us to simplify the definitions of the trig functions.

The Unit Circle

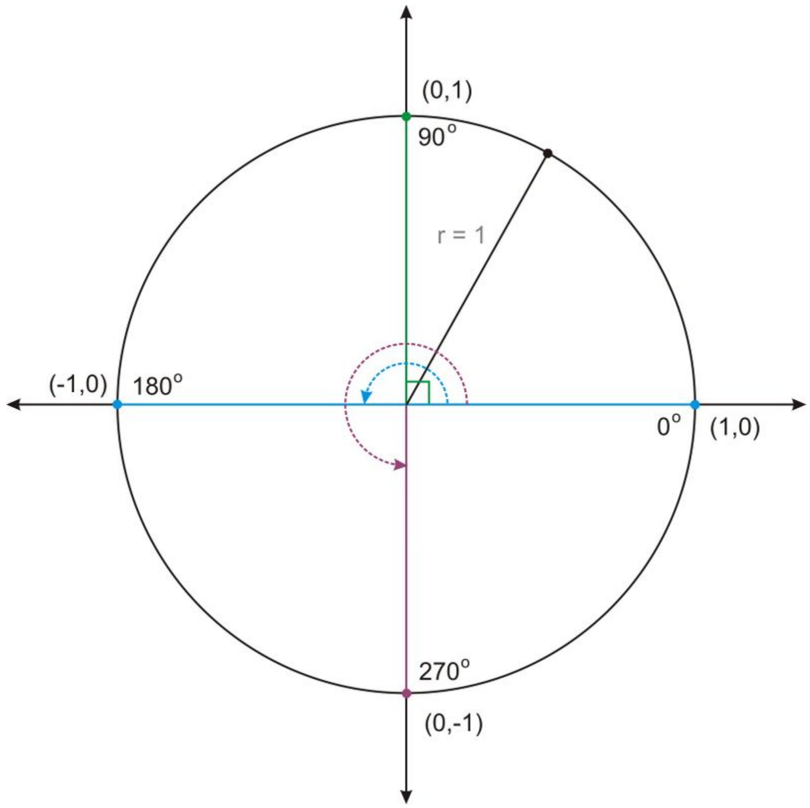
Consider an angle in standard position, such that the point (x,y) on the terminal side of the angle is a point on a circle with radius 1.



This circle is called the **unit circle**. With $r = 1$, we can define the trigonometric functions in the unit circle:

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sec\theta = \frac{r}{x} = \frac{1}{x}$$
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \csc\theta = \frac{r}{y} = \frac{1}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Notice that in the unit circle, the sine and cosine of an angle are the x and y coordinates of the point on the terminal side of the angle. Now we can find the values of the trigonometric functions of any angle of rotation, even the quadrantal angles, which are not angles in triangles.



We can use the figure above to determine values of the trig functions for the quadrantal angles. For example, $\sin 90^\circ = y = 1$.

Example 2: Use the unit circle above to find each value:

a. $\cos 90^\circ$

b. $\cot 180^\circ$

c. $\sec 0^\circ$

Solution:

a. $\cos 90^\circ = 0$

The ordered pair for this angle is (0, 1). The cosine value is the x coordinate, 0.

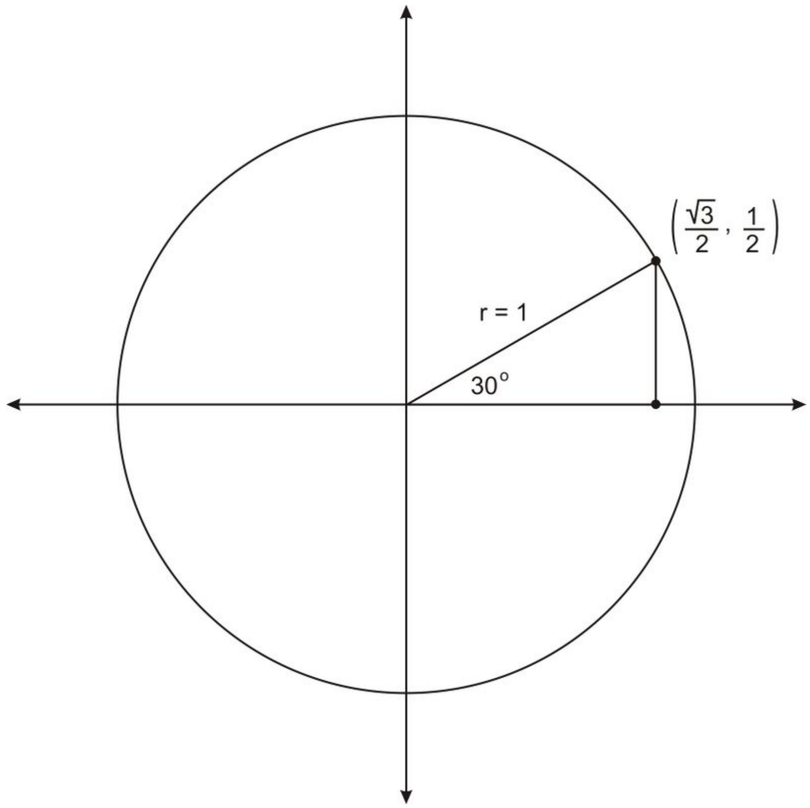
b. $\cot 180^\circ$ is undefined

The ordered pair for this angle is (-1, 0). The ratio $\frac{x}{y}$ is $\frac{-1}{0}$, which is undefined.

c. $\sec 0^\circ = 1$

The ordered pair for this angle is (1, 0). The ratio $\frac{1}{x}$ is $\frac{1}{1} = 1$.

There are several important angles in the unit circle that you will work with extensively in your study of trigonometry, primarily 30°, 45°, and 60°. Recall section 1.2 to find the values of the trigonometric functions of these angles. First, we need to know the ordered pairs. Let's begin with 30°.



This triangle is identical to #8 from 1.3. If you look back at this problem, you will recall that you found the sine, cosine and tangent of 30° and 60°. It is no coincidence that the endpoint on the unit circle is the same as your answer from #8.

The terminal side of the angle intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore we can find the values of any of the trig functions of 30°. For example, the cosine value is the $x-$coordinate, so $\cos(30^\circ) = \frac{\sqrt{3}}{2}$. Because the coordinates are fractions, we have to do a bit more work in order to find the tangent value:

$$\tan 30^\circ = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

In the review exercises you will find the values of the remaining four trig functions of this angle. The table below summarizes the ordered pairs for 30°, 45°, and 60° on the unit circle.

TABLE 1.1:

Angle	$x-$**coordinate**	$y-$**coordinate**
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

We can use these values to find the values of any of the six trig functions of these angles.

Example 3: Find the value of each function.

a. $\cos 45^\circ$

b. $\sin 60^\circ$

c. $\tan 45^\circ$

Solution:

a. $\cos 45^\circ = \frac{\sqrt{2}}{2}$ The cosine value is the $x-$ coordinate of the point.

b. $\sin 60^\circ = \frac{\sqrt{3}}{2}$ The sine value is the $y-$ coordinate of the point.

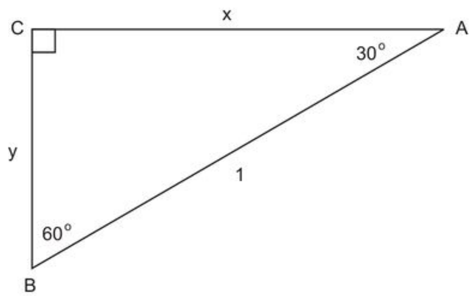
c. $\tan 45^\circ = 1$ The tangent value is the ratio of the $y-$ coordinate to the $x-$ coordinate. Because the $x-$ and $y-$ coordinates are the same for this angle, the tangent ratio is 1.

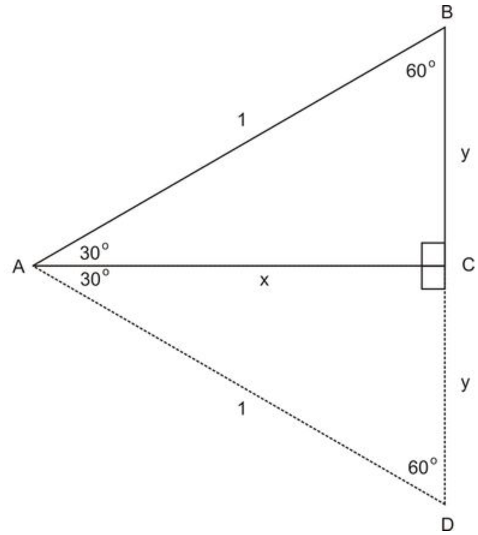
Points to Consider

- How can some values of the trig functions be negative? How is it that some are undefined?
- Why is the unit circle and the trig functions defined on it useful, even when the hypotenuses of triangles in the problem are not 1?

Review Questions

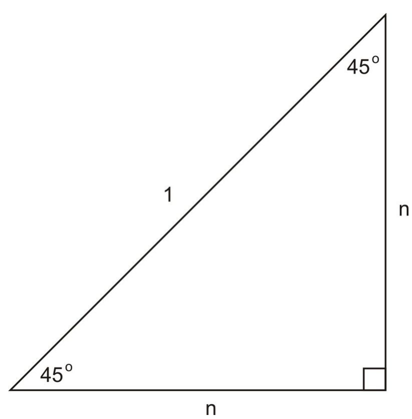
1. The point (3, -4) is a point on the terminal side of an angle θ in standard position.
   a. Determine the radius of the circle.
   b. Determine the values of the six trigonometric functions of the angle.
2. The point (-5, -12) is a point on the terminal side of an angle θ in standard position.
   a. Determine the radius of the circle.
   b. Determine the values of the six trigonometric functions of the angle.
3. $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$. Find $\sin\theta$.
4. $\csc\theta = -4$ and $\tan\theta > 0$. Find the exact values of the remaining trigonometric functions.
5. (2, 6) is a point on the terminal side of θ. Find the exact values of the six trigonometric functions.
6. The terminal side of the angle 270° intersects the unit circle at (0, -1). Use this ordered pair to find the six trig functions of 270°.
7. In the lesson you learned that the terminal side of the angle 30° intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Here you will prove that this is true.





a. Explain why Triangle ABD is an equiangular triangle. What is the measure of angle DAB?
b. What is the length of BD? How do you know?
c. What is the length of BC and CD? How do you know?
d. Now explain why the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
e. Why does this tell you that the ordered pair for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$?

8. In the lesson you learned that the terminal side of the angle 45° is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Use the figure below and the Pythagorean Theorem to show that this is true.



9. In what quadrants will an angle in standard position have a positive tangent value? Explain your thinking.
10. Sketch the angle 150° on the unit circle is. How is this angle related to 30°? What do you think the ordered pair is?
11. We now know that $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = \frac{y}{x}$. First, explain how it looks as though sine, cosine, and tangent are related. Second, can you rewrite tangent in terms of sine and cosine?

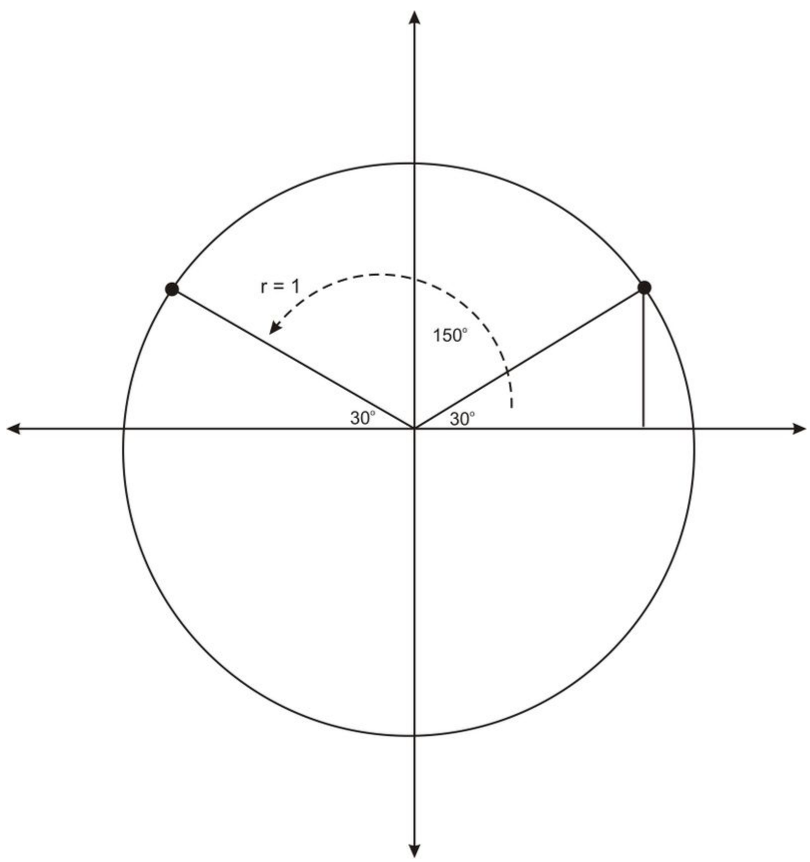
1.7 Trigonometric Functions of Any Angle

Learning Objectives

- Identify the reference angles for angles in the unit circle.
- Identify the ordered pair on the unit circle for angles whose reference angle is 30°, 45°, and 60°, or a quadrantal angle, including negative angles, and angles whose measure is greater than 360°.
- Use these ordered pairs to determine values of trig functions of these angles.
- Use calculators to find values of trig functions of any angle.

Reference Angles and Angles in the Unit Circle

In the previous lesson, one of the review questions asked you to consider the angle 150°. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of 30°, across the $y-$axis.



Notice that 150° makes a 30° angle with the negative $x-$axis. Therefore we say that 30° is the **reference angle** for 150°. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the $x-$axis. Notice that 30° is the reference angle for many angles. For example, it is the reference angle for 210° and for -30°.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.

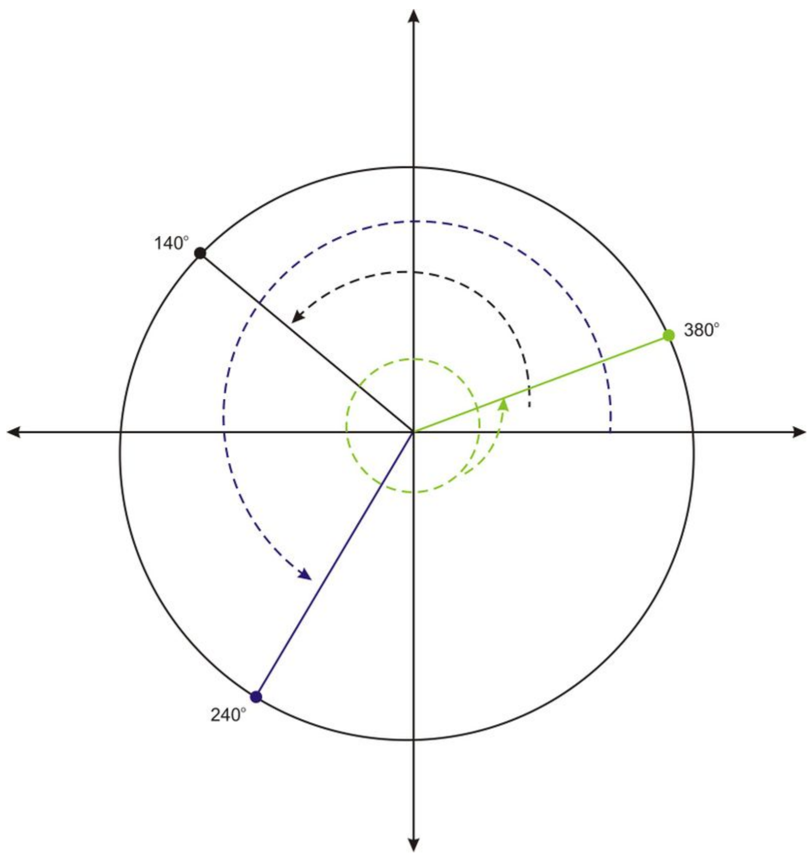
Example 1: Graph each angle and identify its reference angle.

a. 140°

b. 240°

c. 380°

Solution:



a. 140° makes a 40° angle with the $x-$axis. Therefore the reference angle is 40°.

b. 240° makes a 60° with the $x-$axis. Therefore the reference angle is 60°.

c. 380° is a full rotation of 360°, plus an additional 20°. So this angle is co-terminal with 20°, and 20° is its reference angle.

If an angle has a reference angle of 30°, 45°, or 60°, we can identify its ordered pair on the unit circle, and so we can find the values of the six trig functions of that angle. For example, above we stated that 150° has a reference angle of 30°. Because of its relationship to 30°, the ordered pair for 150° is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Now we can find the values of the six trig functions of 150°:

$$\cos 150 = x = \frac{-\sqrt{3}}{2} \qquad \sec 150 = \frac{1}{x} = \frac{1}{\frac{-\sqrt{3}}{2}} = \frac{-2}{\sqrt{3}}$$

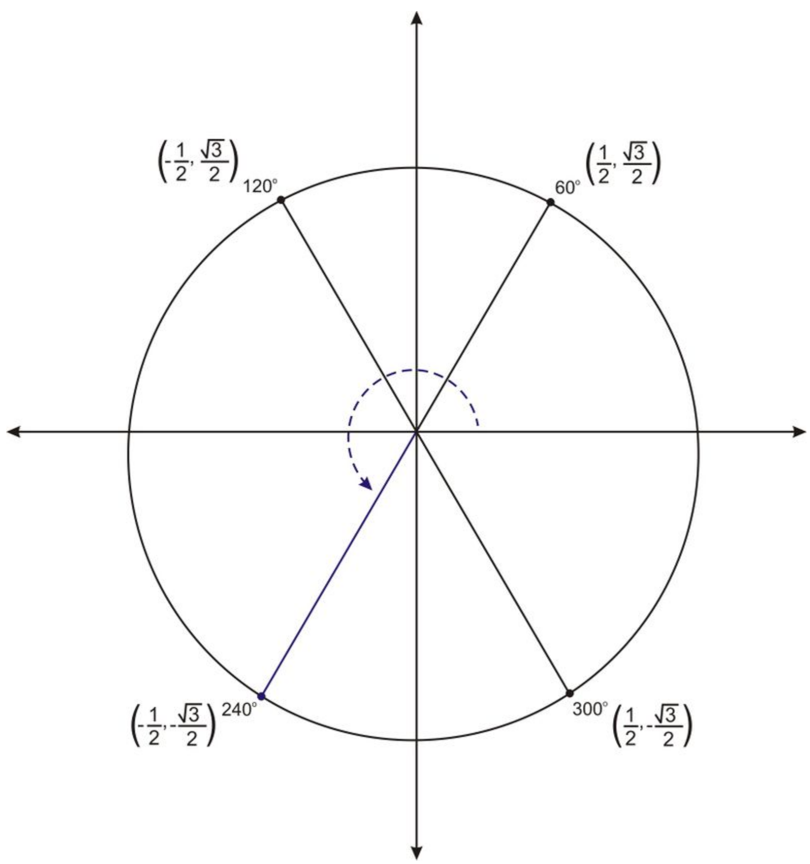
$$\sin 150 = y = \frac{1}{2} \qquad \csc 150 = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\tan 150 = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{-\sqrt{3}} \qquad \cot 150 = \frac{x}{y} = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Example 2: Find the ordered pair for 240° and use it to find the value of sin 240°.

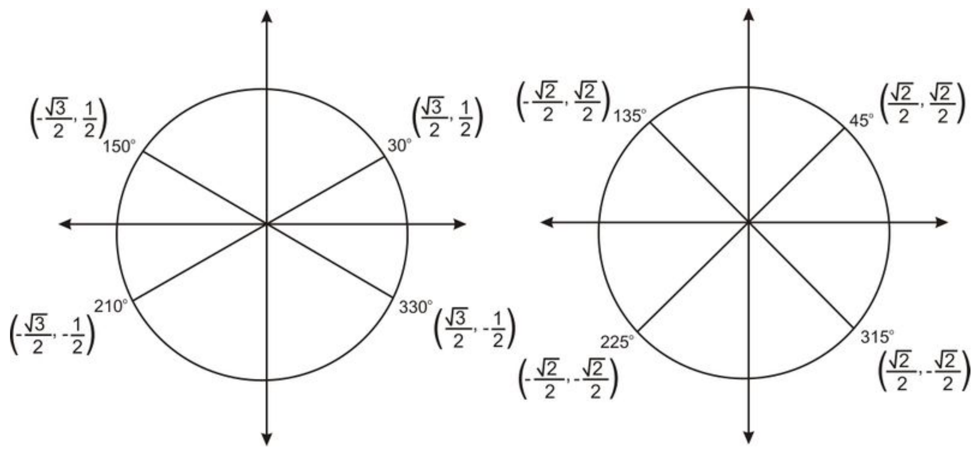
Solution: $\sin 240^\circ = \frac{-\sqrt{3}}{2}$

As we found in example 1, the reference angle for 240° is 60°. The figure below shows 60° and the three other angles in the unit circle that have 60° as a reference angle.



The terminal side of the angle 240° represents a reflection of the terminal side of 60° over both axes. So the coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The $y-$coordinate is the sine value, so $\sin 240^\circ = \frac{-\sqrt{3}}{2}$.

Just as the figure above shows 60° and three related angles, we can make similar graphs for 30° and 45°.



Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

Example 3: Find the value of $\cot 300^\circ$

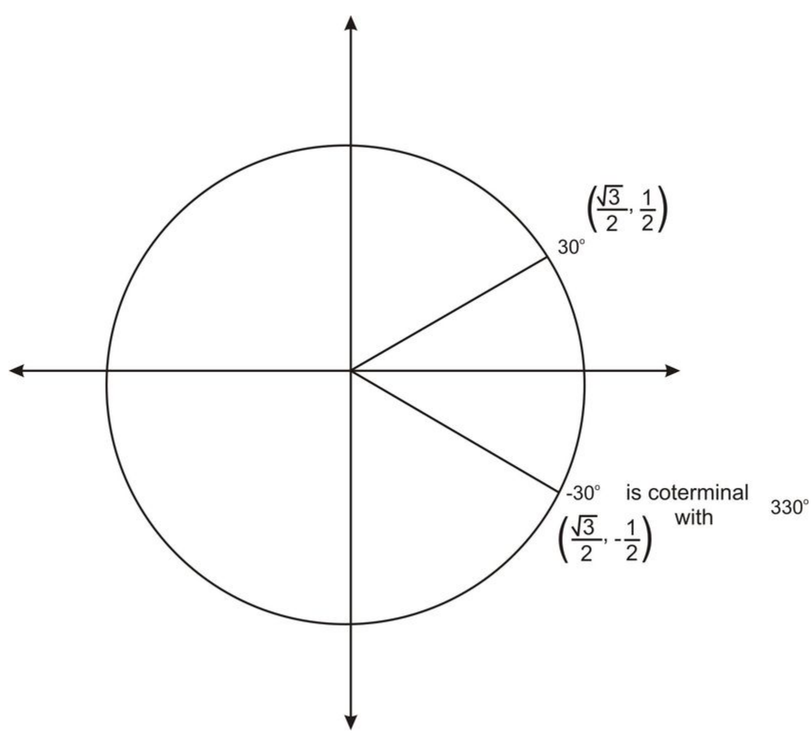
Solution: $\cot 300^\circ = -\frac{1}{\sqrt{3}}$

Using the graph above, you will find that the ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is $\cot 300 = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Trigonometric Functions of Negative Angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows -30°.



Notice that this angle is coterminal with 330°. So the ordered pair is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. We can use this ordered pair to find the values of any of the trig functions of -30°. For example, $\cos(-30^\circ) = x = \frac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.

Example 4: Find the value of each expression.

a. $\sin(-45^\circ)$

b. $\sec(-300^\circ)$

c. $\cos(-90^\circ)$

Solution:

a. $\sin(-45^\circ) = -\frac{\sqrt{2}}{2}$

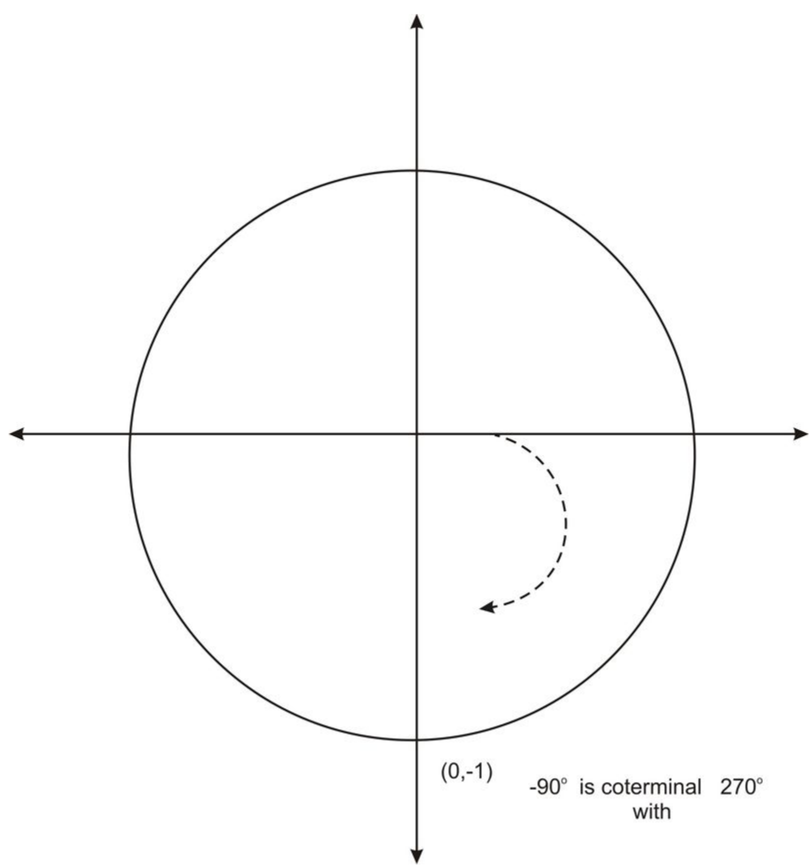
-45° is in the 4^{th} quadrant, and has a reference angle of 45°. That is, this angle is coterminal with 315°. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the sine value is $-\frac{\sqrt{2}}{2}$.

b. $\sec(-300^\circ) = 2$

The angle -300° is in the 1^{st} quadrant and has a reference angle of 60°. That is, this angle is coterminal with 60°. Therefore the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the secant value is $\frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$.

c. $\cos(-90^\circ) = 0$

The angle -90° is coterminal with 270°. Therefore the ordered pair is (0, -1) and the cosine value is 0.

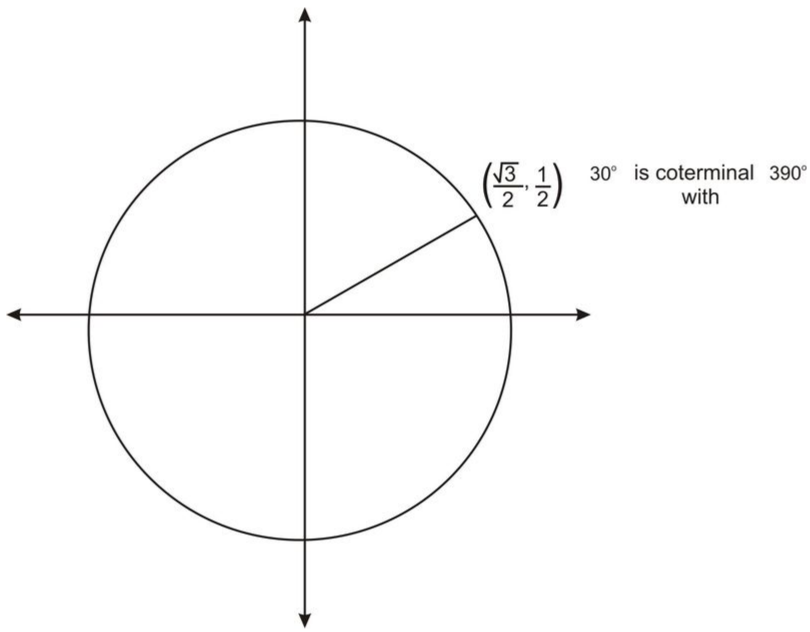


We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Trigonometric Functions of Angles Greater than 360 Degrees

Consider the angle 390°. As you learned previously, you can think of this angle as a full 360 degree rotation, plus an additional 30 degrees. Therefore 390° is coterminal with 30°. As you saw above with negative angles, this means that 390° has the same ordered pair as 30°, and so it has the same trig values. For example,

$$\cos 390^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$



In general, if an angle whose measure is greater than 360 has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can find the values of any of the trig functions of the angle. Again, determine the reference angle first.

Example 5: Find the value of each expression.

a. $\sin 420^\circ$

b. $\tan 840^\circ$

c. $\cos 540^\circ$

Solution:

a. $\sin 420^\circ = \frac{\sqrt{3}}{2}$

420° is a full rotation of 360 degrees, plus an additional 60 degrees. Therefore the angle is coterminal with 60°, and so it shares the same ordered pair, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The sine value is the $y-$coordinate.

b. $\tan 840^\circ = -\sqrt{3}$

840° is two full rotations, or 720 degrees, plus an additional 120 degrees:

$$840 = 360 + 360 + 120$$

Therefore 840° is coterminal with 120°, so the ordered pair is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The tangent value can be found by the following:

$$\tan 840^\circ = \tan 120^\circ = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \times -\frac{2}{1} = -\sqrt{3}$$

c. $\cos 540^\circ = -1$

540° is a full rotation of 360 degrees, plus an additional 180 degrees. Therefore the angle is coterminal with 180°, and the ordered pair is (-1, 0). So the cosine value is -1.

So far all of the angles we have worked with are multiples of 30, 45, 60, and 90. Next we will find approximate values of the trig functions of other angles.

Using a Calculator to Find Values

If you have a scientific calculator, you can determine the value of any trig function for any angle. Here we will focus on using a TI graphing calculator to find values.

First, your calculator needs to be in the correct "mode." In chapter 2 you will learn about a different system for measuring angles, known as radian measure. In this chapter, we are measuring angles in degrees. We need to make sure that the calculator is in degrees. To do this, press [MODE]. In the third row, make sure that Degree is highlighted. If Radian is highlighted, scroll down to this row, scroll over to Degree, and press [ENTER]. This will highlight Degree. Then press [2nd] [MODE] to return to the main screen.

Now you can calculate any value. For example, we can verify the values from the table above. To find $\sin 130^\circ$, press [Sin] [130] [ENTER]. The calculator should return the value .7660444431.

Example 6: Find the approximate value of each expression. Round your answer to 4 decimal places.

a. $\sin 130^\circ$

b. $\cos 15^\circ$

c. $\tan 50^\circ$

Solution:

a. $\sin 130^\circ \approx 0.7660$

b. $\cos 15^\circ \approx 0.9659$

c. $\tan 50^\circ \approx 1.1918$

You may have noticed that the calculator provides a "(" after the SIN. In the previous calculations, you can actually leave off the ")". However, in more complicated calculations, leaving off the closing ")" can create problems. It is a good idea to get in the habit of closing parentheses.

You can also use a calculator to find values of more complicated expressions.

Example 7: Use a calculator to find an approximate value of $\sin 25^\circ + \cos 25^\circ$. Round your answer to 4 decimal places.

Solution: $\sin 25^\circ + \cos 25^\circ \approx 1.3289$

*This is an example where you need to close the parentheses.

Points to Consider

- What is the difference between the measure of an angle, and its reference angle? In what cases are these measures the same value?
- Which angles have the same cosine value, or the same sine value? Which angles have opposite cosine and sine values?

Review Questions

1. State the reference angle for each angle.
    a. 190°
    b. -60°
    c. 1470°
    d. -135°
2. State the ordered pair for each angle.
    a. 300°
    b. -150°
    c. 405°
3. Find the value of each expression.
    a. $\sin 210^\circ$
    b. $\tan 270^\circ$
    c. $\csc 120^\circ$
4. Find the value of each expression.
    a. $\sin 510^\circ$
    b. $\cos 930^\circ$
    c. $\csc 405^\circ$
5. Find the value of each expression.
    a. $\cos -150^\circ$
    b. $\tan -45^\circ$
    c. $\sin -240^\circ$
6. Use a calculator to find each value. Round to 4 decimal places.
    a. $\sin 118^\circ$
    b. $\tan 55^\circ$
    c. $\cos 100^\circ$
7. Recall, in lesson 1.4, we introduced inverse trig functions. Use your calculator to find the measure of an angle whose sine value is 0.2.
8. In example 6c, we found that $\tan 50^\circ \approx 1.1918$. Use your knowledge of a special angle to explain why this value is reasonable. *HINT: You will need to change the tangent of this angle into a decimal.*

9. Use the table below or a calculator to explore sum and product relationships among trig functions. Consider the following functions:

$$f(x) = \sin(x+x) \text{ and } g(x) = \sin(x) + \sin(x)$$
$$h(x) = \sin(x) * \sin(x) \text{ and } j(x) = \sin(x^2)$$

Do you observe any patterns in these functions? Are there any equalities among the functions? Can you make a general conjecture about $\sin(a)+\sin(b)$ and $\sin(a+b)$ for all values of a,b? What about $\sin(a)\sin(a)$ and $\sin(a^2)$?

TABLE 1.2:

$a°$	$b°$	$\sin a + \sin b$	$\sin(a+b)$
10	30	.6736	.6428
20	60	1.2080	.9848
55	78	1.7973	.7314
122	25	1.2707	.5446
200	75	.6239	-.9962

10. Use a calculator or your knowledge of special angles to fill in the values in the table, then use the values to make a conjecture about the relationship between $(\sin a)^2$ and $(\cos a)^2$. If you use a calculator, round all values to 4 decimal places.

TABLE 1.3:

a	$(\sin a)^2$	$(\cos a)^2$
0		
25		
45		
80		
90		
120		
250		

1.8 Relating Trigonometric Functions

Learning Objectives

- State the reciprocal relationships between trig functions, and use these identities to find values of trig functions.
- State quotient relationships between trig functions, and use quotient identities to find values of trig functions.
- State the domain and range of each trig function.
- State the sign of a trig function, given the quadrant in which an angle lies.
- State the Pythagorean identities and use these identities to find values of trig functions.

Reciprocal identities

The first set of identities we will establish are the reciprocal identities. A **reciprocal** of a fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.

First, consider the definition of the sine function for angles of rotation: $\sin\theta = \frac{y}{r}$. Now consider the cosecant function: $\csc\theta = \frac{r}{y}$. In the unit circle, these values are $\sin\theta = \frac{y}{1} = y$ and $\csc\theta = \frac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin\theta = \frac{1}{2}$, then $\csc\theta = \frac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{or} \quad \cos\theta = \frac{1}{\sec\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \quad \text{or} \quad \tan\theta = \frac{1}{\cot\theta}$$

Example 1: Find the value of each expression using a reciprocal identity.

a. $\cos\theta = .3, \sec\theta = ?$

b. $\cot\theta = \frac{4}{3}, \tan\theta = ?$

Solution:

a. $\sec\theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos\theta = .3$, then $\sec\theta = \frac{1}{.3}$. It is easier to find the reciprocal if we express the values as fractions: $\cos\theta = .3 = \frac{3}{10} \Rightarrow \sec\theta = \frac{10}{3}$.

b. $\tan\theta = \frac{3}{4}$

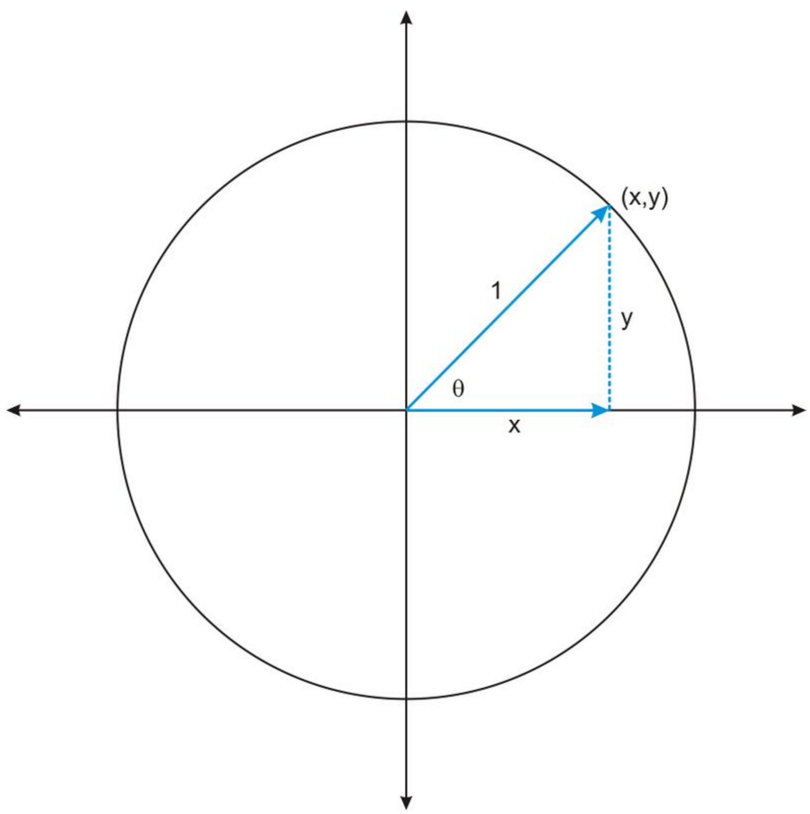
These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the domain and range of functions.

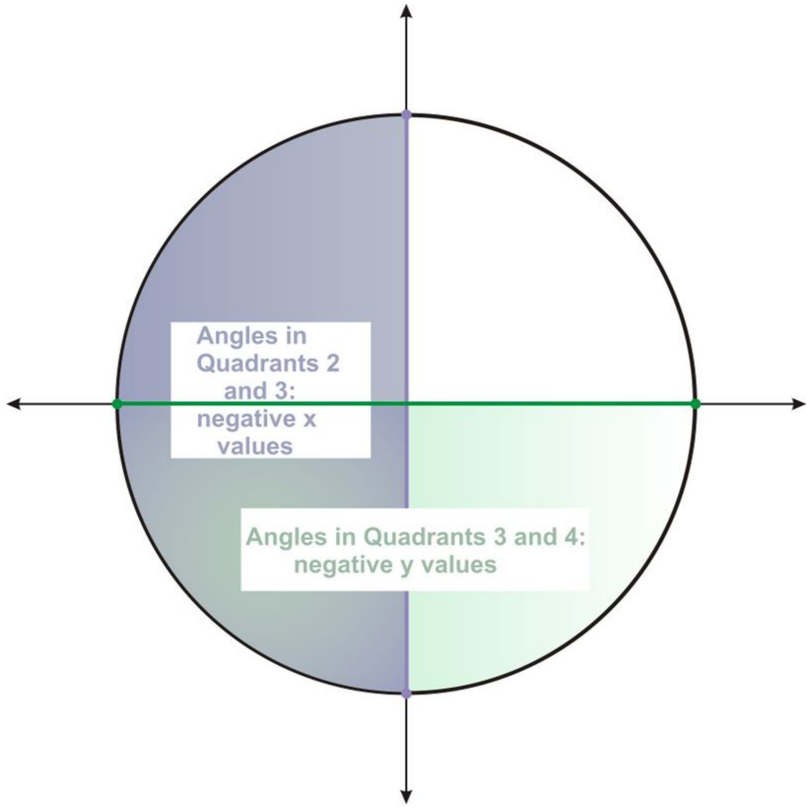
Domain, Range, and Signs of Trig Functions

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of "input" and "output." The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.

Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous sections, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, R. We can determine the range of the functions if we think about the fact that the sine of an angle is the $y-$coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the $x-$coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.



In this right triangle, x and y are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however, contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative $y-$coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative $x-$coordinate.



In either case, the minimum value is -1. For example, $\cos 180^\circ = -1$ and $\sin 270^\circ = -1$. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

TABLE 1.4:

	Domain	Range
Sine	$\theta = R$	$-1 \leq y \leq 1$
Cosine	$\theta = R$	$-1 \leq y \leq 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are 0°, 180°, 360°, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function in order to begin to verify this range, as well as the domain and range of the secant function.

TABLE 1.5:

	Domain	Range
Cosecant	$\theta \varepsilon R, \theta \neq 0, 180, 360 \ldots$	$\csc\theta \leq -1$ or $\csc\theta \geq 1$
Secant	$\theta \varepsilon R, \theta \neq 90, 270, 450 \ldots$	$\sec\theta \leq -1$ or $\sec\theta \geq 1$

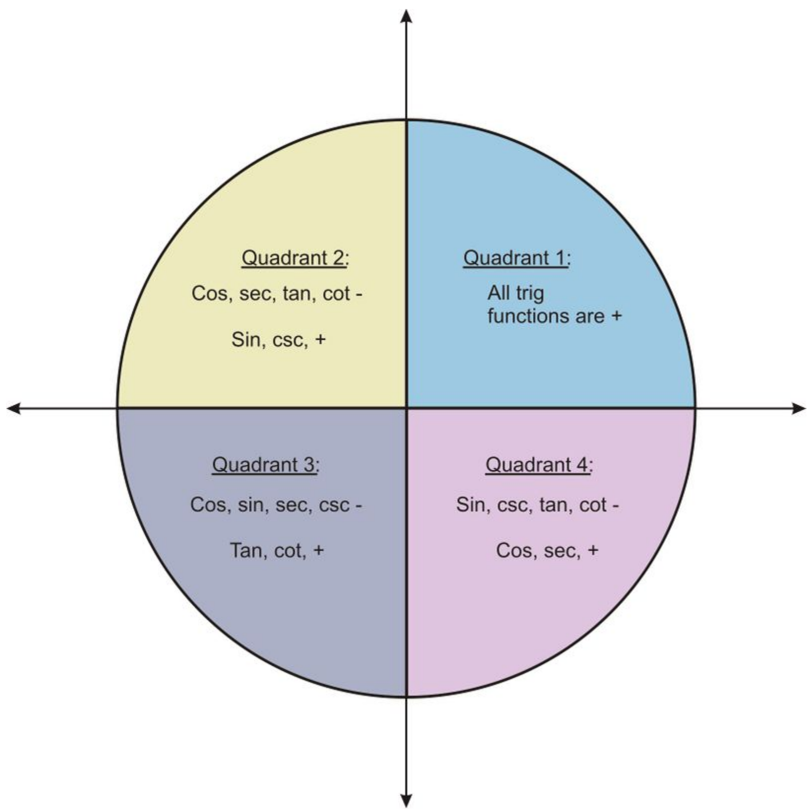
Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan\theta = \frac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an $x-$coordinate of $0 : 90^\circ, 270^\circ$, etc. The cotangent function is defined as $\cot\theta = \frac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a $y-$coordinate of $0:0^\circ$, 180°, 360°, etc.

TABLE 1.6:

Function	Domain	Range
Tangent	$\theta \varepsilon R, \theta \neq 90, 270, 450\ldots$	All reals
Cotangent	$\theta \varepsilon R, \theta \neq 0, 180, 360\ldots$	All reals

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.



An easy way to remember this is "All Students Take Calculus." Quadrant I: All values are positive, Quadrant II: Sine is positive, Quadrant III: Tangent is positive, and Quadrant IV: Cosine is positive. This simple memory device will help you remember which trig functions are positive and where.

Example 2: State the sign of each expression.

a. $\cos 100^\circ$

b. $\csc 220^\circ$

c. $\tan 370^\circ$

Solution:

a. The angle 100° is in the second quadrant. Therefore the $x-$coordinate is negative and so $\cos 100^\circ$ is negative.

b. The angle 220° is in the third quadrant. Therefore the $y-$coordinate is negative. So the sine, and the cosecant are negative.

c. The angle 370° is in the first quadrant. Therefore the tangent value is positive.

So far we have considered relationships between pairs of functions: the six trig functions can be grouped in pairs as reciprocals. Now we will consider relationships among three trig functions.

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities above. They also lead us to another set of identities, the quotient identities.

Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta.$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Example 3: If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

Solution: $\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

Example 4: Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Solution:

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

This is also an identity that you can use to find the value of the cotangent function, given values of sine and cosine. Both of the quotient identities will also be useful in chapter 3, in which you will prove other identities.

Cofunction Identities and Reflection

These identities relate to the problems you did in section 1.3. Recall, #3 and #4 from the review questions, where $\sin X = \cos Z$ and $\cos X = \sin Z$, where X and Z were complementary angles. These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$
$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

Example 5: Find the value of each trig function.

a. $\cos 120^\circ$

b. $\cos(-120^\circ)$

c. $\sin 135^\circ$

d. $\sin(-135^\circ)$

Solution: Because these angles have reference angles of 60° and 45°, the values are:

a. $\cos 120^\circ = -\frac{1}{2}$

b. $\cos(-120^\circ) = \cos 240^\circ = -\frac{1}{2}$

c. $\sin 135^\circ = \frac{\sqrt{2}}{2}$

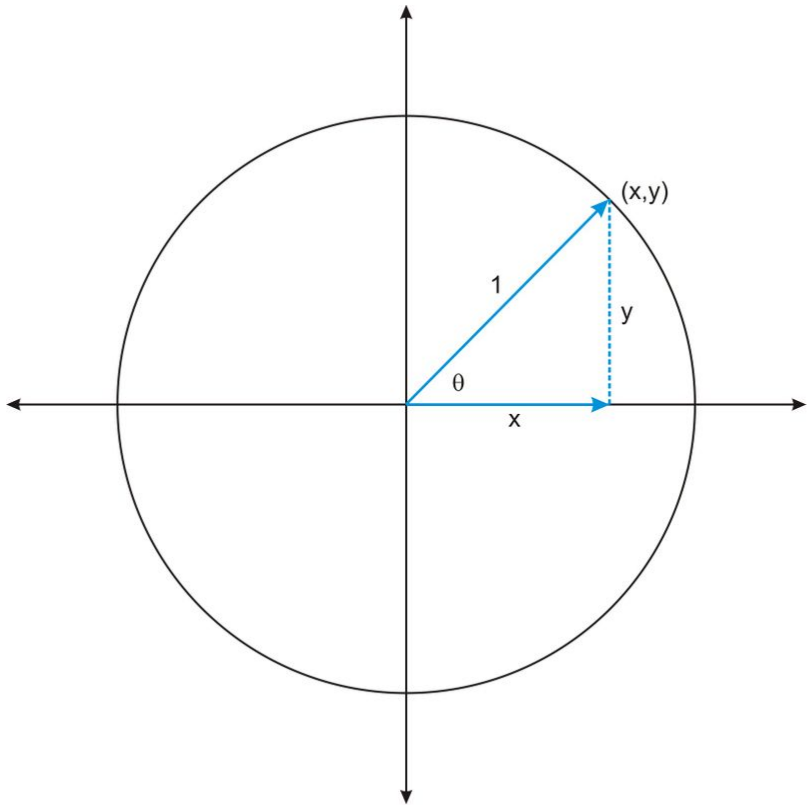
d. $\sin(-135^\circ) = \sin 225^\circ = -\frac{\sqrt{2}}{2}$

These values show us that sine and cosine also reflect over the x axis. This allows us to generate three more identities.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

Pythagorean Identities

The final set of identities are called the Pythagorean Identities because they rely on the Pythagorean Theorem. In previous lessons we used the Pythagorean Theorem to find the sides of right triangles. Consider once again the way that we defined the trig functions in 1.3. Let's look at the unit circle:



The legs of the right triangle are x, and y. The hypotenuse is 1. Therefore the following equation is true for all x and y on the unit circle:

$$x^2 + y^2 = 1$$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Note: Writing the exponent 2 after the cos and sin is the standard way of writing exponents. Just keeping mind that $\cos^2\theta$ *means* $(\cos\theta)^2$ *and* $\sin^2\theta$ *means* $(\sin\theta)^2$.

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example 6: If $\cos\theta = \frac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \frac{\sqrt{15}}{4}$

$$\begin{aligned}
\cos^2\theta+\sin^2\theta &= 1\\
\left(\frac{1}{4}\right)^2+\sin^2\theta &= 1\\
\frac{1}{16}+\sin^2\theta &= 1\\
\sin^2\theta &= 1-\frac{1}{16}\\
\sin^2\theta &= \frac{15}{16}\\
\sin\theta &= \pm\sqrt{\frac{15}{16}}\\
\sin\theta &= \pm\frac{\sqrt{15}}{4}
\end{aligned}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \frac{\sqrt{15}}{4}$.

Example 7: Use the identity $\cos^2\theta+\sin^2\theta = 1$ to show that $\cot^2\theta+1 = \csc^2\theta$

Solution:

$$\begin{aligned}
\cos^2\theta+\sin^2\theta &= 1 && \text{Divide both sides by} \sin^2\theta.\\
\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\
\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} && \frac{\sin^2\theta}{\sin^2\theta}=1\\
\frac{\cos^2\theta}{\sin^2\theta}+1 &= \frac{1}{\sin^2\theta}\\
\frac{\cos\theta}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}+1 &= \frac{1}{\sin\theta}\times\frac{1}{\sin\theta} && \text{Write the squared functions in terms of their factors.}\\
\cot\theta\times\cot\theta+1 &= \csc\theta\times\csc\theta && \text{Use the quotient and reciprocal identities.}\\
\cot^2\theta+1 &= \csc^2\theta && \text{Write the functions as squared functions.}
\end{aligned}$$

Points to Consider

1. How do you know if an equation is an identity? *HINT: you could consider using a the calculator and graphing a related function, or you could try to prove it mathematically.*
2. How can you verify the domain or range of a function?

Review Questions

1. Use reciprocal identities to give the value of each expression.
   a. $\sec\theta = 4, \cos\theta = ?$
   b. $\sin\theta = \frac{1}{3}, \csc\theta = ?$
2. In the lesson, the range of the cosecant function was given as: $\csc\theta \leq -1$ or $\csc\theta \geq 1$.
   a. Use a calculator to fill in the table below. Round values to 4 decimal places.
   b. Use the values in the table to explain in your own words what happens to the values of the cosecant function as the measure of the angle approaches 0 degrees.
   c. Explain what this tells you about the range of the cosecant function.
   d. Discuss how you might further explore values of the sine and cosecant to better understand the range of the cosecant function.

TABLE 1.7:

Angle	Sin	Csc
10		
5		
1		
0.5		
0.1		
0		
-.1		
-.5		
-1		
-5		
-10		

3. In the lesson the domain of the secant function were given: Domain: $\theta \varepsilon^\circ, \theta \neq 90, 270, 450\ldots$ Explain why certain values are excluded from the domain.
4. State the quadrant in which each angle lies, and state the sign of each expression
   a. $\sin 80^\circ$
   b. $\cos 200^\circ$
   c. $\cot 325^\circ$
   d. $\tan 110^\circ$
5. If $\cos\theta = \frac{6}{10}$ and $\sin\theta = \frac{8}{10}$, what is the value of $\tan\theta$?
6. Use quotient identities to explain why the tangent and cotangent function have positive values for angles in the third quadrant.
7. If $\sin\theta = 0.4$, what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.
8. If $\cot\theta = 2$, what is the value of $\csc\theta$? Assume that θ is an angle in the first quadrant.
9. Show that $1 + \tan^2\theta = \sec^2\theta$.
10. Explain why it is necessary to state the quadrant in which the angle lies for problems such as #7.

Chapter Summary

In this chapter students learned about right triangles and special right triangles. Through the special right triangles and the Pythagorean Theorem, the study of trigonometry was discovered. Sine, cosine, tangent, secant, cosecant, and cotangent are all functions of angles and the result is the ratio of the sides of a right triangle. We learned that only our special right triangles generate sine, cosine, tangent values that can be found without the use of a scientific calculator. When incorporating the trig ratios and the Pythagorean Theorem, we discovered the first of many trig identities. This concept will be explored further in Chapter 3.

Vocabulary

Adjacent
A side adjacent to an angle is the side next to the angle. In a right triangle, it is the leg that is next to the angle.

Angle of depression
The angle between the horizontal line of sight, and the line of sight down to a given point.

Angle of elevation
The angle between the horizontal line of sight, and the line of sight up to a given point.

Bearings
The direction from one object to another, usually measured as an angle.

Clinometer
A device used to measure angles of elevation or depression.

Coterminal angles
Two angles in standard position are coterminal if they share the same terminal side.

Distance Formula
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Hypotenuse
The hypotenuse is the longest side in a right triangle, opposite the right angle.

Identity
An identity is an equation that is always true, as long as the variables and expressions involved are defined.

Included Angle
The angle inbetween two sides of a polygon.

Leg
The legs of a right triangle are the two shorter sides.

Nautical Mile
A nautical mile is a unit of length that corresponds approximately to one minute of latitude along any meridian. A nautical mile is equal to 1.852 meters.

Pythagorean Theorem
$a^2 + b^2 = c^2$

Pythagorean Triple
A set whole numbers for which the Pythagorean Theorem holds true.

Quadrantal angle
A quadrantal angle is an angle in standard position whose terminal side lies on an axis.

Radius
The radius of a circle is the distance from the center of the circle to the edge. The radius defines the circle.

Reference angle
The reference angle of an angle in standard position is the measure of the angle between the terminal side and the closest portion of the $x-$axis.

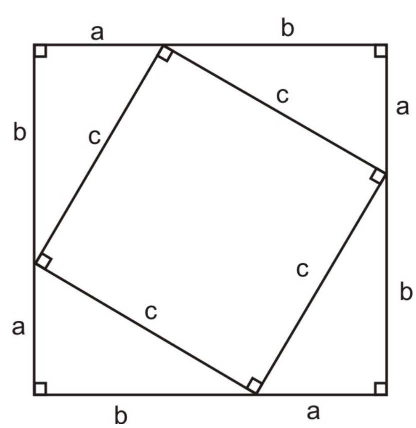
Standard position
An angle in standard position has its initial side on the positive $x-$axis, its vertex at the origin, and its terminal side anywhere in the plane. A positive angle means a counterclockwise rotation. A negative angle means a clockwise rotation.

Theodolite
A device used to measure angles of elevation or depression.

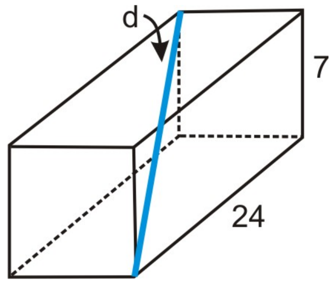
Unit Circle
The unit circle is the circle with radius 1 and center (0, 0). The equation of the unit circle is $x^2 + y^2 = 1$

Review Questions

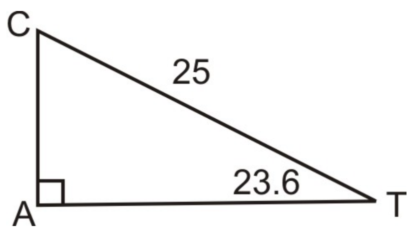
1. One way to prove the Pythagorean Theorem is through the picture below. Determine the area of the square two different ways and set each equal to each other.



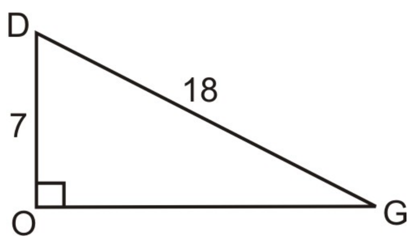
2. A flute is resting diagonally, d, in the rectangular box (prism) below. Find the length of the flute.



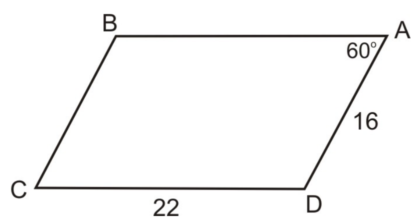
3. Solve the right triangle.



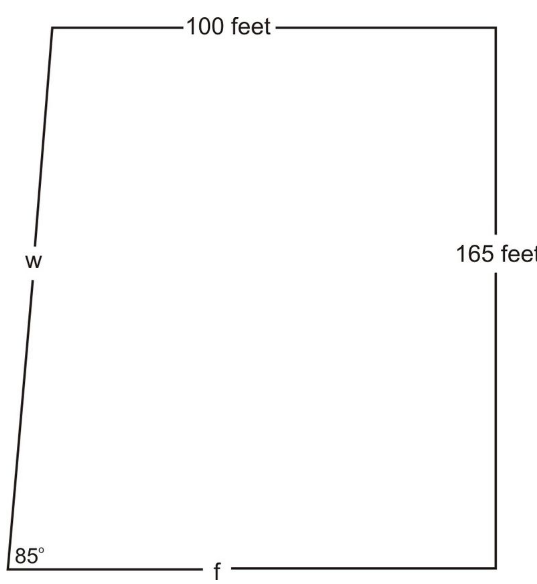
4. Solve the right triangle.



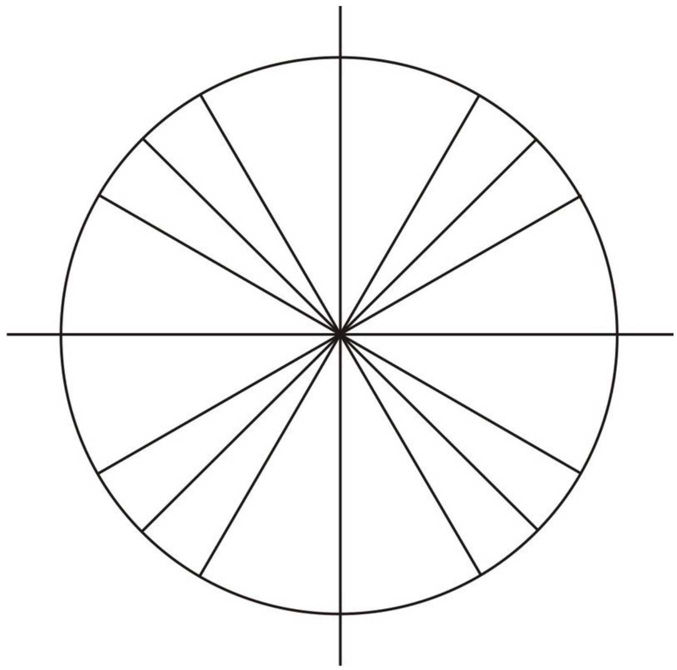
5. Find the *exact* value of the area of the parallelogram below.



6. The modern building shown below is built with an outer wall (shown on the left) that is not at a 90-degree angle with the floor. The wall on the right is perpendicular to both the floor and ceiling. Find the length of w.



7. Given that $\cos(90^\circ - x) = \frac{2}{7}$, find the $\sin x$.
8. If $\cos(-x) = \frac{3}{4}$ and $\tan x = \frac{\sqrt{7}}{3}$, find $\sin(-x)$.
9. If $\sin y = \frac{1}{3}$, what is $\cos y$?
10. $\sin\theta = \frac{1}{3}$ find the value(s) of $\cos\theta$.
11. $\cos\theta = -\frac{2}{5}$, and θ is a second quadrant angle. Find the exact values of remaining trigonometric functions.
12. (3, -4) is a point on the terminal side of θ. Find the exact values of the six trigonometric functions.
13. Determine reference angle and two coterminal angles for 165°. Plot the angle in standard position.
14. It is very helpful to have the unit circle with all the special values on one circle. Fill out the unit circle below with all of the endpoints for each special value and quadrantal value.



Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9699 .

CHAPTER 2 Graphing Trigonometric Functions

Chapter Outline

2.1 RADIAN MEASURE

2.2 APPLICATIONS OF RADIAN MEASURE

2.3 CIRCULAR FUNCTIONS OF REAL NUMBERS

2.4 TRANSLATING SINE AND COSINE FUNCTIONS

2.5 AMPLITUDE, PERIOD AND FREQUENCY

2.6 GENERAL SINUSOIDAL GRAPHS

2.7 GRAPHING TANGENT, COTANGENT, SECANT, AND COSECANT

2.8 REFERENCES

2.1 Radian Measure

Introduction

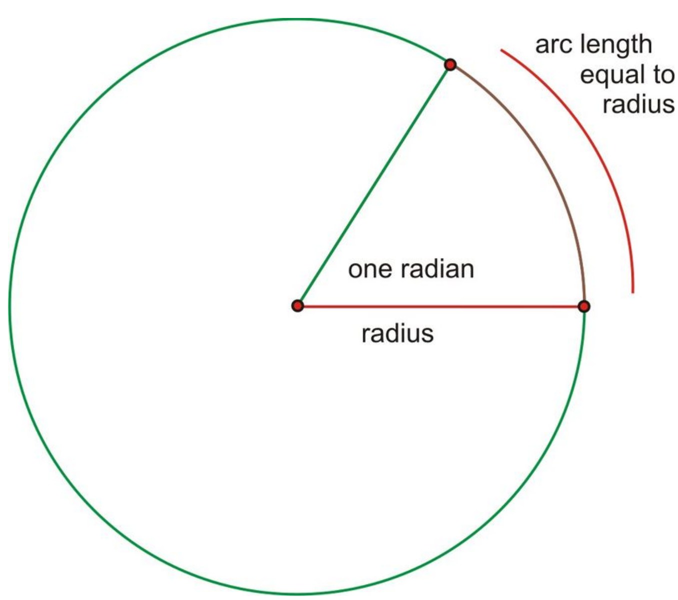
Now that we know how to find the sine, cosine and tangent of any angle, we can extend this concept to the $x-y$ plane. First, we need to derive a different way to measure angles, called radians. Radians are much like arc length. This way, we can take the "length" of a degree measurement and plot it like x. Then, the value of the function is the y value on the graph. In this manner we will be able to see the six trigonometric functions in a whole new way.

Learning Objectives

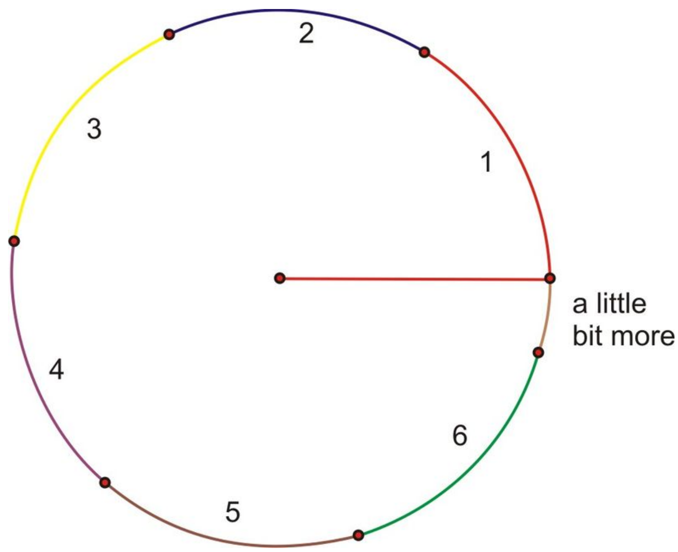
- Define radian measure.
- Convert angle measure from degrees to radians and radians to degrees.
- Calculate the values of the 6 trigonometric functions for special angles in terms of radians or degrees.

Understanding Radian Measure

Until now, we have used degrees to measure angles. But, what exactly is a degree? A **degree** is $\frac{1}{360^{th}}$ of a complete rotation around a circle. **Radians** are alternate units used to measure angles in trigonometry. Just as it sounds, a radian is based on the *radius* of a circle. One **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle. Because a radian is based on an actual part of the circle rather than an arbitrary division, it is a much more natural unit of angle measure for upper level mathematics.



What if we were to rotate all the way around the circle? Continuing to add radius lengths, we find that it takes a little more than 6 of them to complete the rotation.



Recall from geometry that the arc length of a complete rotation is the circumference, where the formula is equal to 2π times the length of the radius. 2π is approximately 6.28, so the circumference is a little more than 6 radius lengths. Or, in terms of radian measure, a complete rotation (360 degrees) is 2π radians.

$$360 \text{ degrees} = 2\pi \text{ radians}$$

With this as our starting point, we can find the radian measure of other angles. Half of a rotation, or 180 degrees, must therefore be π radians, and 90 degrees must be $\frac{1}{2}\pi$, written $\frac{\pi}{2}$.

Example 1: Find the radian measure of these angles.

TABLE 2.1:

Angle in Degrees	Angle in Radians
90	$\frac{\pi}{2}$
45	
30	
60	
75	

Solution: Because 45 is half of 90, half of $\frac{1}{2}\pi$ is $\frac{1}{4}\pi$. 30 is one-third of a right angle, so multiplying gives:

$$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$$

and because 60 is twice as large as 30:

$$2 \times \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Here is the completed table:

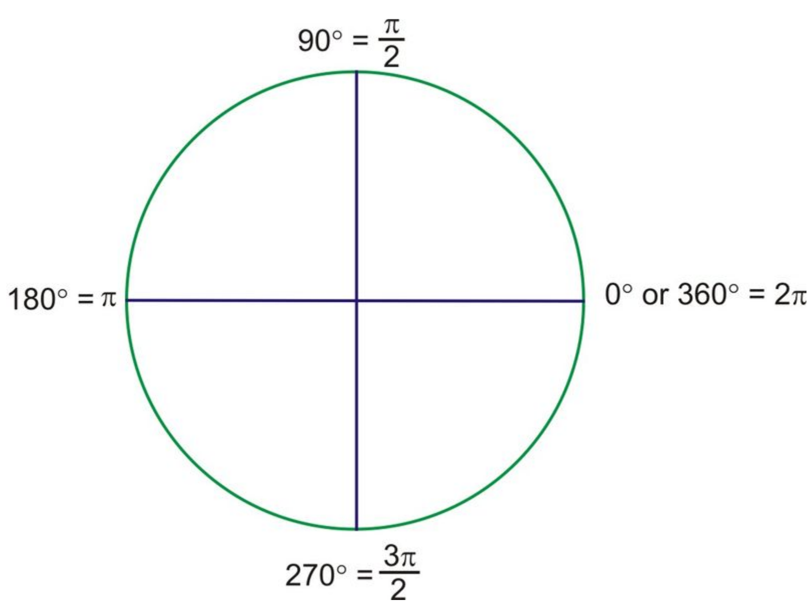
TABLE 2.2:

Angle in Degrees	Angle in Radians
90	$\frac{\pi}{2}$
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$
60	$\frac{\pi}{3}$

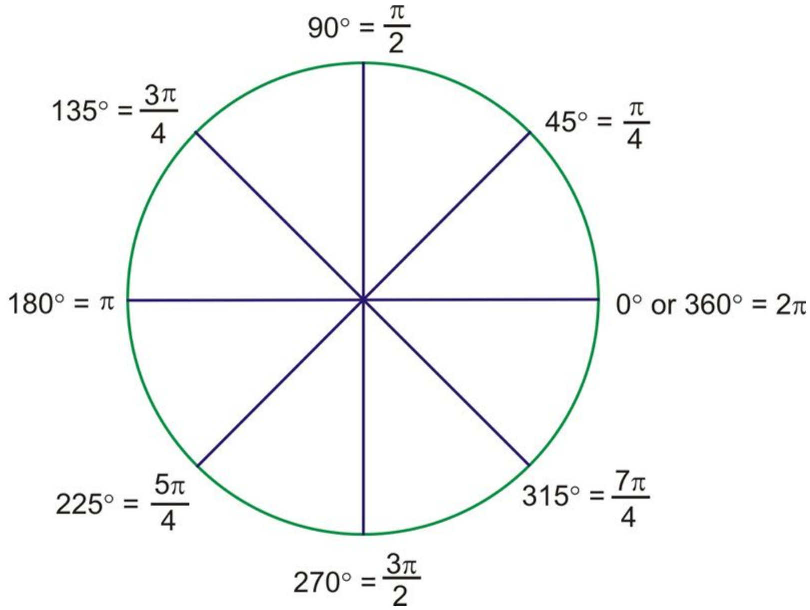
There is a formula to convert between radians and degrees that you may already have discovered while doing this example. However, many angles that are commonly used can be found easily from the values in this table. For example, most students find it easy to remember 30 and 60. 30 is π over ***6*** and 60 is π over ***3***. Knowing these angles, you can find any of the special angles that have reference angles of 30 and 60 because they will all have the same denominators. The same is true of multiples of $\frac{\pi}{4}$ (45 degrees) and $\frac{\pi}{2}$ (90 degrees).

Critical Angles in Radians

Extending the radian measure past the first quadrant, the quadrantal angles have been determined, except 270°. Because 270° is halfway between 180°, π, and 360°, 2π, it must be 1.5π, usually written $\frac{3\pi}{2}$.

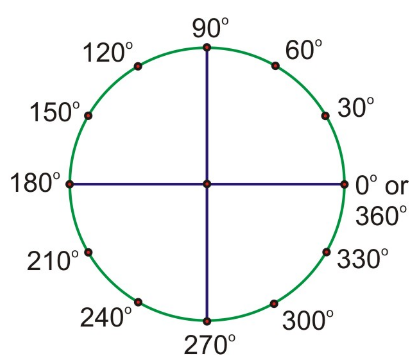


For the 45° angles, the radians are all multiples of $\frac{\pi}{4}$. For example, 135° is $3 \cdot 45°$. Therefore, the radian measure should be $3 \cdot \frac{\pi}{4}$, or $\frac{3\pi}{4}$. Here are the rest of the multiples of 45°, in radians:

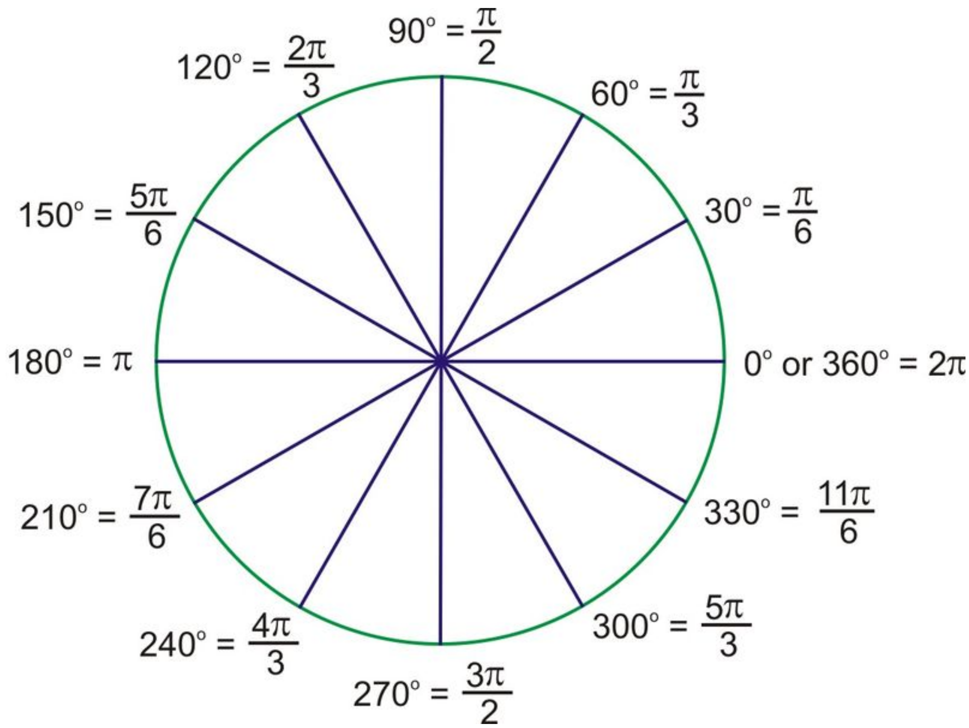


Notice that the additional angles in the drawing all have reference angles of 45 degrees and their radian measures are all multiples of $\frac{\pi}{4}$. All of the even multiples are the quadrantal angles and are reduced, just like any other fraction.

Example 2: Complete the following radian measures by counting in multiples of $\frac{\pi}{3}$ and $\frac{\pi}{6}$:



Solution:



Notice that all of the angles with 60-degree reference angles are multiples of $\frac{\pi}{3}$, and all of those with 30-degree reference angles are multiples of $\frac{\pi}{6}$. Counting in these terms based on this pattern, rather than converting back to degrees, will help you better understand radians.

Converting Any Degree to Radians

For all examples there is a conversion formula. This formula works for all degrees and radians. Remember that: π radians $= 180^\circ$. If you divide both sides of this equation by π, you will have the conversion formula:

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

If we have a degree measure and wish to convert it to radians, then manipulating the equation above gives:

$$\text{degrees} \times \frac{\pi}{180} = \text{radians}$$

Example 3: Convert $\frac{11\pi}{3}$ to degree measure.

From the last section, you should recognize that this angle is a multiple of $\frac{\pi}{3}$ (or 60 degrees), so there are 11, $\frac{\pi}{3}$'s in this angle, $\frac{\pi}{3} \times 11 = 60^\circ \times 11 = 660^\circ$.

Here is what it would look like using the formula:

radians $\times \frac{180}{\pi} =$ degrees

$$\frac{11\cancel{\pi}}{\cancel{3}} \times \frac{\cancel{180}^{\,60}}{\cancel{\pi}} = 11 \times 60 = 660^\circ$$

Example 4: Convert -120° to radian measure. Leave the answer in terms of π.

$$\text{degrees} \times \frac{\pi}{180} = \text{radians}$$
$$-120^\circ \times \frac{\pi}{180} = \frac{-120^\circ \pi}{180}$$

and reducing to lowest terms gives us $-\frac{2\pi}{3}$

You could also have noticed that 120 is 2×60. Since 60° is $\frac{\pi}{3}$ radians, then 120 is 2, $\frac{\pi}{3}$'s, or $\frac{2\pi}{3}$. Make it negative and you have the answer, $-\frac{2\pi}{3}$.

Example 5: Express $\frac{11\pi}{12}$ radians terms of degrees.

$\text{radians} \times \frac{180}{\pi} = \text{degrees}$

$$\frac{11\pi}{12} \times \frac{180}{\pi} = 11 \times 15 = 165^\circ$$

Note: Sometimes students have trouble remembering if it is $\frac{180}{\pi}$ or $\frac{\pi}{180}$. It might be helpful to remember that radian measure is almost always expressed in terms of π. If you want to convert from radians to degrees, you want the π to cancel out when you multiply, so it must be in the denominator.

The Six Trig Functions and Radians

Even though you are used to performing the trig functions on degrees, they still will work on radians. The only difference is the way the problem looks. If you see $\sin \frac{\pi}{6}$, that is still $\sin 30^\circ$ and the answer is still $\frac{1}{2}$.

Example 6: Find $\tan \frac{3\pi}{4}$.

Solution: If needed, convert $\frac{3\pi}{4}$ to degrees. Doing this, we find that it is 135°. So, this is $\tan 135^\circ$, which is -1.

Example 7: Find the value of $\cos \frac{11\pi}{6}$.

Solution: If needed, convert $\frac{11\pi}{6}$ to degrees. Doing this, we find that it is 330°. So, this is $\cos 330^\circ$, which is $\frac{\sqrt{3}}{2}$.

Example 8: Convert 1 radian to degree measure.

Solution: Many students get so used to using π in radian measure that they incorrectly think that ***1*** radian means 1π radians. While it is more convenient and common to express radian measure in terms of π, don't lose sight of the fact that π radians is a number. It specifies an angle created by a rotation of approximately 3.14 radius lengths. So 1 radian is a rotation created by an arc that is only a single radius in length.

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

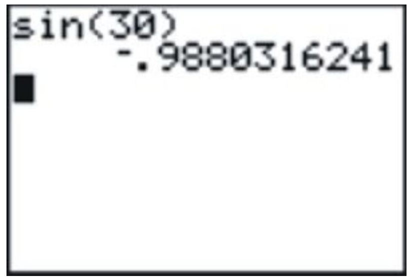
So 1 radian would be $\frac{180}{\pi}$ degrees. Using any scientific or graphing calculator will give a reasonable approximation for this degree measure, approximately 57.3°.

Example 9: Find the radian measure of an acute angle, θ, with $\sin\theta = \frac{\sqrt{2}}{2}$.

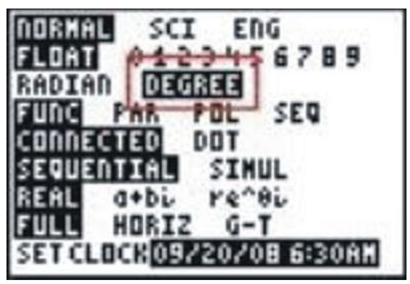
Solution: Here, we are working backwards. From last chapter, you may recognize that $\frac{\sqrt{2}}{2}$ goes with 45°. Because the example is asking for an acute angle, we just need to convert 45° to radians. 45° in radians is $\frac{\pi}{4}$.

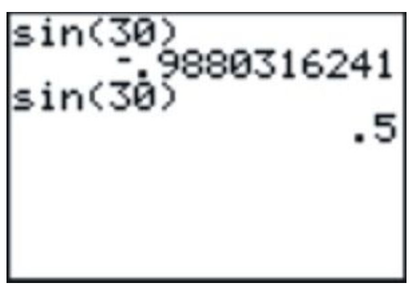
Check the Mode

Most scientific and graphing calculators have a **MODE** setting that will allow you to either convert between the two, or to find approximations for trig functions using either measure. It is important that if you are using your calculator to estimate a trig function that you know which mode you are using. Look at the following screen:



If you entered this expecting to find the sine of 30 *degrees* you would realize based on the last chapter that something is wrong because it should be $\frac{1}{2}$. In fact, as you may have suspected, the calculator is interpreting this as 30 ***radians***. In this case, changing the mode to degrees and recalculating will give the expected result.





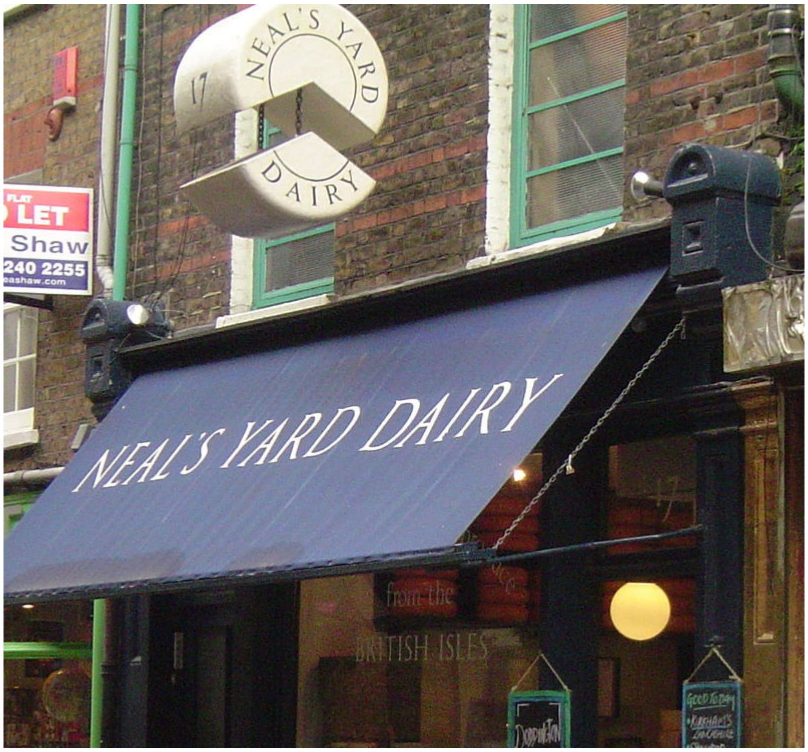
Scientific calculators will usually have a 3-letter display that shows either **DEG** or **RAD** to tell you which mode the calculator is in.

Points to Consider

- In certain cases, why are radians more useful than degrees?
- Think about the steps you would take to solve $\sin \frac{11\pi}{6}$. Are these step similar for finding any trig function for any angle in radians?

Review Questions

1. The following picture is a sign for a store that sells cheese.



   a. Estimate the degree measure of the angle of the circle that is missing.
   b. Convert that measure to radians.
   c. What is the radian measure of the part of the cheese that remains?

2. Convert the following degree measures to radians. All answers should be in terms of π.

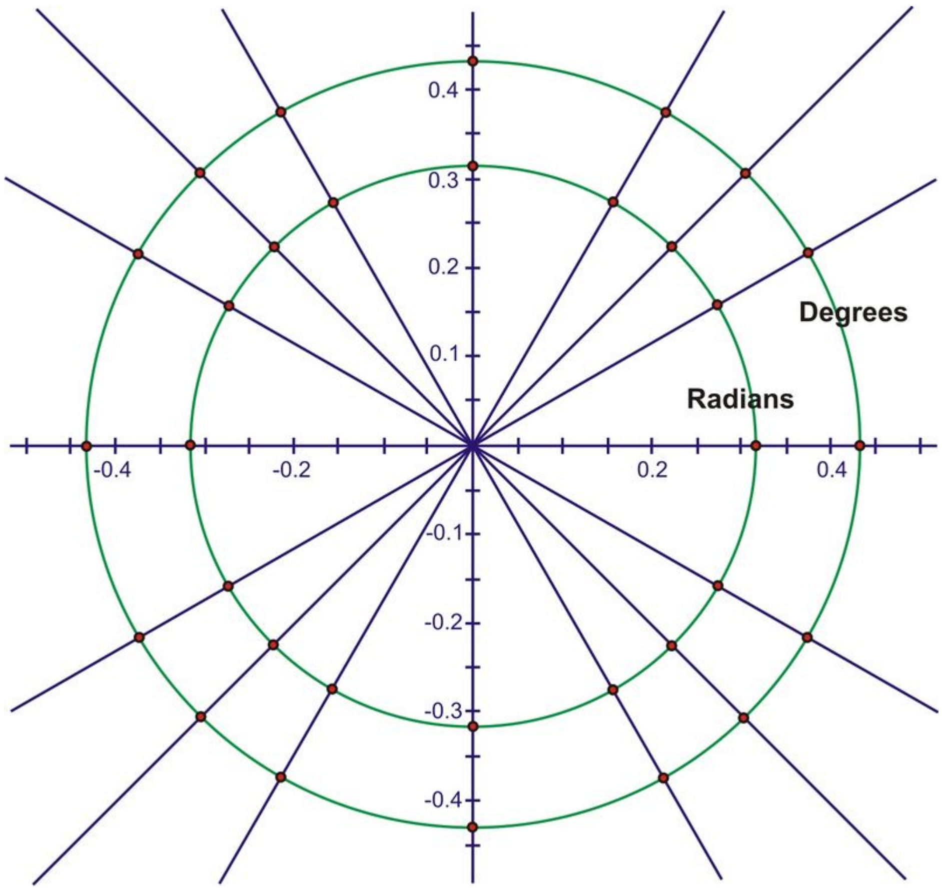
   a. $240°$
   b. $270°$
   c. $315°$
   d. $-210°$
   e. $120°$
   f. $15°$
   g. $-450°$
   h. $72°$
   i. $720°$
   j. $330°$

3. Convert the following radian measures to degrees:

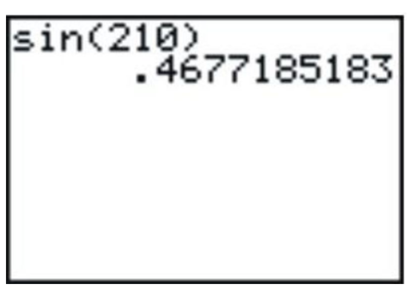
   a. $\frac{\pi}{2}$
   b. $\frac{11\pi}{5}$
   c. $\frac{2\pi}{3}$
   d. 5π
   e. $\frac{7\pi}{2}$
   f. $\frac{3\pi}{10}$
   g. $\frac{5\pi}{12}$
   h. $-\frac{13\pi}{6}$

i. $\frac{8}{\pi}$
j. $\frac{4\pi}{15}$

4. The drawing shows all the quadrant angles as well as those with reference angles of 30°, 45°, and 60°. On the inner circle, label all angles with their radian measure in terms of π and on the outer circle, label all the angles with their degree measure.



5. Using a calculator, find the approximate degree measure (to the nearest tenth) of each angle expressed in radians.
   a. $\frac{6\pi}{7}$
   b. 1 radian
   c. 3 radian
   d. $\frac{20\pi}{11}$
6. Gina wanted to calculate the $\sin 210^\circ$ and got the following answer on her calculator:



Fortunately, Kylie saw her answer and told her that it was obviously incorrect.

   a. Write the correct answer, in simplest radical form.
   b. Explain what she did wrong.

7. Complete the following chart. Write your answers in simplest radical form.

TABLE 2.3:

x	Sin(x)	Cos(x)	Tan(x)
$\frac{5\pi}{4}$			
$\frac{11\pi}{6}$			
$\frac{2\pi}{3}$			
$\frac{\pi}{2}$			
$\frac{7\pi}{2}$			

2.2 Applications of Radian Measure

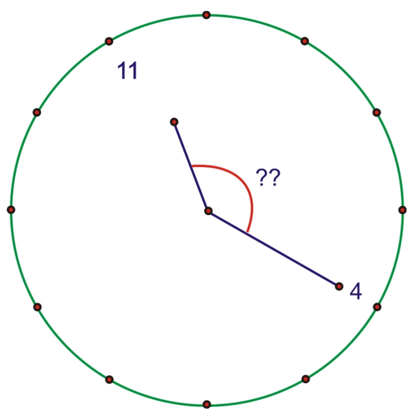
Learning Objectives

- Solve problems involving angles of rotation using radian measure.
- Calculate the length of an arc and the area of a sector.
- Approximate the length of a chord given the central angle and radius.
- Solve problems about angular speed.

Rotations

Example 1: The hands of a clock show 11:20. Express the obtuse angle formed by the hour and minute hands in radian measure.

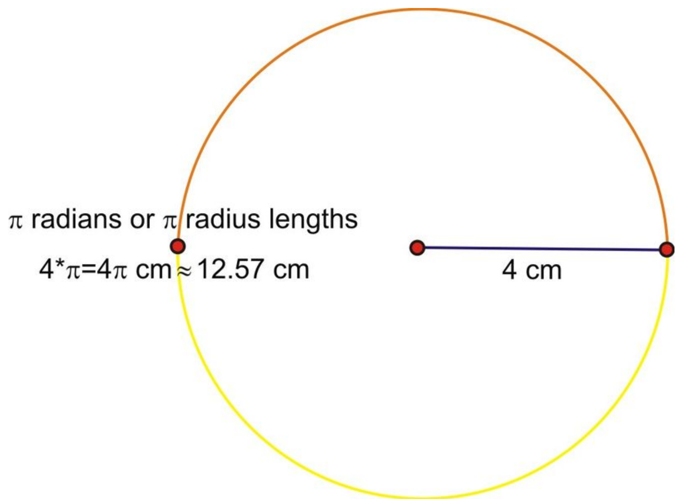
Solution: The following diagram shows the location of the hands at the specified time.



Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or 30°). So, the angle between the 12 and the 4 is $4 \times \frac{\pi}{6} = \frac{2\pi}{3}$ (or 120°). Because the rotation from 12 to 4 is one-third of a complete rotation, it seems reasonable to assume that the hour hand is moving continuously and has therefore moved one-third of the distance between the 11 and the 12. This means that the angle between the hour hand and the 12 is two-thirds of the distance between the 11 and the 12. So, $\frac{2}{3} \times \frac{\pi}{6} = \frac{2\pi}{18} = \frac{\pi}{9}$, and the total measure of the angle is therefore $\frac{\pi}{9} + \frac{2\pi}{3} = \frac{\pi}{9} + \frac{6\pi}{9} = \frac{7\pi}{9}$.

Length of Arc

The length of an arc on a circle depends on both the angle of rotation and the radius length of the circle. If you recall from the last lesson, the measure of an angle in radians is defined as the length of the arc cut off by one radius length. What if the radius is 4 cm? Then, the length of the half-circle arc would be π multiplied by the radius length, or 4π cm in length.



This results in a formula that can be used to calculate the length of any arc.

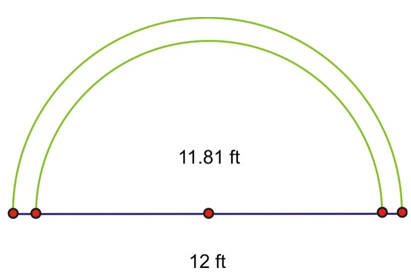
$$s = r\theta,$$

where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.

Solving this equation for θ will give us a formula for finding the radian measure given the arc length and the radius length:

$$\theta = \frac{s}{r}$$

Example 2: The free-throw line on an NCAA basketball court is 12 ft wide. In international competition, it is only about 11.81 ft. How much longer is the half circle above the free-throw line on the NCAA court?



Solution: Find both arc lengths.

NCAA	INTERNATIONAL
$s_1 = r\theta$	$s_2 = r\theta$
$s_1 = \frac{12}{2}(\pi)$	$s_2 \approx \frac{11.81}{2}(\pi)$
$s_1 = 6\pi$	$s_2 \approx 5.905\pi$

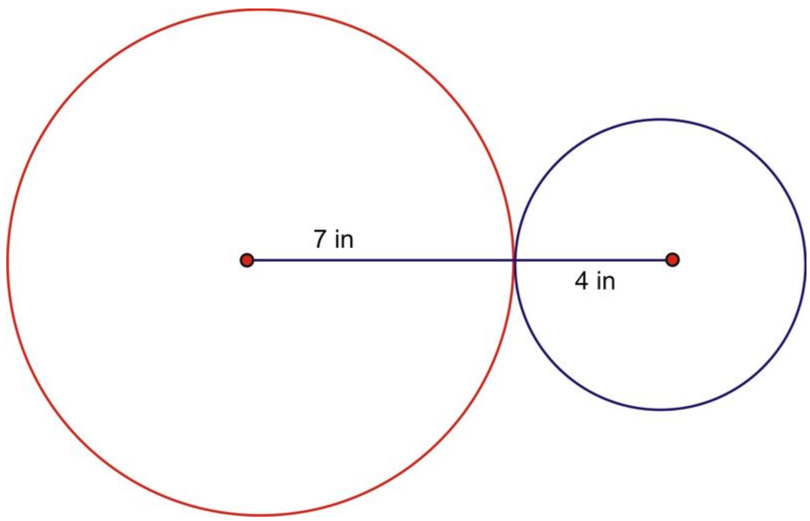
So the answer is approximately $6\pi - 5.905\pi \approx 0.095\pi$

.095π

.2984513021

This is approximately 0.3 ft, or about 3.6 inches longer.

Example 3: Two connected gears are rotating. The smaller gear has a radius of 4 inches and the larger gear's radius is 7 inches. What is the angle through which the larger gear has rotated when the smaller gear has made one complete rotation?



Solution: Because the blue gear performs one complete rotation, the length of the arc traveled is:

$$s = r\theta$$
$$s = 4 \times 2\pi$$

So, an 8π arc length on the larger circle would form an angle as follows:

$$\theta = \frac{s}{r}$$
$$\theta = \frac{8\pi}{7}$$
$$\theta \approx 3.6$$

So the angle is approximately 3.6 radians.

$$3.6 \times \frac{180}{\pi} \approx 206^\circ$$

Area of a Sector

One of the most common geometric formulas is the area of a circle:

$$A = \pi r^2$$

In terms of angle rotation, this is the area created by 2π radians.

$$2\pi \text{ rad} = \pi r^2 \text{ area}$$

A half-circle, or π radian rotation would create a section, or **sector** of the circle equal to half the area or:

$$\frac{1}{2}\pi r^2$$

So an angle of 1 radian would define an area of a sector equal to:

$$\frac{2\pi}{2\pi} = \frac{\pi r^2}{2\pi}$$

$1 = \frac{1}{2}r^2$

From this we can determine the area of the sector created by any angle, θ radians, to be:

$$A = \frac{1}{2}r^2\theta$$

Example 4: Crops are often grown using a technique called center pivot irrigation that results in circular shaped fields.

Here is a satellite image taken over fields in Kansas that use this type of irrigation system.

If the irrigation pipe is 450 m in length, what is the area that can be irrigated after a rotation of $\frac{2\pi}{3}$ radians?

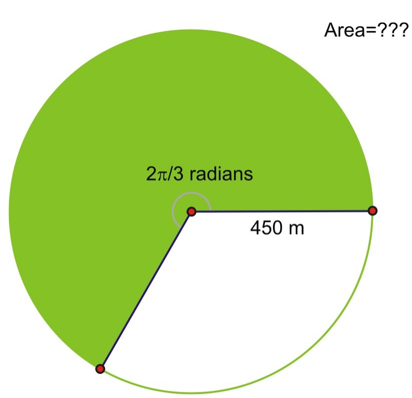
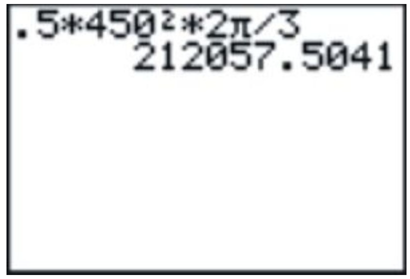


FIGURE 2.1

FIGURE 2.2

Solution: Using the formula:

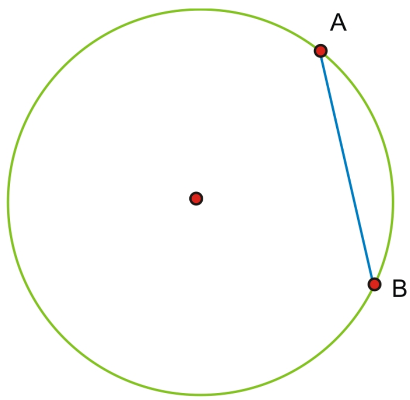
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(450)^2\left(\frac{2\pi}{3}\right)$$



The area is approximately 212,058 square meters.

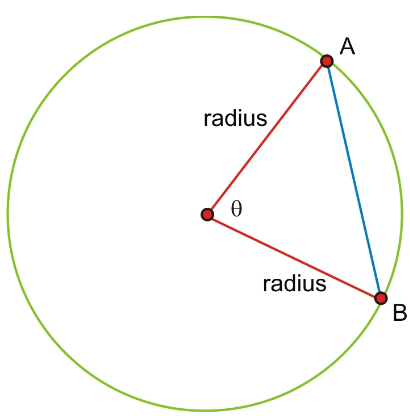
Length of a Chord

You may recall from your Geometry studies that a chord is a segment that begins and ends on a circle.

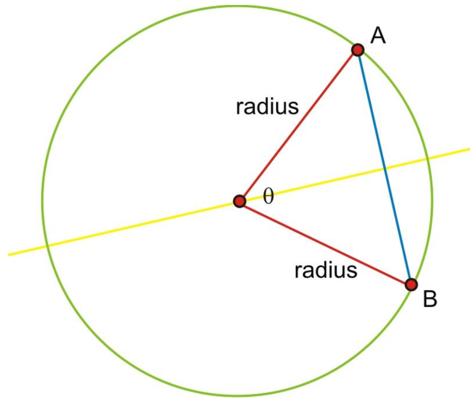


$\overline{AB}$ is a chord in the circle.

We can calculate the length of any chord if we know the angle measure and the length of the radius. Because each endpoint of the chord is on the circle, the distance from the center to A and B is the same as the radius length.



Next, if we bisect the angle, the angle bisector must be perpendicular to the chord and bisect it (we will leave the proof of this to your Geometry class). This forms a right triangle.



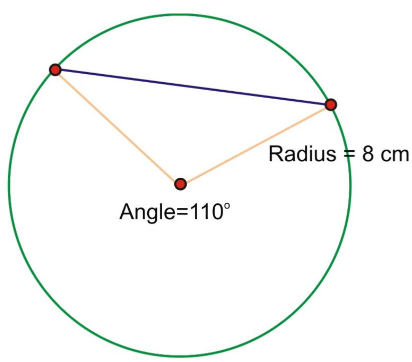
We can now use a simple sine ratio to find half the chord, called c here, and double the result to find the length of the chord.

$$\sin\frac{\theta}{2} = \frac{c}{r}$$
$$c = r \times \sin\frac{\theta}{2}$$

So the length of the chord is:

$$2c = 2r\sin\frac{\theta}{2}$$

Example 5: Find the length of the chord of a circle with radius 8 cm and a central angle of 110°. Approximate your answer to the nearest mm.



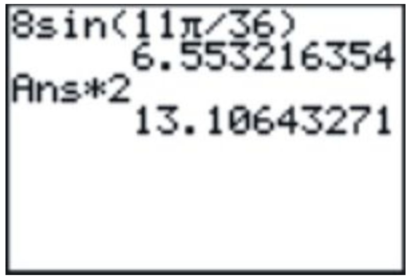
Solution: We must first convert the angle measure to radians:

$$110 \times \frac{\pi}{180} = \frac{11\pi}{18}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{11\pi}{18} \times \frac{1}{2} = \frac{11\pi}{36}$$
$$8 \times \sin\frac{11\pi}{36}$$

Multiply this result by 2.



So, the length of the arc is approximately 13.1 cm.

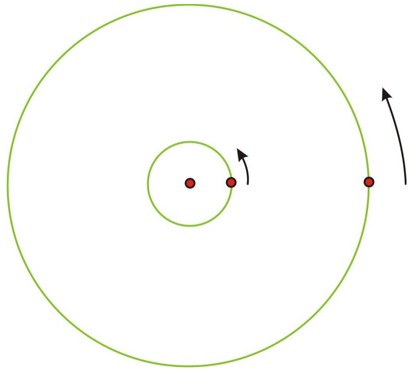
Angular Velocity

What about objects that are traveling on a circular path? Do you remember playing on a merry-go-round when you were younger?

FIGURE 2.3

If two people are riding on the outer edge, their velocities should be the same. But, what if one person is close to the center and the other person is on the edge? They are on the same object, but their speed is actually not the same.

Look at the following drawing.



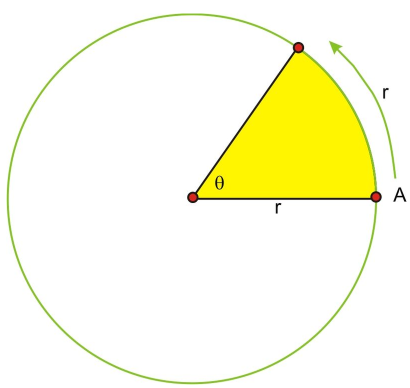
Imagine the point on the larger circle is the person on the edge of the merry-go-round and the point on the smaller circle is the person towards the middle. If the merry-go-round spins exactly once, then both individuals will also make one complete revolution in the same amount of time.

However, it is obvious that the person in the center did not travel nearly as far. The *circumference* (and of course the radius) of that circle is much smaller and therefore the person who traveled a greater distance in the same amount of time is actually traveling faster, even though they are on the same object. So the person on the edge has a greater *linear velocity* (recall that linear velocity is found using distance $=$ rate $\cdot$ time). If you have ever actually ridden on a merry-go-round, you know this already because it is much more fun to be on the edge than in the center! But, there is something about the two individuals traveling around that is the same. They will both cover the same rotation in the same period of time. This type of speed, measuring the angle of rotation over a given amount of time is called the **angular velocity.**

The formula for angular velocity is:

$$\omega = \frac{\theta}{t}$$

ω is the last letter in the Greek alphabet, omega, and is commonly used as the symbol for angular velocity. θ is the angle of rotation expressed in radian measure, and t is the time to complete the rotation.



In this drawing, θ is exactly one radian, or the length of the radius bent around the circle. If it took point A exactly 2 seconds to rotate through the angle, the *angular velocity* of A would be:

$$\omega = \frac{\theta}{t}$$
$$\omega = \frac{1}{2} \text{ radians per second}$$

In order to know the *linear speed* of the particle, we would have to know the actual distance, that is, the length of the radius. Let's say that the radius is 5 cm.

If linear velocity is $v = \frac{d}{t}$ then, $v = \frac{5}{2}$ or 2.5 cm per second.

If the angle were not exactly 1 radian, then the distance traveled by the point on the circle is the length of the arc, $s = r\theta$, or, the radius length times the measure of the angle in radians.

Substituting into the formula for linear velocity gives: $v = \frac{r\theta}{t}$ or $v = r \cdot \frac{\theta}{t}$.

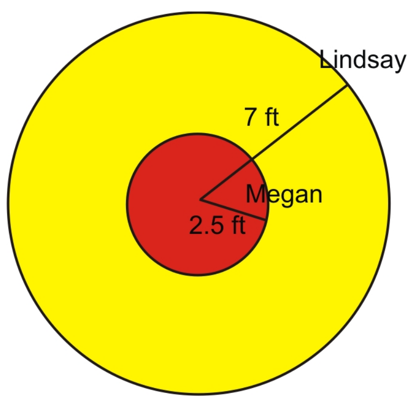
Look back at the formula for angular velocity. Substituting ω gives the following relationship between linear and angular velocity, $v = r\omega$. So, the *linear velocity* is equal to the radius times the *angular velocity.*

Remember in a unit circle, the radius is 1 unit, so in this case the linear velocity is the same as the angular velocity.

$$v = r\omega$$
$$v = 1 \times \omega$$
$$v = \omega$$

Here, the distance traveled around the circle is the same for a given unit of time as the angle of rotation, measured in radians.

Example 6: Lindsay and Megan are riding on a Merry-go-round. Megan is standing 2.5 feet from the center and Lindsay is riding on the outside edge 7 feet from the center. It takes them 6 seconds to complete a rotation. Calculate the linear *and* angular velocity of each girl.



Solution: We are told that it takes 6 seconds to complete a rotation. A complete rotation is the same as 2π radians. So the angular velocity is:

$\omega = \frac{\theta}{t} = \frac{2\pi}{6} = \frac{\pi}{3}$ radians per second, which is slightly more than 1 (about 1.05), radian per second. Because both girls cover the same angle of rotation in the same amount of time, their *angular speed* is the same. In this case they rotate through approximately 60 degrees of the circle every second.

As we discussed previously, their linear velocities are different. Using the formula, Megan's linear velocity is:

$$v = r\omega = (2.5)\left(\frac{\pi}{3}\right) \approx 2.6 \text{ ft per sec}$$

Lindsay's linear velocity is:

$$v = r\omega = (7)\left(\frac{\pi}{3}\right) \approx 7.3 \text{ ft per sec}$$

Points to Consider

- What is the difference between finding arc length and the area of a sector?
- What is the difference between linear velocity and angular velocity?
- How are linear and angular velocity related?

Review Questions

1. The following **Figure** 2.4 shows a 24-hour clock.

    a. What is the angle between each number of the clock expressed in:

        a. exact radian measure in terms of π ?
        b. to the nearest tenth of a radian?
        c. in degree measure?

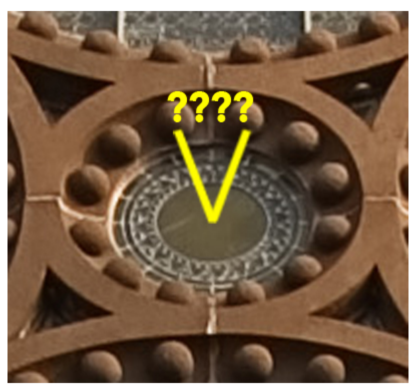
    b. Estimate the measure of the angle between the hands at the time shown in:

        a. to the nearest whole degree
        b. in radian measure in terms of π

FIGURE 2.4

2. The following picture is a window of a building on the campus of Princeton University in Princeton, New Jersey.

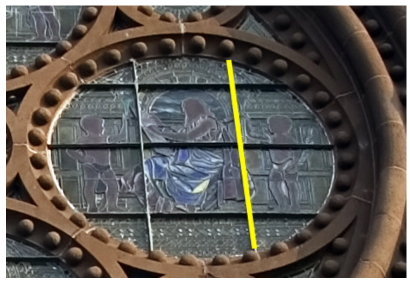
a. What is the exact radian measure in terms of π between two consecutive circular dots on the small circle in the center of the window?



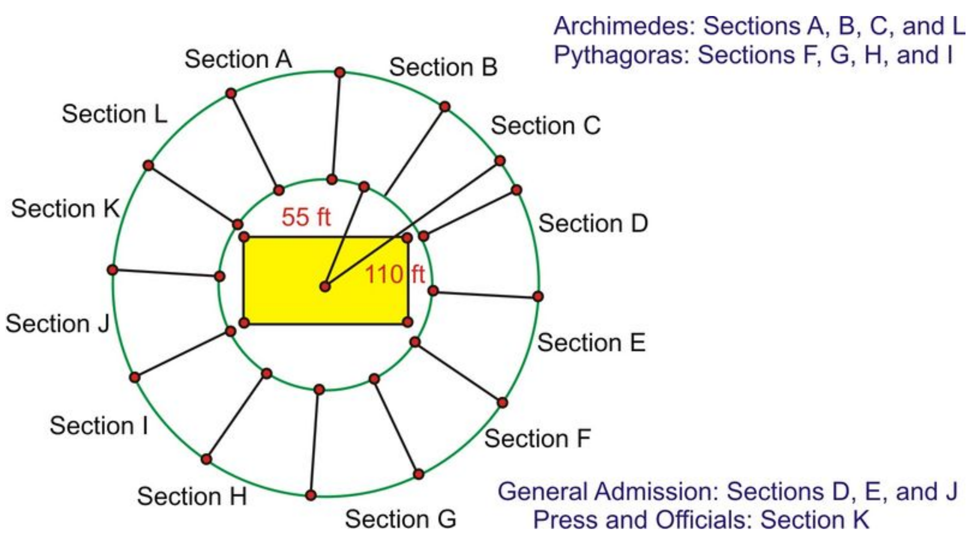
b. If the radius of this circle is about 0.5 m, what is the length of the arc between the centers of each consecutive dot? Round your answer to the nearest cm.

3. Now look at the next larger circle in the window.

a. Find the exact radian measure in terms of π between two consecutive dots in this window.
b. The radius of the glass portion of this window is approximately 1.20 m. Calculate an estimate of the length of the highlighted chord to the nearest cm. Explain the reasoning behind your solution.



4. The state championship game is to be held at Ray Diaz Memorial Arena. The seating forms a perfect circle around the court. The principal of Archimedes High School is sent the following diagram showing the seating allotted to the students at her school.



It is 55 ft from the center of the court to the beginning of the stands and 110 ft from the center to the end. Calculate the approximate number of square feet each of the following groups has been granted:

a. the students from Archimedes.
b. general admission.
c. the press and officials.

5. Doris and Lois go for a ride on a carousel. Doris rides on one of the outside horses and Lois rides on one of the smaller horses near the center. Lois' horse is 3 m from the center of the carousel, and Doris' horse is 7 m farther away from the center than Lois'. When the carousel starts, it takes them 12 seconds to complete a rotation.

a. Calculate the linear velocity of each girl.
b. Calculate the angular velocity of the horses on the carousel.

6. The Large Hadron Collider near Geneva, Switzerland began operation in 2008 and is designed to perform experiments that physicists hope will provide important information about the underlying structure of the universe. The LHC is circular with a circumference of approximately 27,000 m. Protons will be accelerated to a speed that is very close to the speed of light ($\approx 3 \times 10^8$ meters per second).
   a. How long does it take a proton to make a complete rotation around the collider?
   b. What is the approximate (to the nearest meter per second) angular speed of a proton traveling around the collider?
   c. Approximately how many times would a proton travel around the collider in one full second?

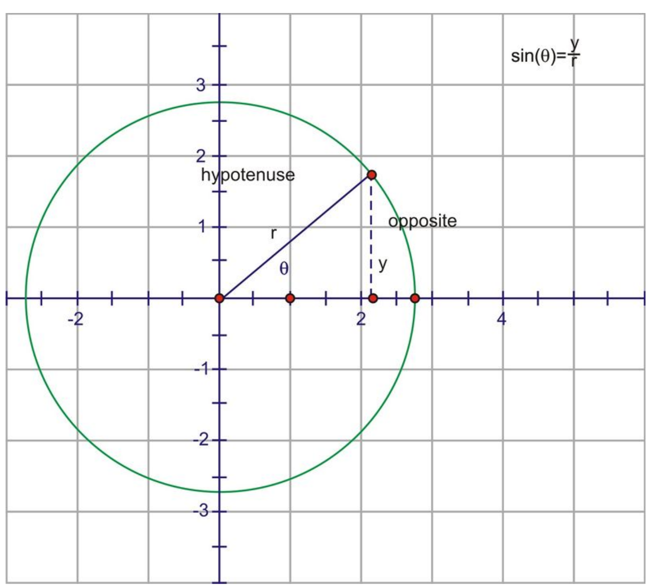
2.3 Circular Functions of Real Numbers

Learning Objectives

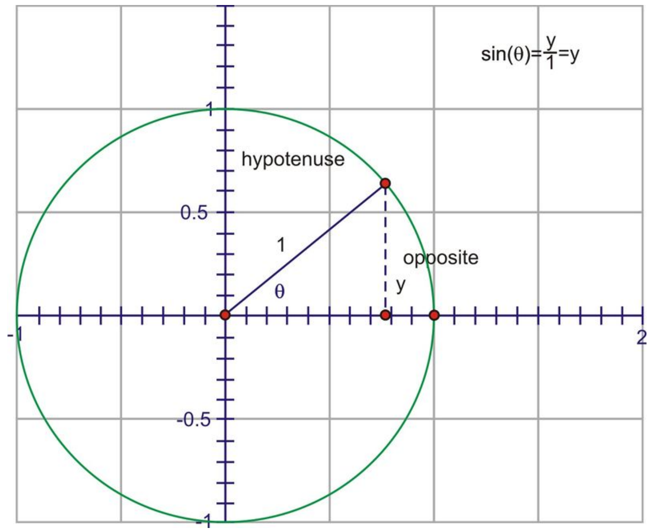
- Graph the six trigonometric ratios as functions on the Cartesian plane.
- Identify the domain and range of these six trigonometric functions.
- Identify the radian and degree measure, as well as the coordinates of points on the unit circle and graph for the critical angles.

The Sine Graph

By now, you have become very familiar with the specific values of sine, cosine, and tangents for certain angles of rotation around the coordinate grid. In mathematics, we can often learn a lot by looking at how one quantity changes as we consistently vary another. We will be looking at the sine value **as a function** of the angle of rotation around the coordinate plane. We refer to any such function as a **circular function**, because they can be defined using the unit circle. Recall from earlier sections that the sine of an angle in standard position is the ratio of $\frac{y}{r}$, where y is the $y-$coordinate of any point on the circle and r is the distance from the origin to that point.

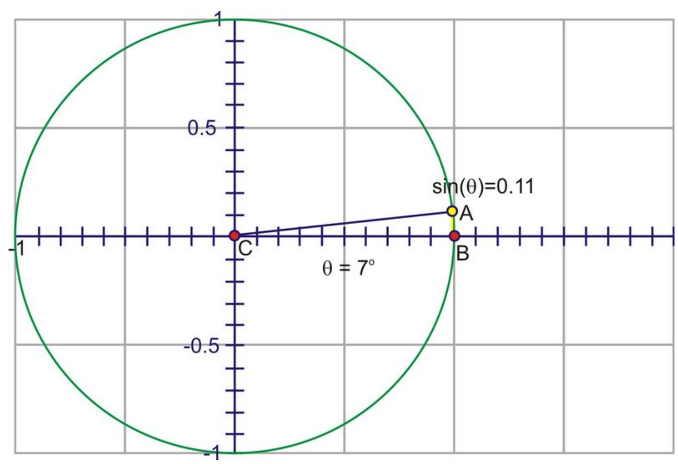


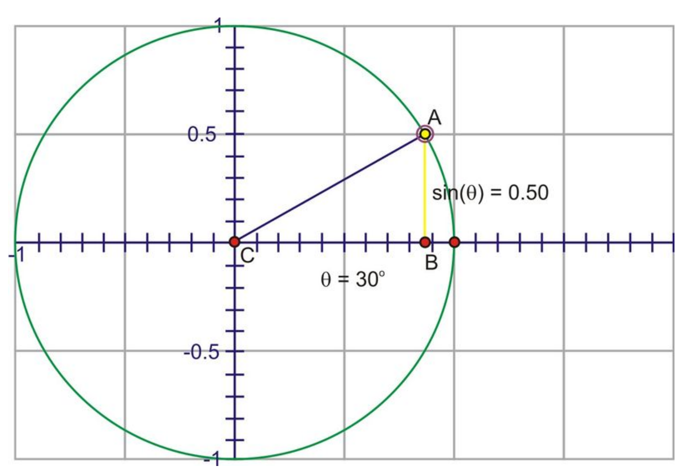
Because the ratios are the same for a given angle, regardless of the length of the radius r, we can use the **unit circle** as a basis for all calculations.

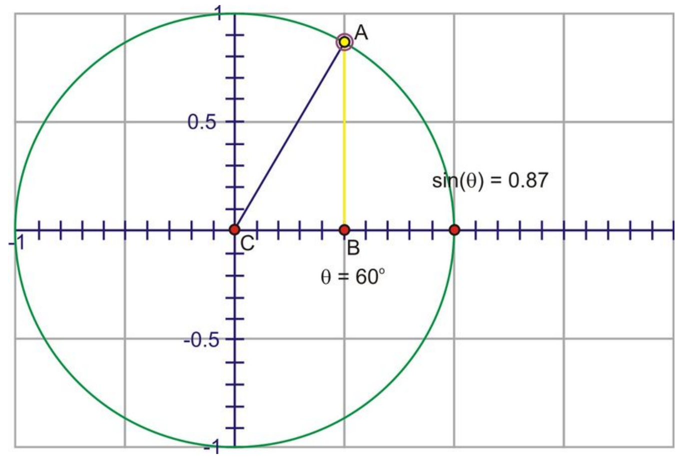


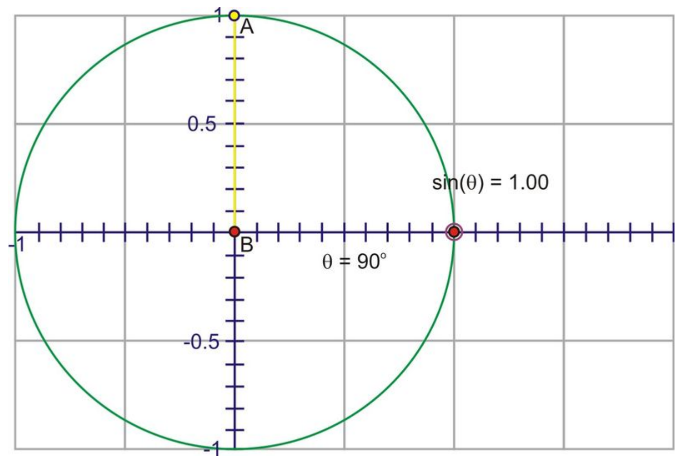
The denominator is now 1, so we have the simpler expression, $\sin x = y$. The advantage to this is that we can use the $y-$coordinate of the point on the unit circle to trace the value of $\sin \theta$ through a complete rotation. Imagine if we start at 0 and then rotate counter-clockwise through gradually increasing angles. Since the $y-$coordinate is the sine value, watch the height of the point as you rotate.

Through Quadrant I that height gets larger, starting at 0, increasing quickly at first, then slower until the angle reaches 90°, at which point, the height is at its maximum value, 1.

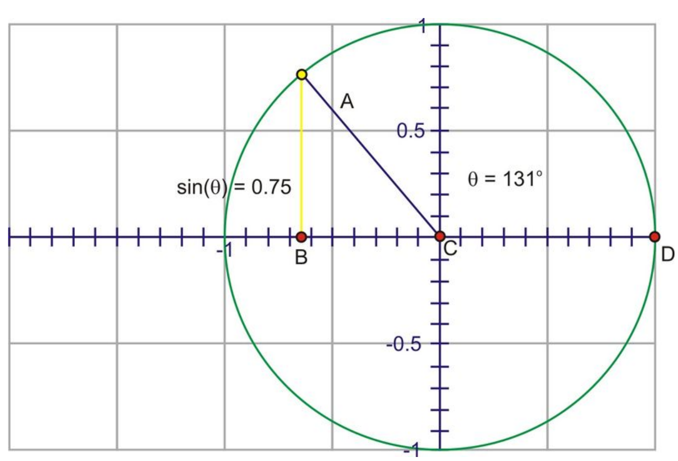


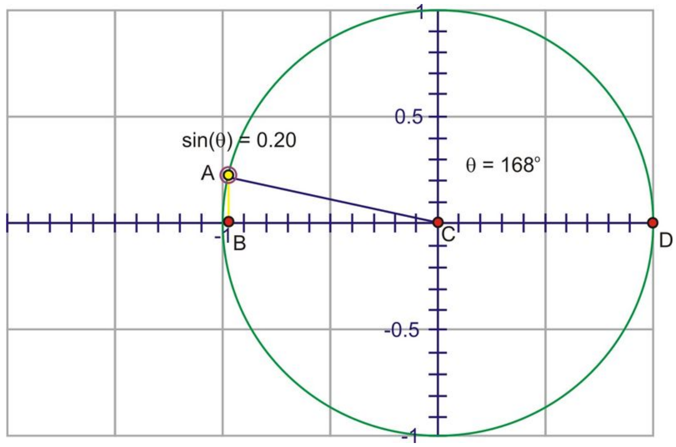




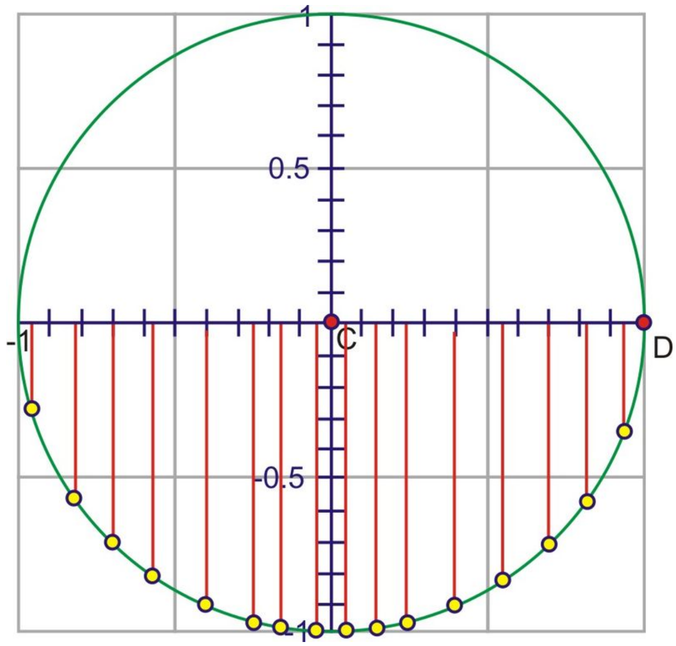


As you rotate into the second quadrant, the height starts to decrease towards zero.



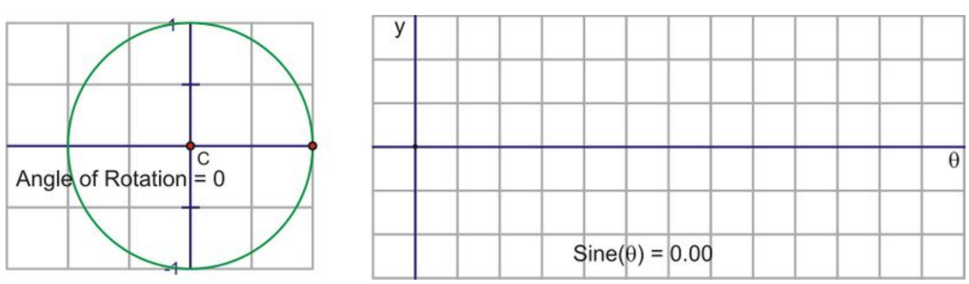


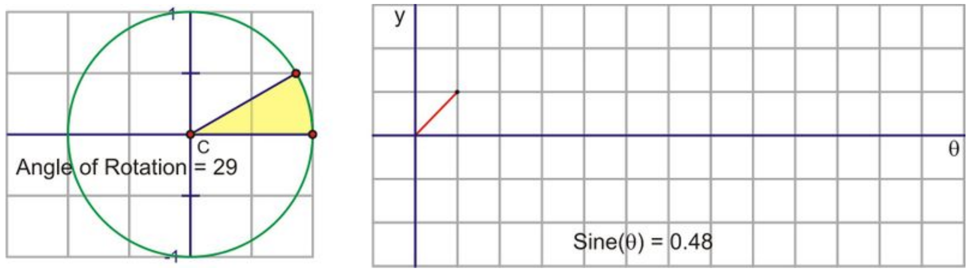
When you start to rotate into the third and fourth quadrants, the length of the segment increases, but this time in a negative direction, growing to -1 at 270° and heading back toward 0 at 360°.

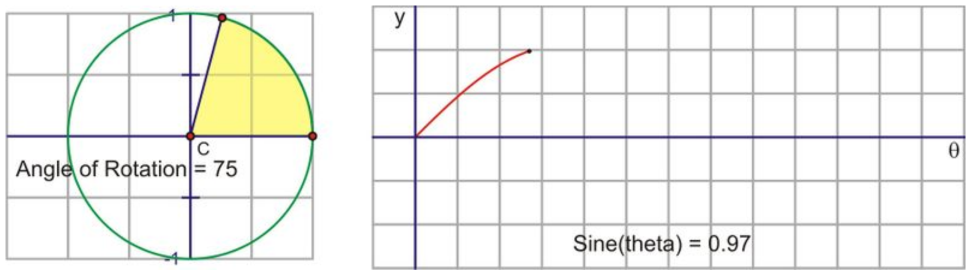


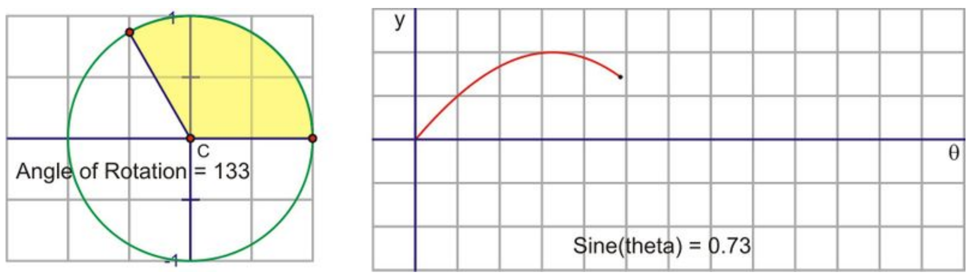
After one complete rotation, even though the angle continues to increase, the sine values will repeat themselves. The same would have been true if we chose to rotate clockwise to investigate negative angles, and this is why the sine function is a periodic function. The period is 2π because that is the angle measure before the sine of the angle will repeat its values.

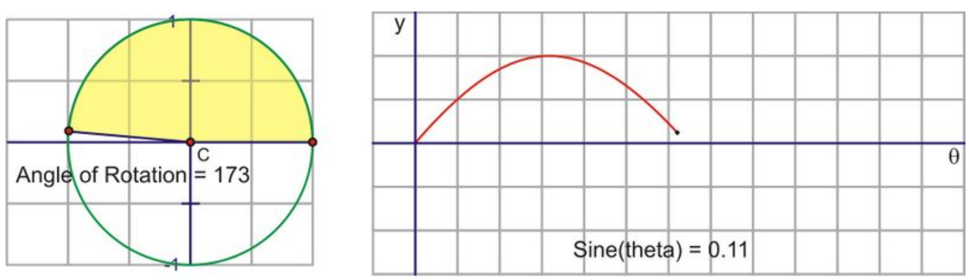
Let's translate this circular motion into a graph of the sine value vs. the angle of rotation. The following sequence of pictures demonstrates the connection. These pictures plot $(\theta, \sin\theta)$ on the coordinate plane as (x, y).

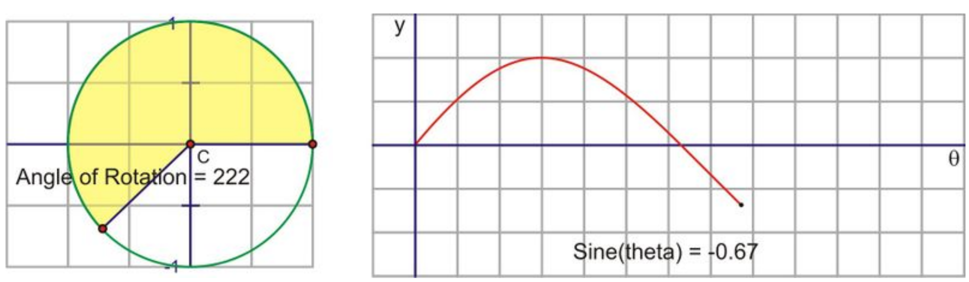


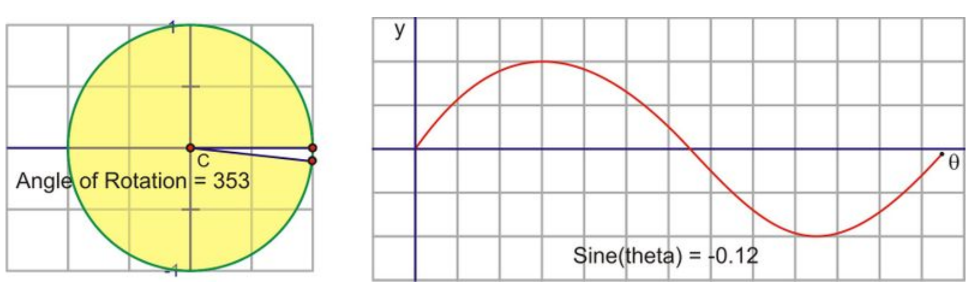

C
Angle of Rotation = 29
y
θ
Sine(θ) = 0.48


C
Angle of Rotation = 75
y
θ
Sine(theta) = 0.97

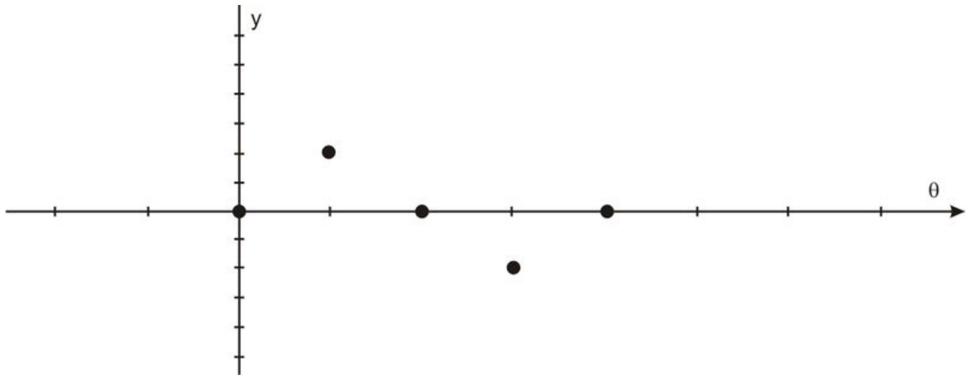

C
Angle of Rotation = 133
y
θ
Sine(theta) = 0.73


C
Angle of Rotation = 173
y
θ
Sine(theta) = 0.11

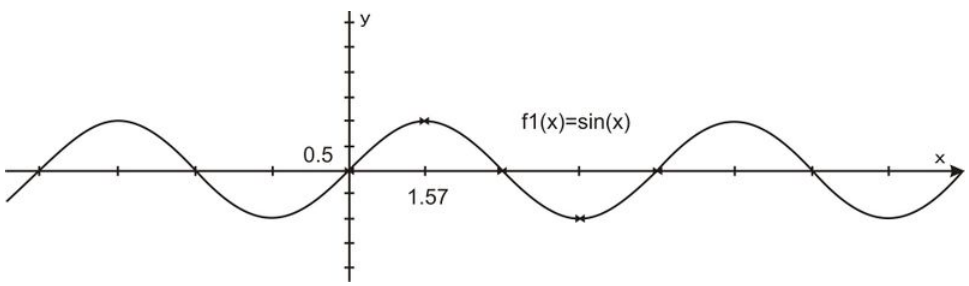

C
Angle of Rotation = 222
y
θ
Sine(theta) = -0.67


C
Angle of Rotation = 353
y
θ
Sine(theta) = -0.12

After we rotate around the circle once, the values start repeating. Therefore, the sine curve, or "wave," also continues to repeat. The easiest way to sketch a sine curve is to plot the points for the quadrant angles. The value of $\sin \theta$ goes from 0 to 1 to 0 to -1 and back to 0. Graphed along a horizontal axis, it would look like this:



Filling in the gaps in between and allowing for multiple rotations as well as negative angles results in the graph of $y = \sin x$ where x is any angle of rotation, in radians.

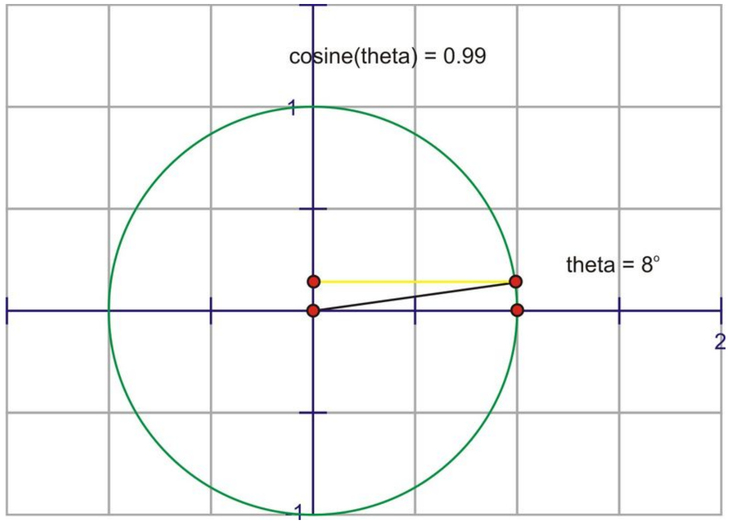


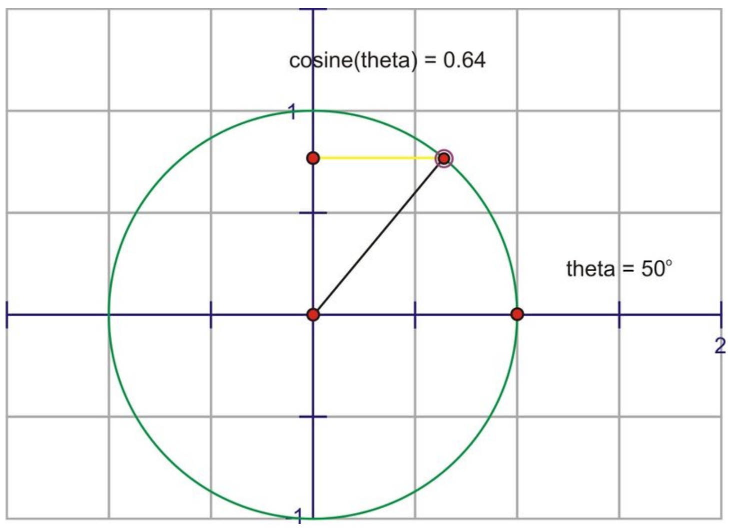
As we have already mentioned, $\sin x$ has a period of 2π. You should also note that the $y-$values never go above 1 or below -1, so the **range** of a sine curve is $\{-1 \leq y \leq 1\}$. Because angles can be any value and will continue to rotate around the circle infinitely, there is no restriction on the angle x, so the **domain** of $\sin x$ is all reals.

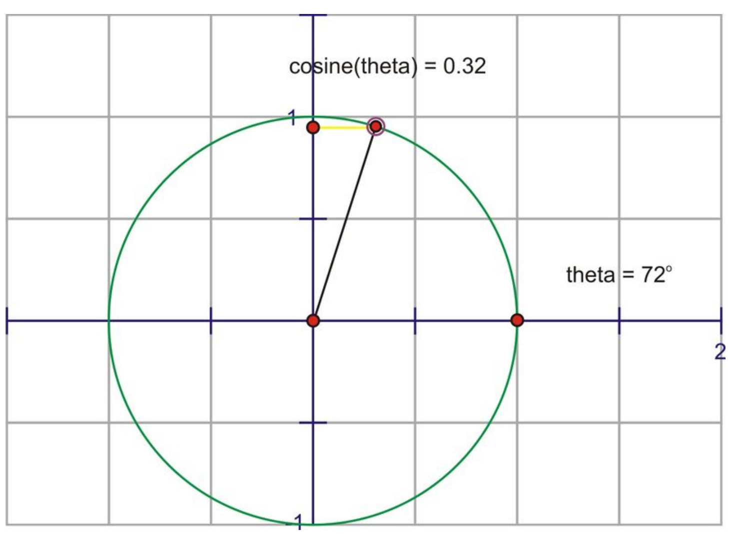
The Cosine Graph

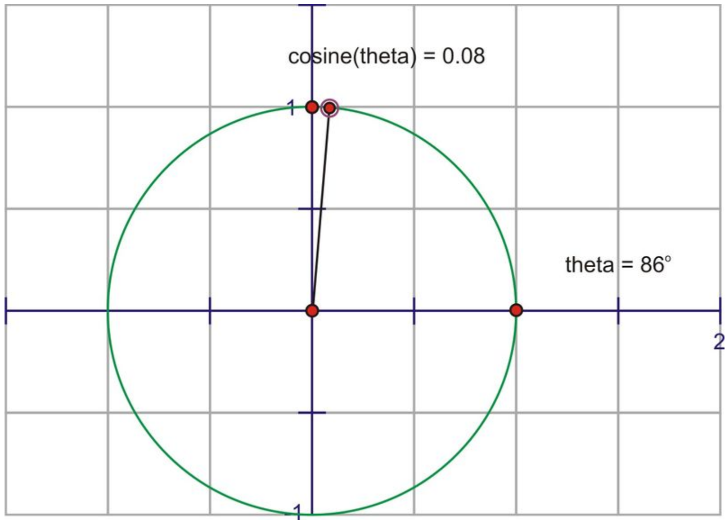
In chapter 1, you learned that sine and cosine are very closely related. The cosine of an angle is the same as the sine of its complementary angle. So, it should not be a surprise that sine and cosine waves are very similar in that they are both periodic with a period of 2π, a range from -1 to 1, and a domain of all real angles.

The cosine of an angle is the ratio of $\frac{x}{r}$, so in the unit circle, the cosine is the $x-$coordinate of the point of rotation. If we trace the $x-$coordinate through a rotation, notice the change in the distance is similar to $\sin x$, but $\cos x$ starts at one instead of zero. The $x-$coordinate at 0° is 1 and the $x-$coordinate for 90° is 0, so the cosine value is decreasing from 1 to 0 through the 1^{st} quadrant.

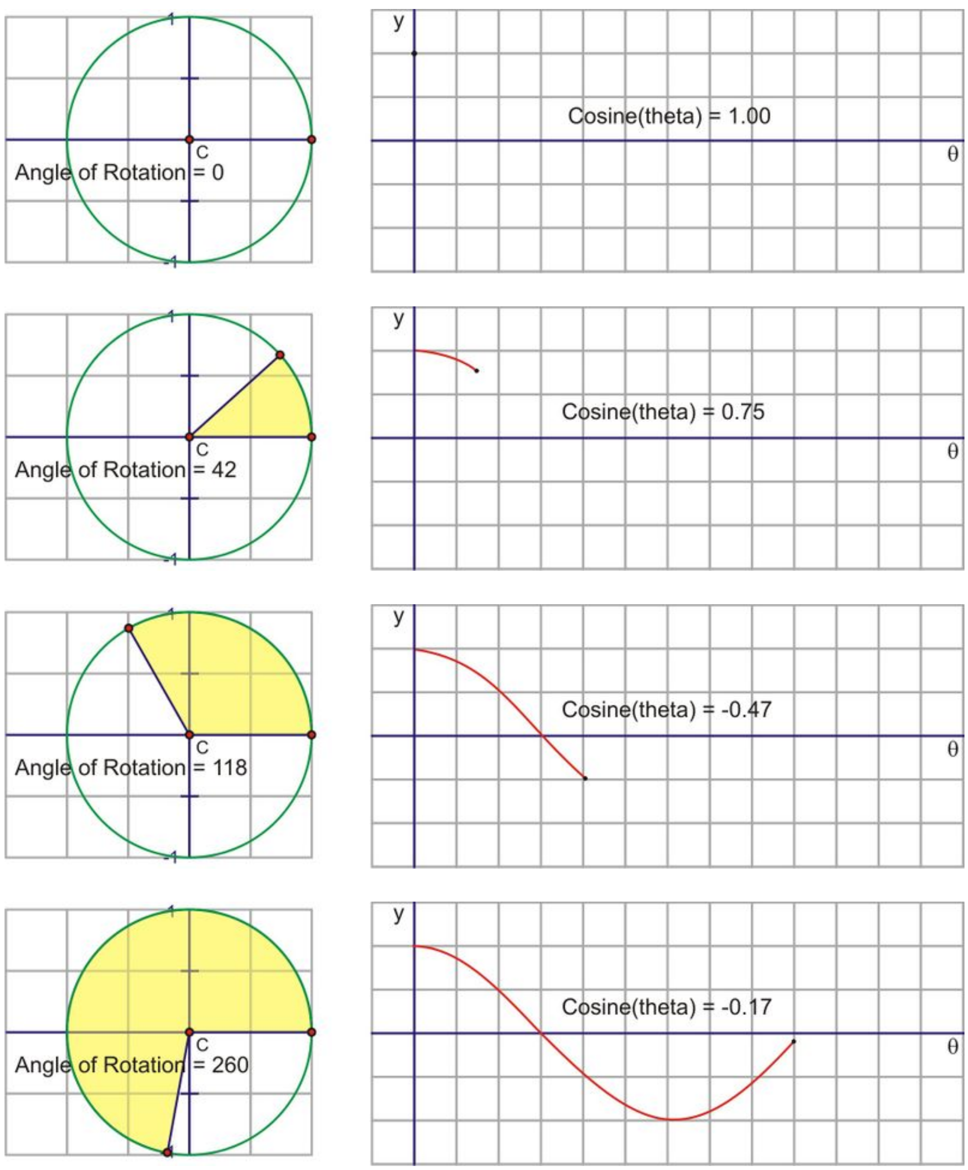

cosine(theta) = 0.99
1
theta = 8°
2
-1

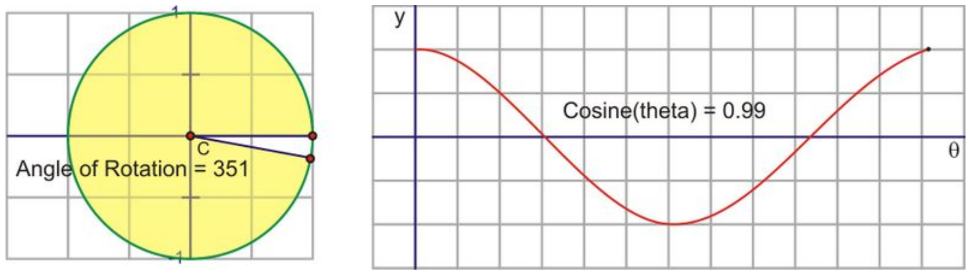

cosine(theta) = 0.64
1
theta = 50°
2
-1


cosine(theta) = 0.32
1
theta = 72°
2
-1

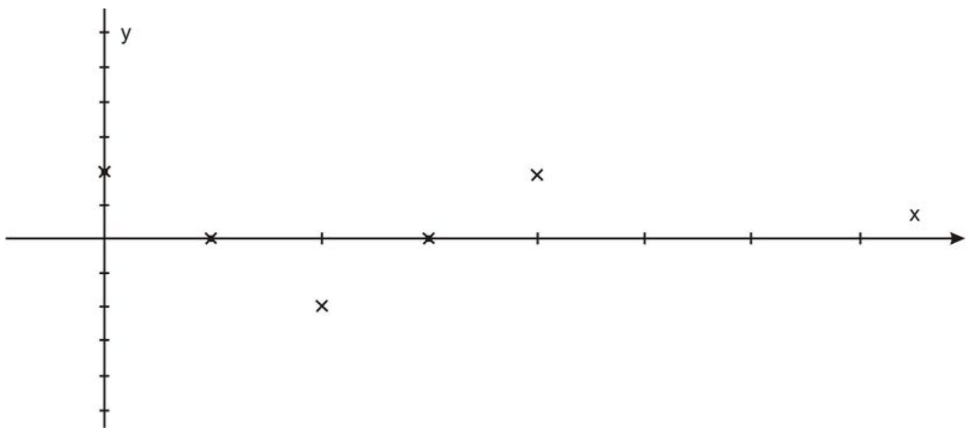


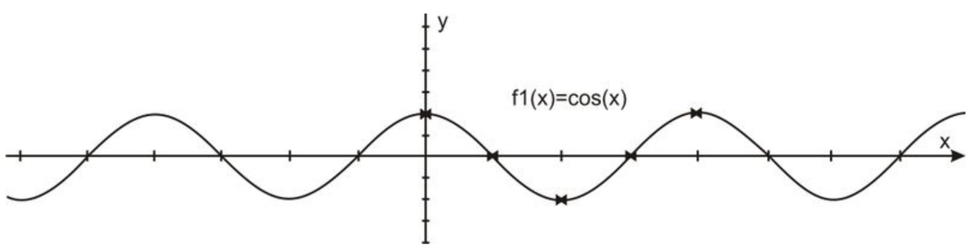
Here is a similar sequence of rotations to the one used for sine. This time compare the $x-$ coordinate of the point of rotation with the height of the point as it traces along the horizontal. These pictures plot $(\theta, \cos\theta)$ on the coordinate plane as (x, y).





Plotting the quadrant angles and filling in the in-between values shows the graph of $y = \cos x$

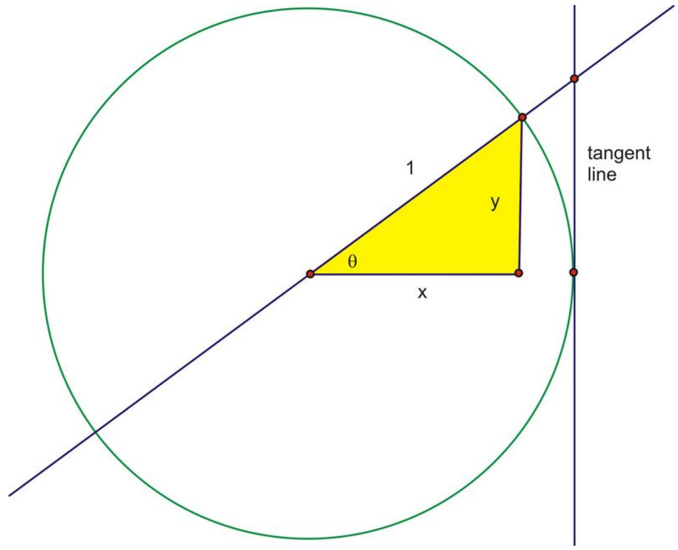




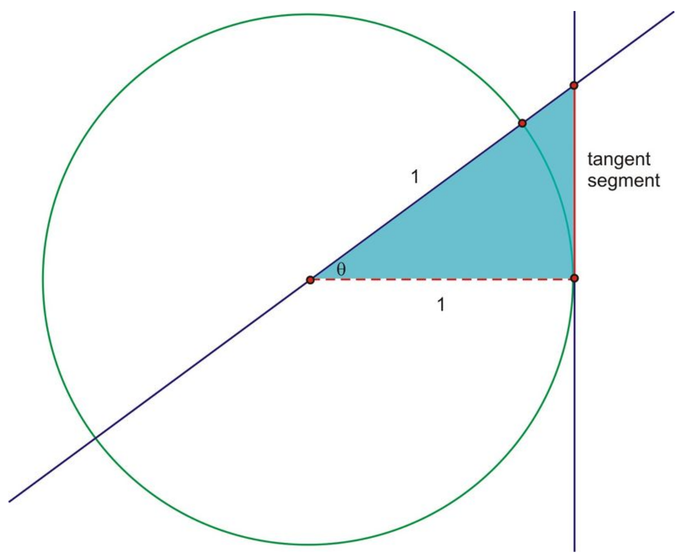
The graph of $y = \cos x$ has a period of 2π. Just like $\sin x$, the **range** of a cosine curve is $\{-1 \leq y \leq 1\}$ and the **domain** of $\cos x$ is all reals. Notice that the shape of the curve is exactly the same, but *shifted* by $\frac{\pi}{2}$.

The Tangent Graph

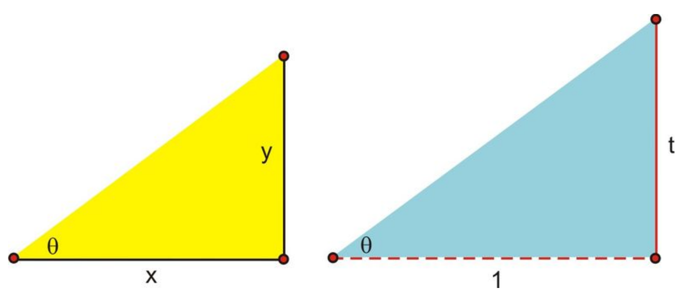
The name of the tangent function comes from the tangent line of a circle. This is a line that is perpendicular to the radius at a point on the circle so that the line touches the circle at exactly one point.



If we extend angle θ through the unit circle so that it intersects with the tangent line, the tangent function is defined as the length of the red segment.

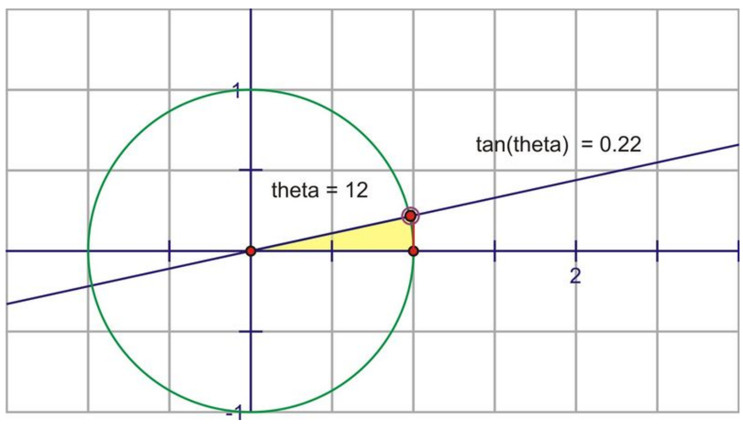


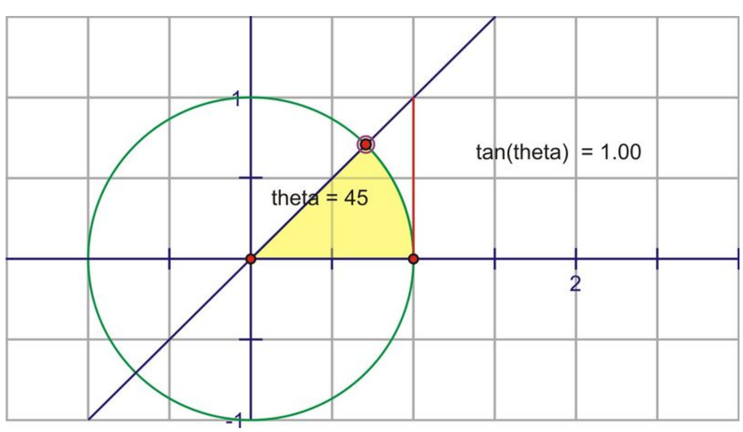
The dashed segment is 1 because it is the radius of the unit circle. Recall that the $\tan\theta = \frac{y}{x}$, and it can be verified that this segment is the tangent by using similar triangles.

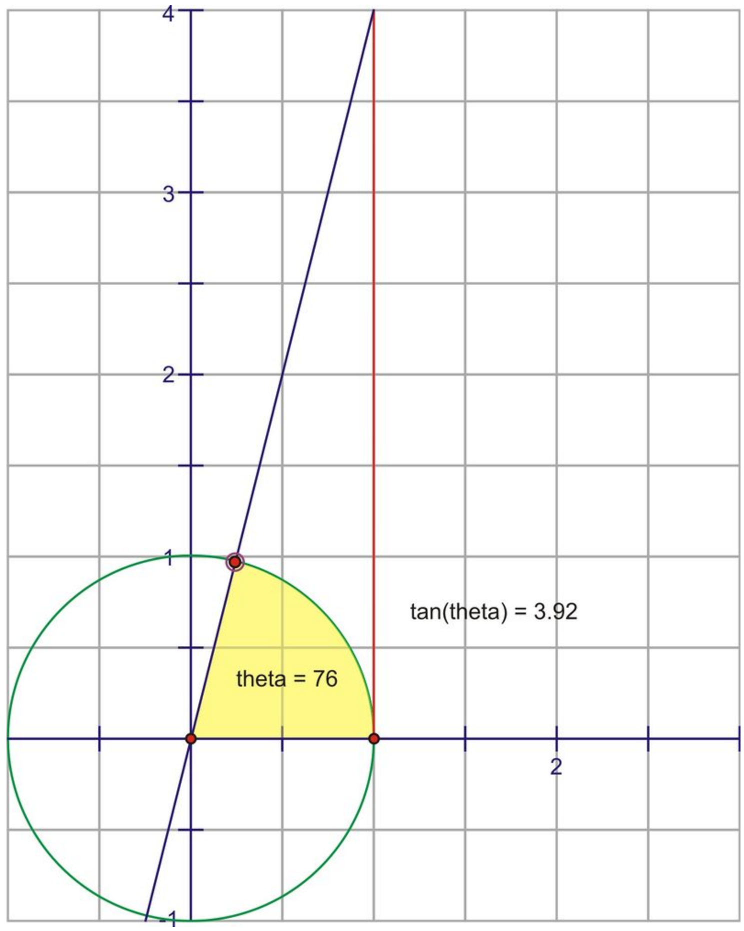


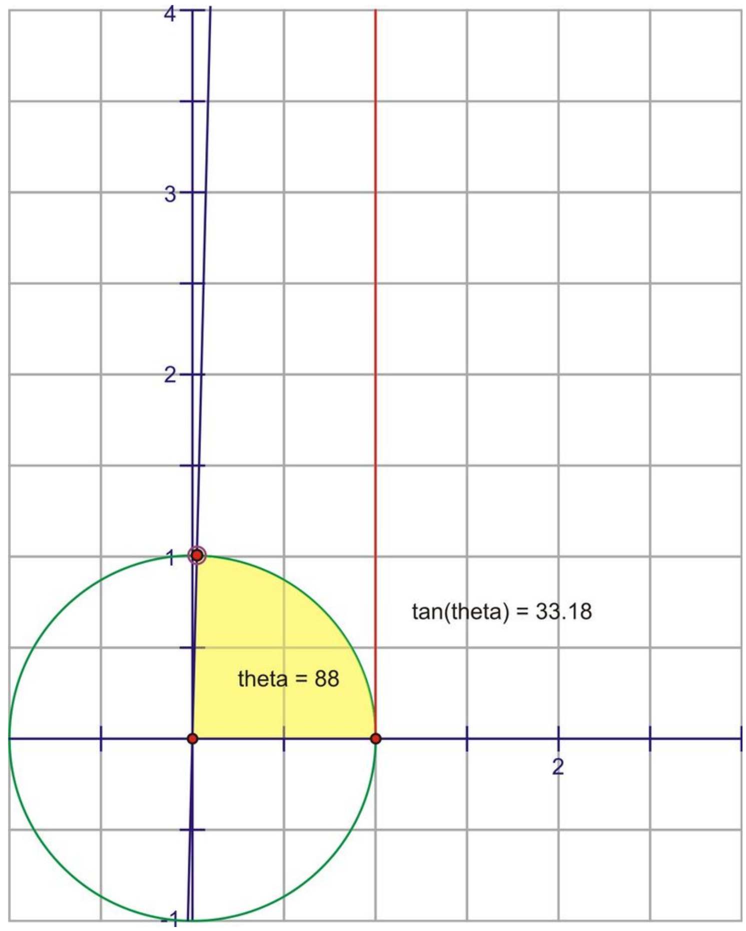
$$\tan\theta = \frac{y}{x} = \frac{t}{1} = t$$
$$\tan\theta = t$$

So, as we increase the angle of rotation, think about how this segment changes. When the angle is zero, the segment has no length. As we rotate through the first quadrant, it will increase very slowly at first and then quickly get very close to one, but never actually touch it.

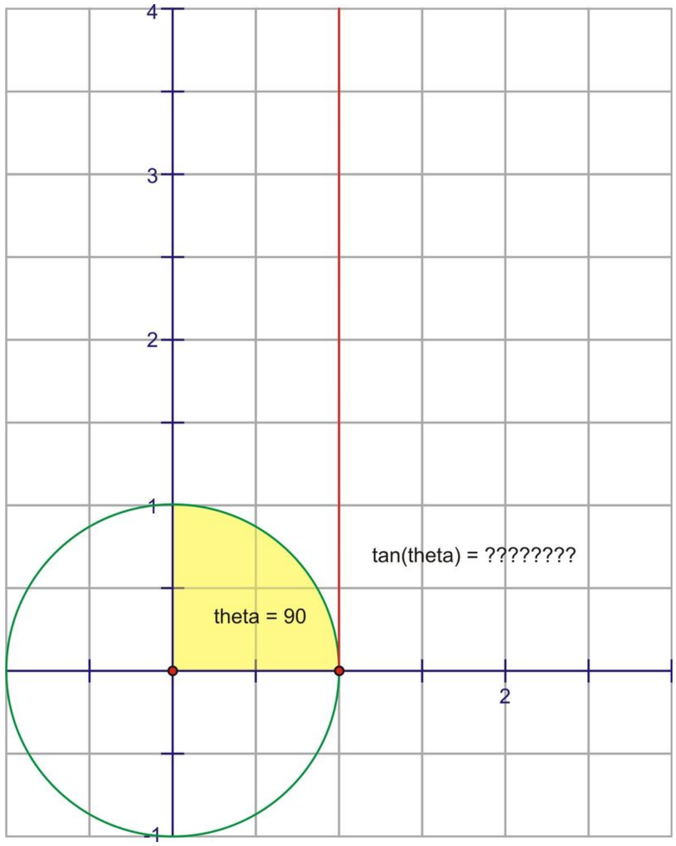





4
3
2
1
-1
2
tan(theta) = 3.92
theta = 76

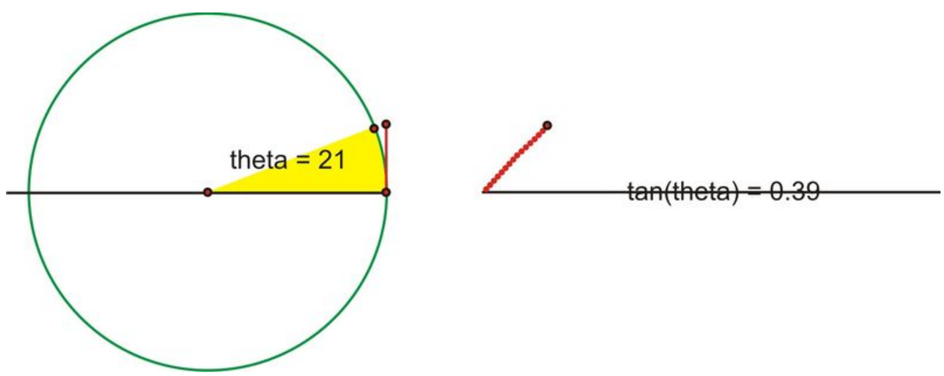


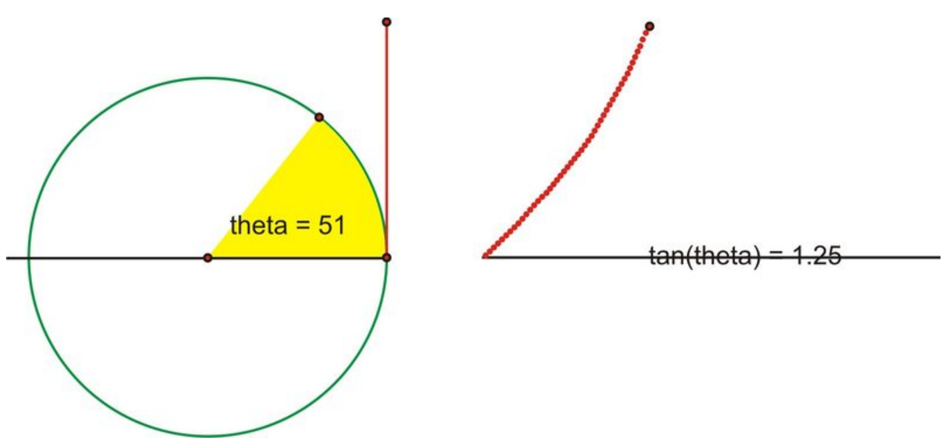
As we get ***very*** close to the $y-$axis the segment gets infinitely large, until when the angle really hits 90°, at which point the extension of the angle and the tangent line will actually be parallel and therefore never intersect.

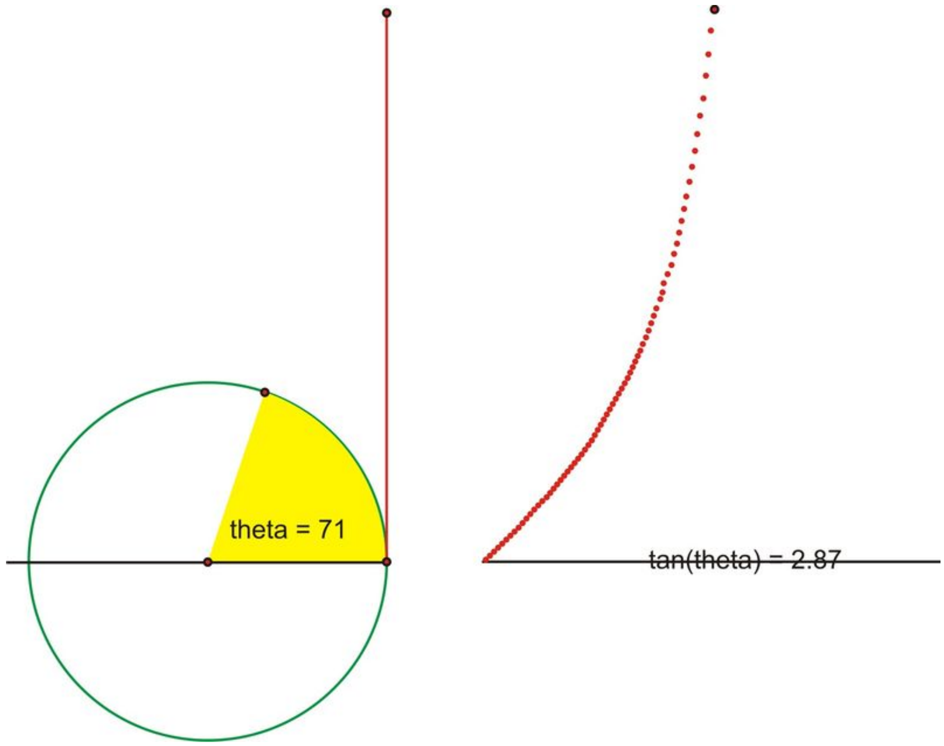


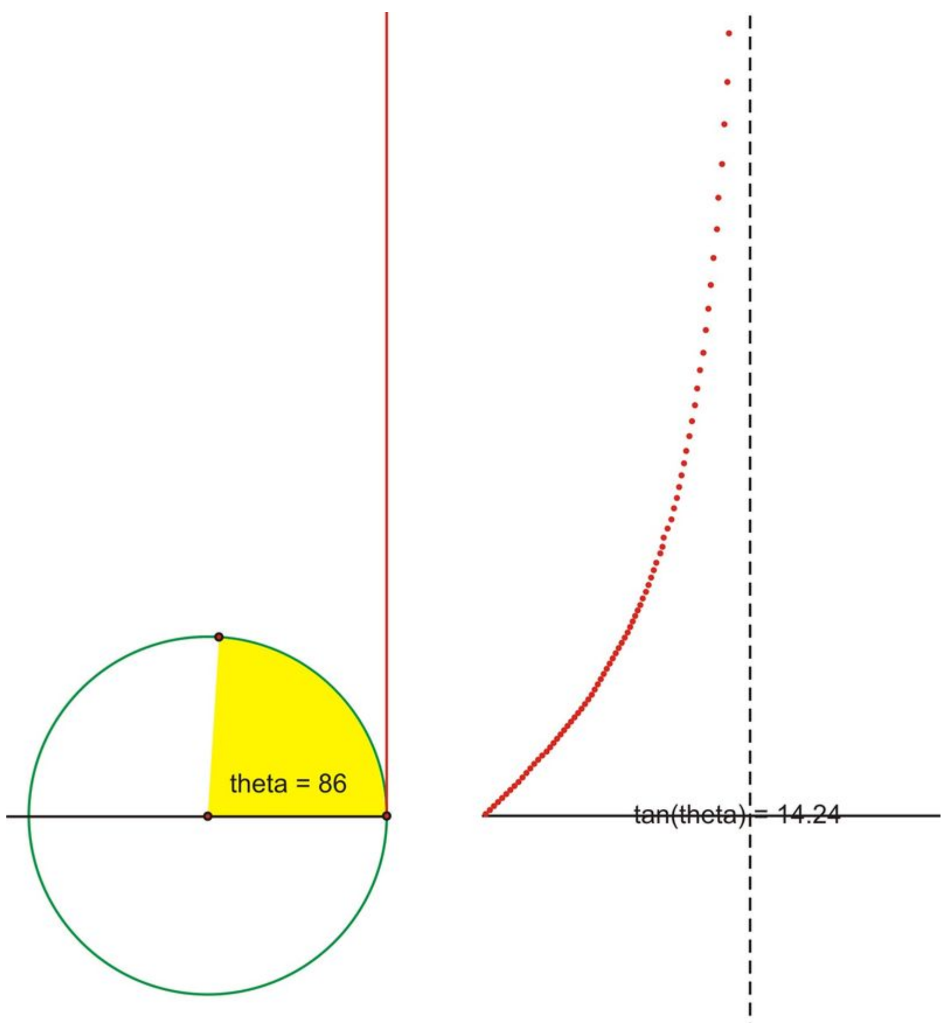
This means there is no finite length of the tangent segment, or the tangent segment is *infinitely large*.

Let's translate this portion of the graph onto the coordinate plane. Plot $(\theta, \tan\theta)$ as (x, y).



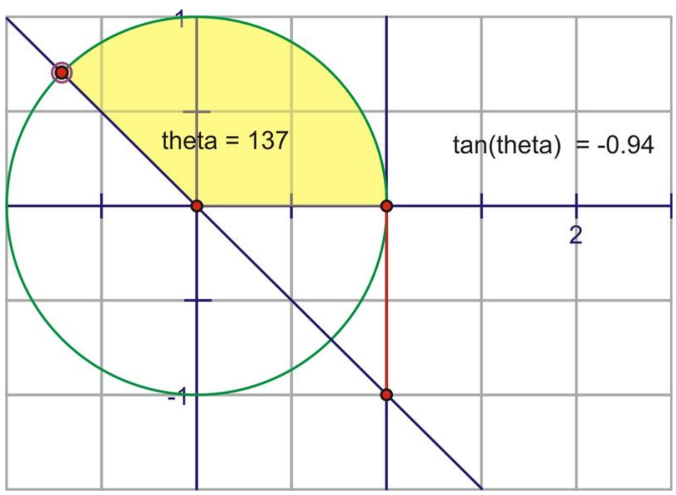


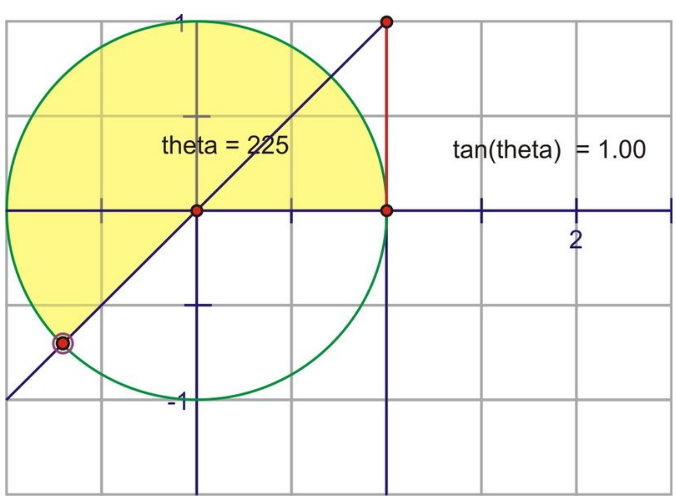

theta = 71
tan(theta) = 2.87

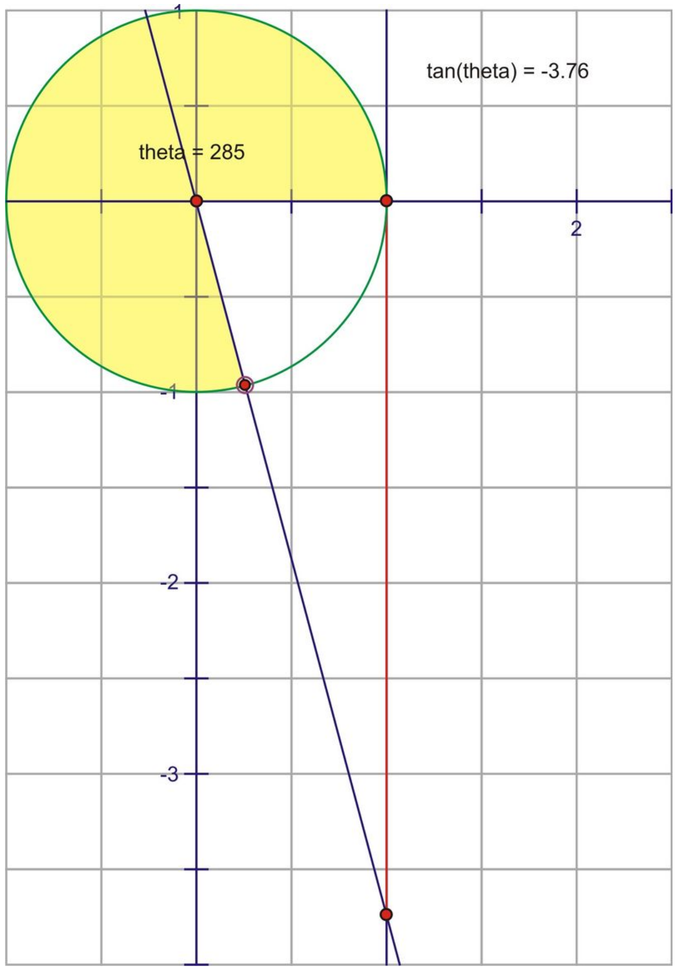

theta = 86
tan(theta) = 14.24

In fact as we get infinitely close to 90°, the tangent value increases without bound, until when we actually reach 90°, at which point the tangent is undefined. Recall there are some angles (90° and 270°, for example) for which the tangent is not defined. Therefore, at these points, there are going to be vertical asymptotes.

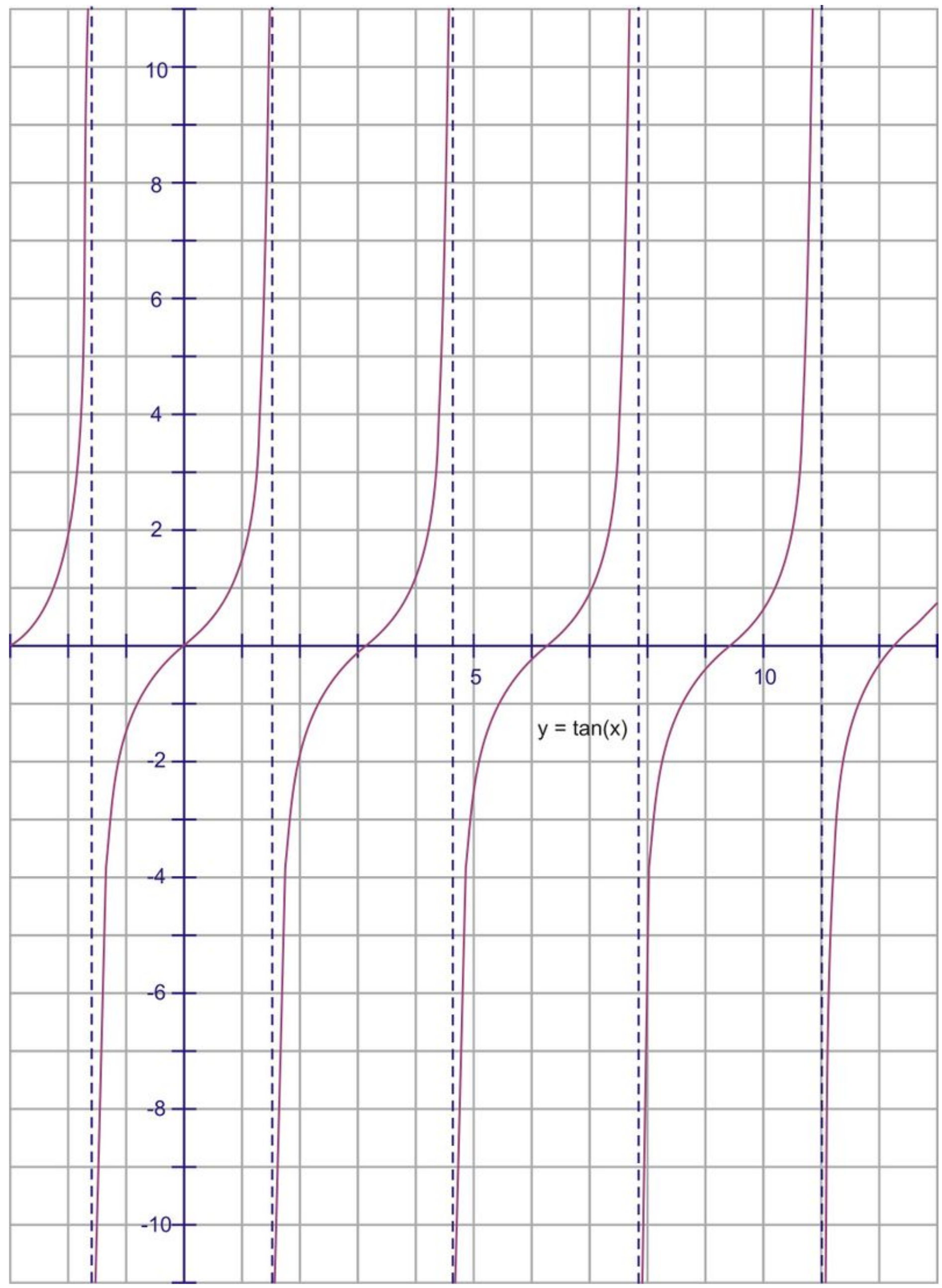
Rotating past 90°, the intersection of the extension of the angle and the tangent line is actually below the $x-$axis. This fits nicely with what we know about the tangent for a 2^{nd} quadrant angle being negative. At first, it will have very large negative values, but as the angle rotates, the segment gets shorter, reaches 0, then crosses back into the positive numbers as the angle enters the 3^{rd} quadrant. The segment will again get infinitely large as it approaches 270°. After being undefined at 270°, the angle crosses into the 4^{th} quadrant and once again changes from being infinitely negative, to approaching zero as we complete a full rotation.







The graph $y = \tan x$ over several rotations would look like this:



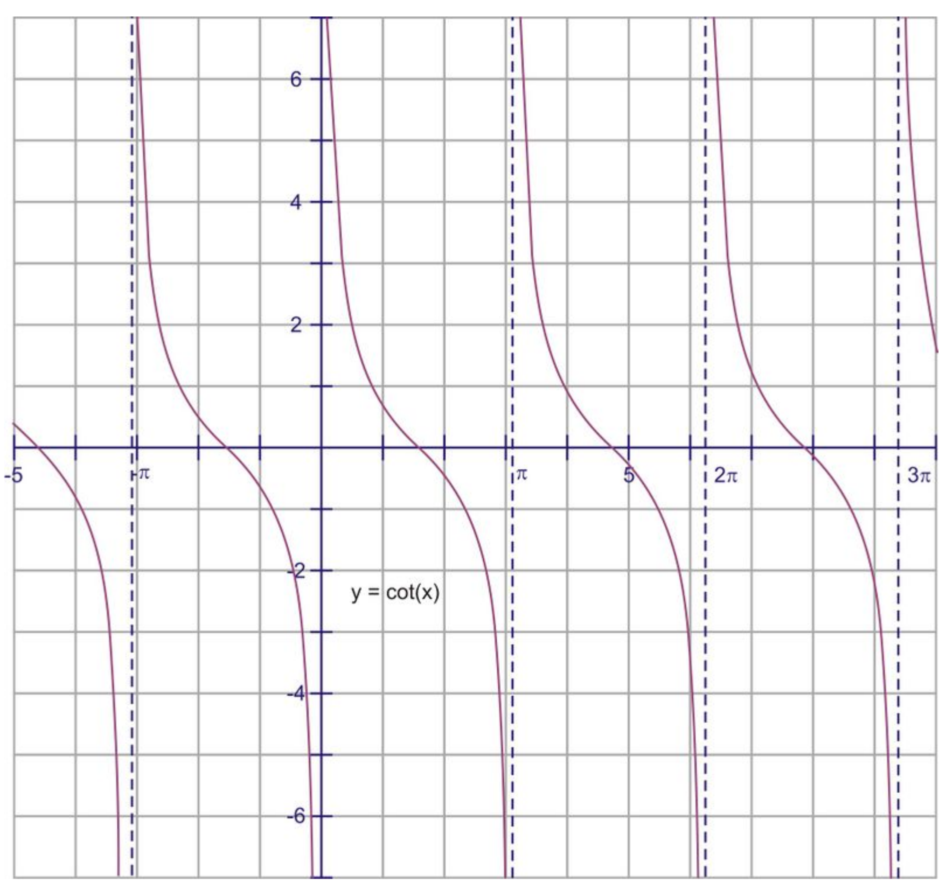
Notice the $x-$axis is measured in radians. Our asymptotes occur every π radians, starting at $\frac{\pi}{2}$. The period of the graph is therefore π radians. The domain is all reals except for the asymptotes at $\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, etc.$ and the range is all real numbers.

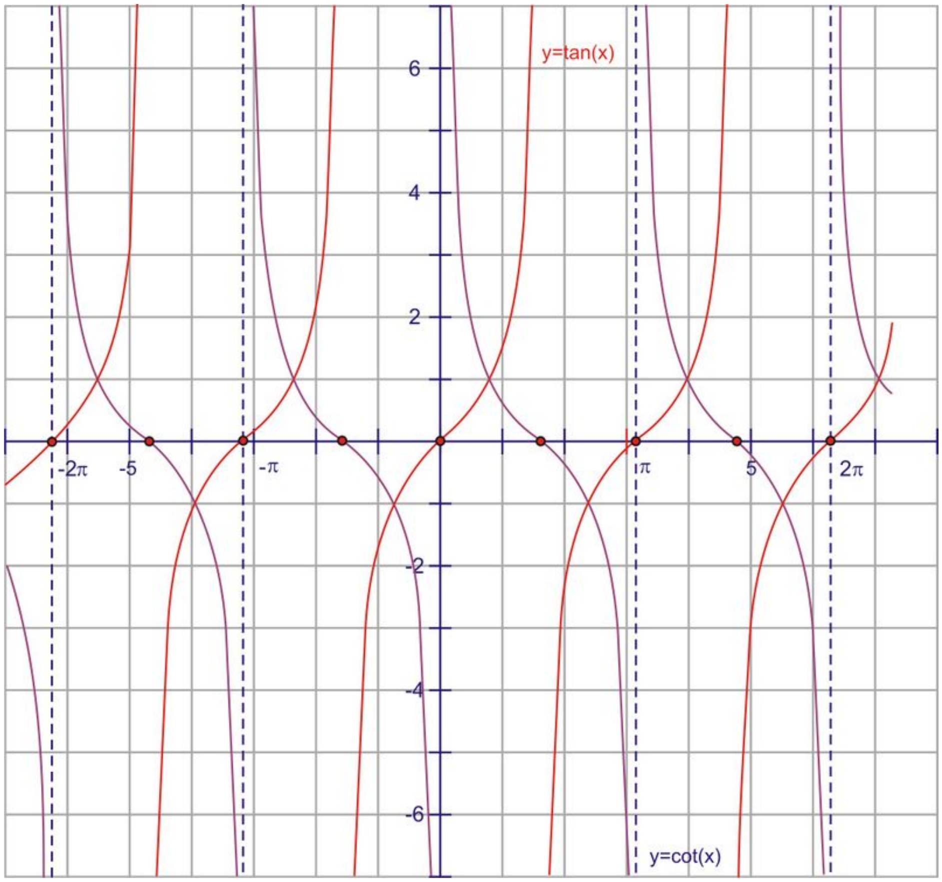
The Three Reciprocal Functions

For the three reciprocal functions, it gets increasingly difficult to show the segment representation on the unit circle. Instead of going through all of this, we will show the $\cot x$, $\csc x$, and $\sec x$ through the graphs of their reciprocal functions, $\tan x$, $\sin x$, and $\cos x$.

Cotangent

Cotangent is the reciprocal of tangent, $\frac{x}{y}$, so it would make sense that where ever the tangent had an asymptote, now the cotangent will be zero. The opposite of this is also true. When the tangent is zero, now the cotangent will have an asymptote. The shape of the curve is generally the same, so the graph looks like this:



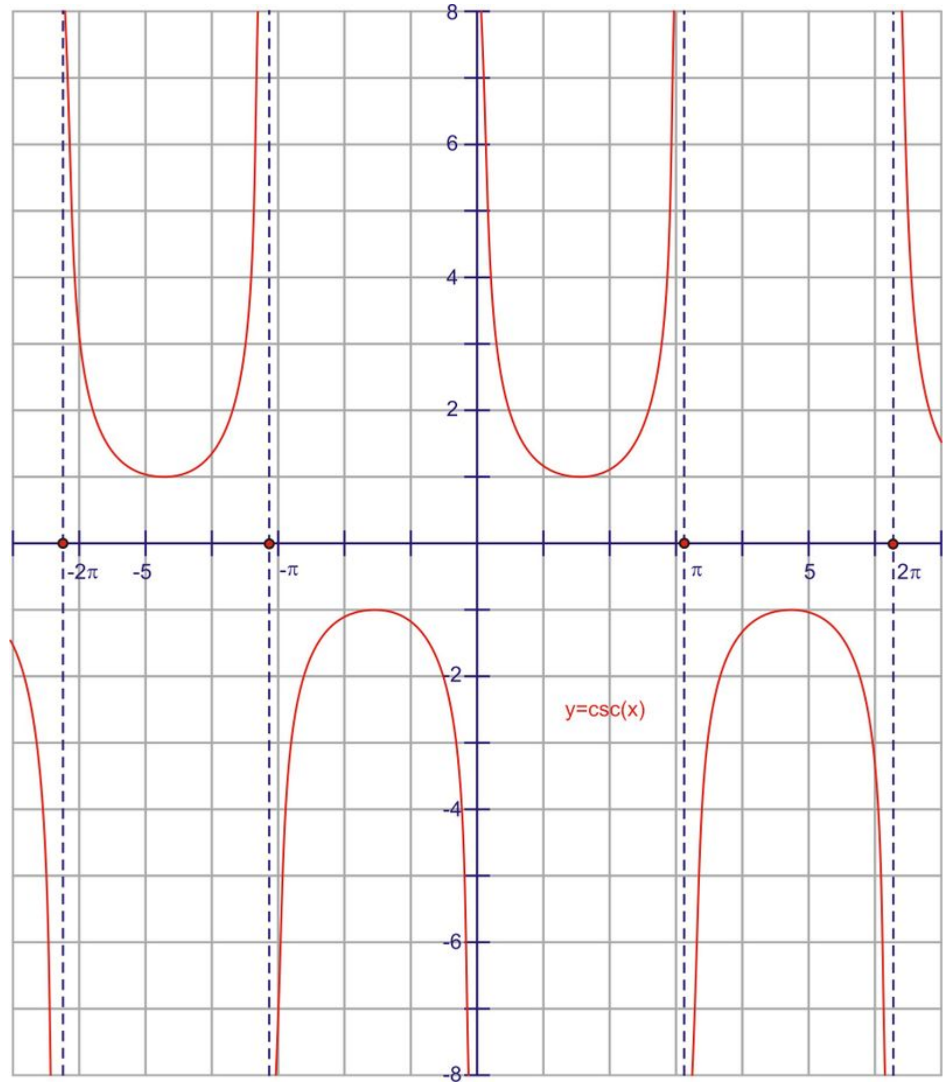


When you overlap the two functions, notice that the graphs consistently intersect at 1 and -1. These are the angles that have 45° as reference angles, which always have tangents and cotangents equal to 1 or -1. It makes sense that 1 and -1 are the only values for which a function and it's reciprocal are the same. Keep this in mind as we look at cosecant and secant compared to their reciprocals of sine and cosine.

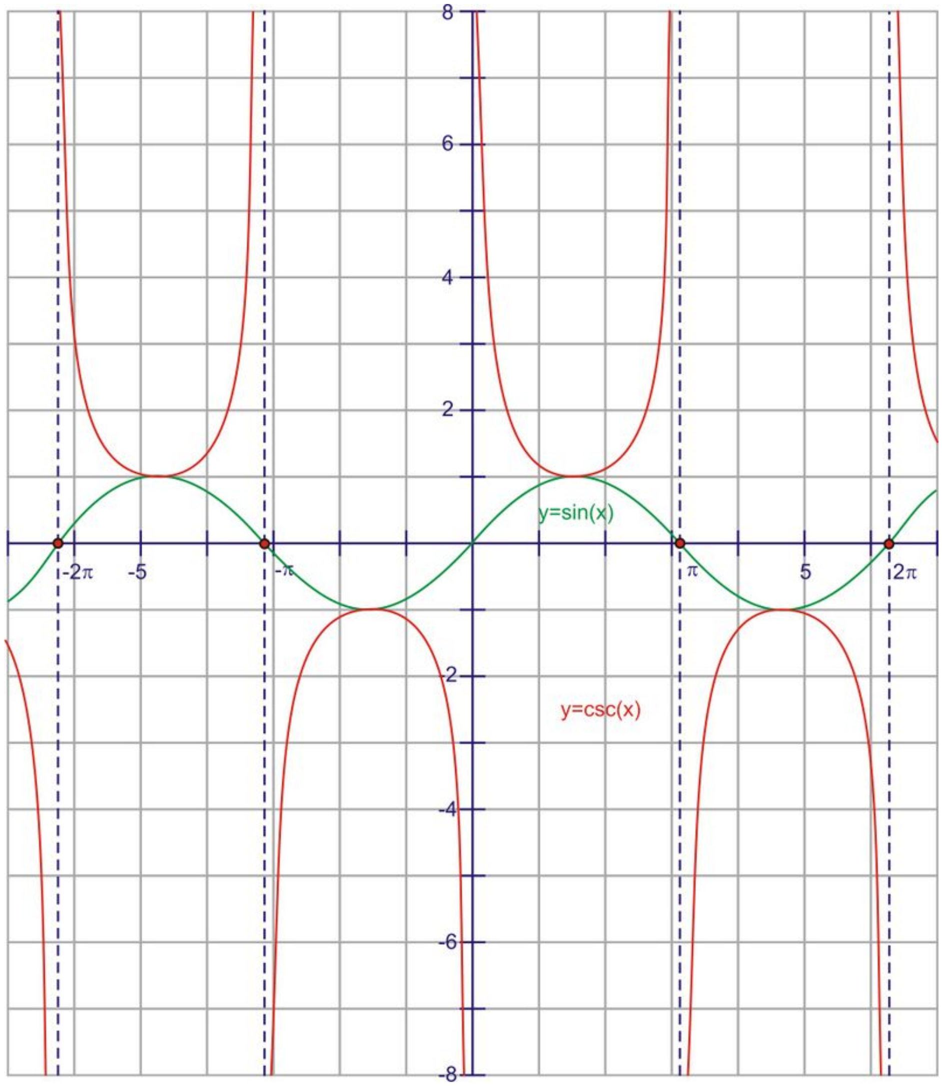
The cotangent function has a domain of all real angles except multiples of $\pi \{\ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots\}$ The range is all real numbers.

Cosecant

Cosecant is the reciprocal of sine, or $\frac{1}{y}$. Therefore, whenever the sine is zero, the cosecant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the sine function in the same quadrants. Here is the graph.



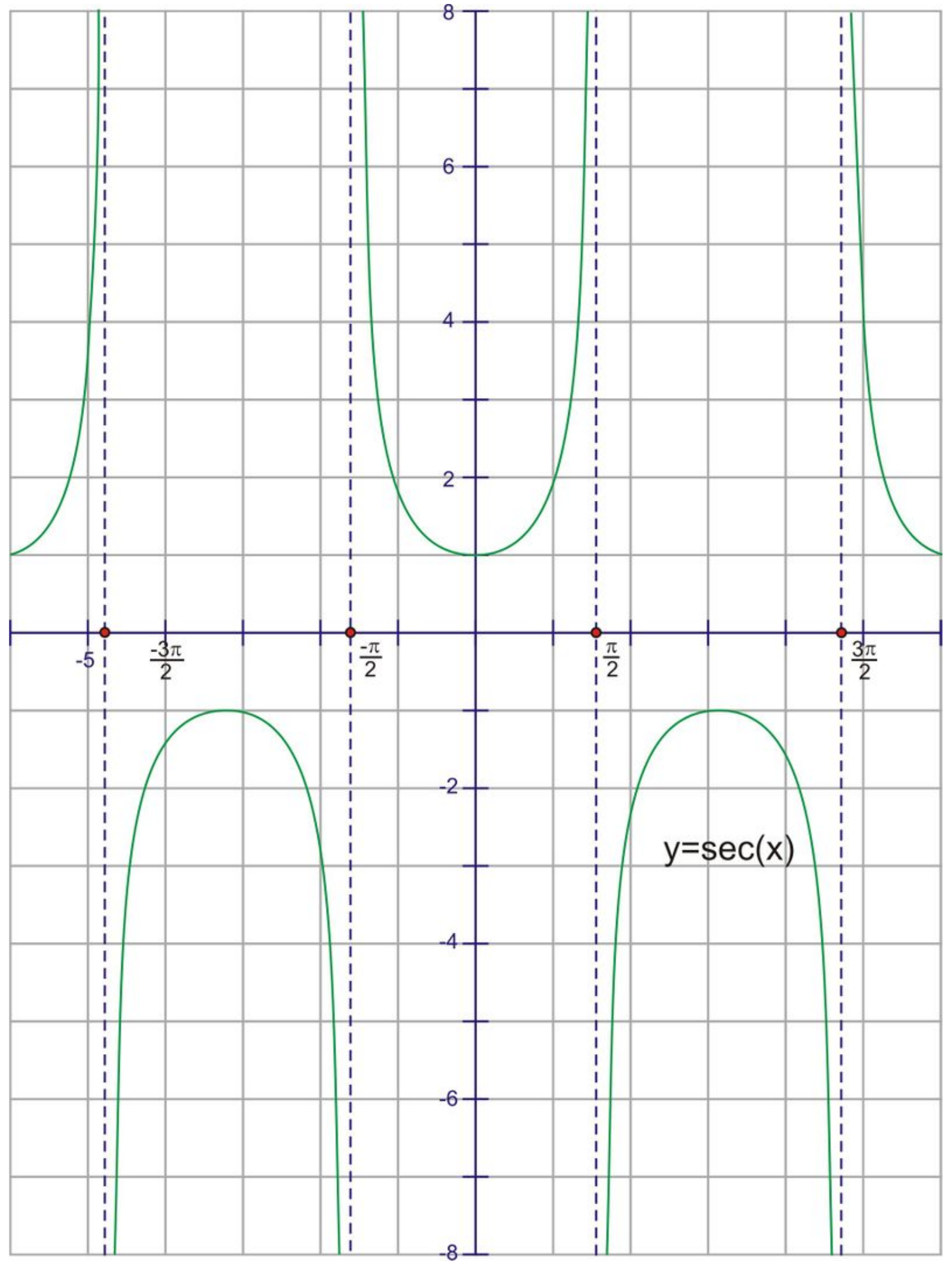
The period of the function is 2π, just like sine. The domain of the function is all real numbers, except multiples of $\pi\{\ldots-2\pi,-\pi,0,\pi,2\pi\ldots\}$. The range is all real numbers greater than or equal to 1, as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where sine is defined (other than the points at the top and bottom of the sine curve).



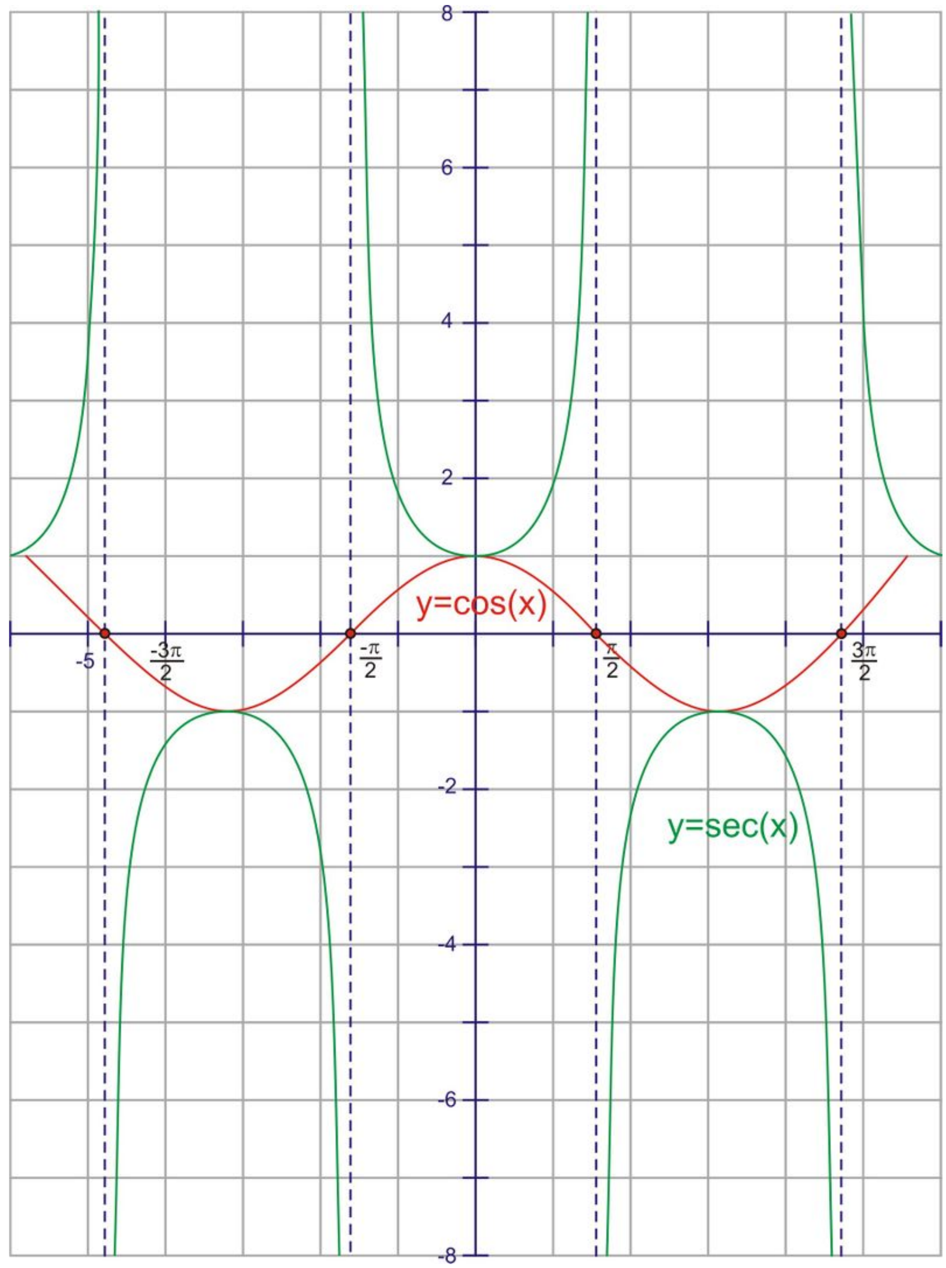
Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Many students are reminded of parabolas when they look at the half-period of the cosecant graph. While they are similar in that they each have a local minimum or maximum and they have the same beginning and ending behavior, the comparisons end there. Parabolas are not restricted by asymptotes, whereas the cosecant curve is.

Secant

Secant is the reciprocal of cosine, or $\frac{1}{x}$. Therefore, whenever the cosine is zero, the secant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the cosine function in the same quadrants. Here is the graph.



The period of the function is 2π, just like cosine. The domain of the function is all real numbers, except multiples of π starting at $\frac{\pi}{2}$. $\left\{\ldots\frac{-\pi}{2},\frac{\pi}{2},0,\frac{3\pi}{2},\frac{5\pi}{2}\ldots\right\}$. The range is all real numbers greater than or equal to 1 as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where cosine is defined (other than the tops and bottoms of the cosine curve).



Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Again, this graph looks parabolic, but it is not.

The table below summarizes the functions with their domains and ranges:

TABLE 2.4:

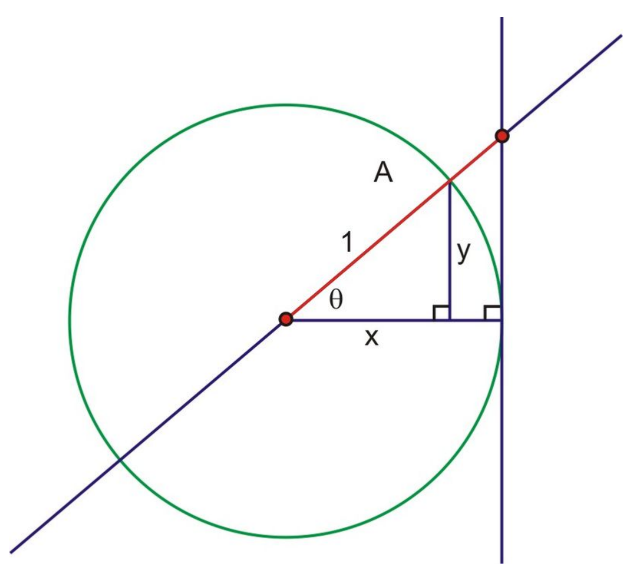
Function	Domain	Range
$\sin x$	**all reals**	$\{y : -1 \leq y \leq 1\}$
$\cos x$	**all reals**	$\{y : -1 \leq y \leq 1\}$
$\tan x$	$\left\{x : x \neq n \times \frac{\pi}{2}, \text{ where n is any odd integer}\right\}$	**all reals**
$\csc x$	$\{x : x \neq n\pi, \text{ where n is any integer}\}$	$\{y : y \geq 1 \text{ or } y \leq -1\}$
$\sec x$	$\left\{x : x \neq n \times \frac{\pi}{2}, \text{where n is any odd integer}\right\}$	$\{y : y \geq 1 \text{ or } y \leq -1\}$
$\cot x$	$\{x : x \neq n\pi, \text{where n is any integer}\}$	**all reals**

Points to Consider

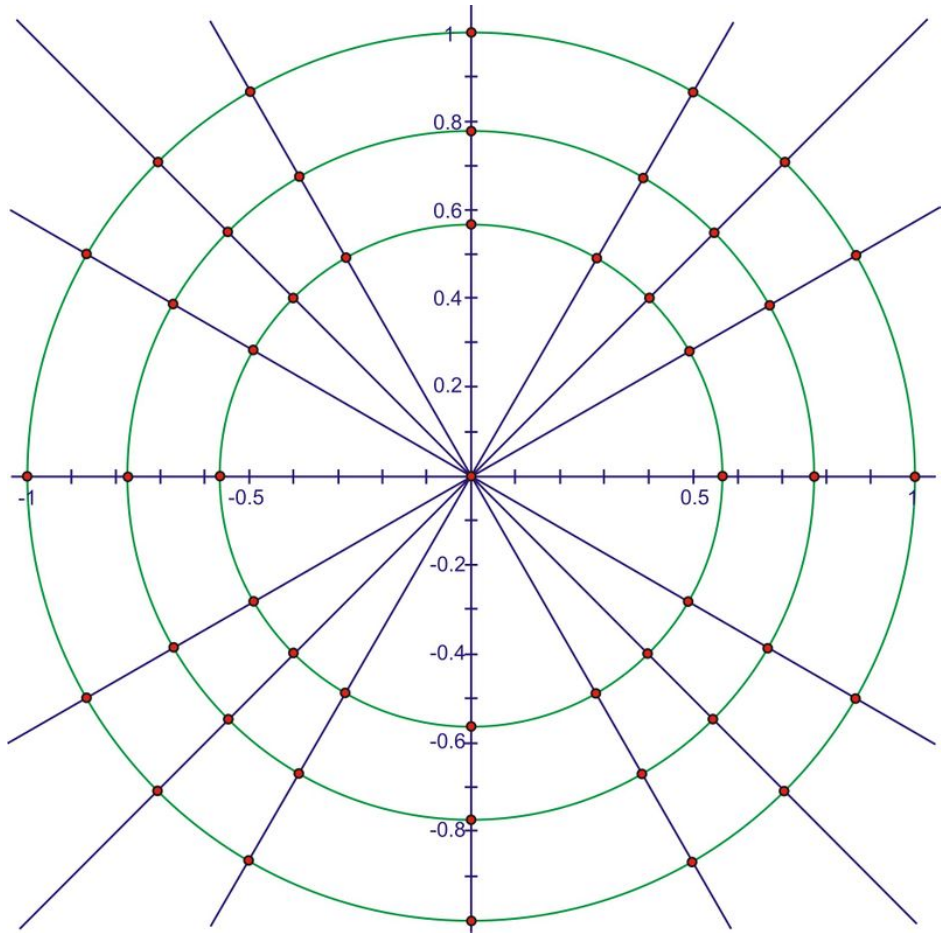
- How are all the reciprocal functions' graphs related to sine, cosine and tangent?
- What would the inverse function of $y = \sin x$ look like?

Review Questions

1. Show that side A (in orange) in this drawing is equal to $\sec \theta$. Use similar triangles in your proof.



2. In Chapter 1, you learned that $\tan^2 \theta + 1 = \sec^2 \theta$. Use the drawing and results from question 1 to demonstrate this identity.
3. This diagram shows a unit circle with all the angles that have reference angles of 30°, 45°, and 60°, as well as the quadrant angles. Label the coordinates of all points on the unit circle. On the smallest circle, label the angles in degrees, and on the middle circle, label the angles in radians.



4. Which of the following shows functions that are both increasing as x increases from 0 to $\frac{\pi}{2}$?
   a. $\sin x$ and $\cos x$
   b. $\tan x$ and $\sec x$
   c. $\sec x$ and $\cot x$
   d. $\csc x$ and $\sec x$
5. Which of the following statements are true as x increases from $\frac{3\pi}{2}$ to 2π?
   a. $\cos x$ approaches 0
   b. $\tan x$ approaches positive infinity
   c. $\cos x < \sin x$
   d. $\cot x$ approaches negative infinity

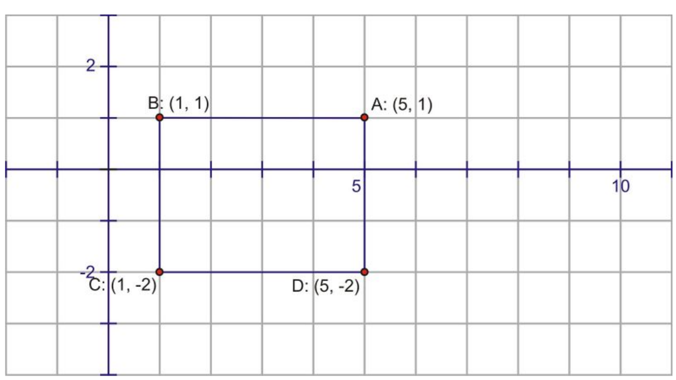
2.4 Translating Sine and Cosine Functions

Learning Objectives

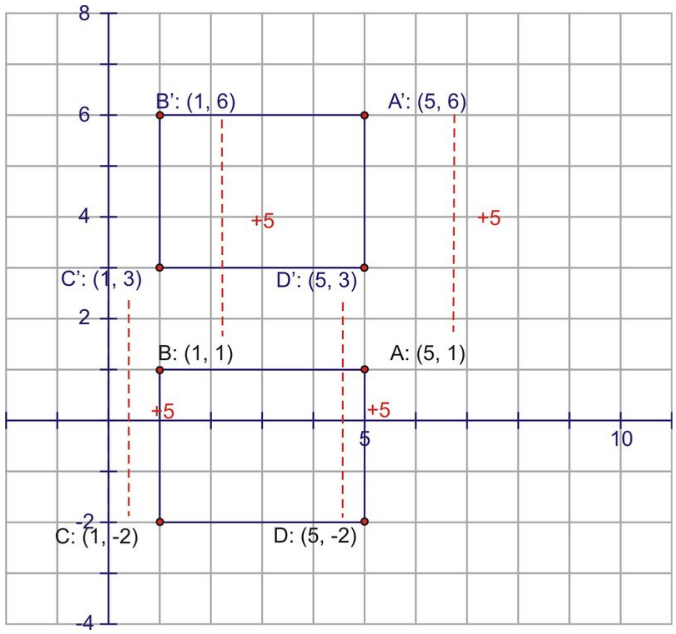
- Translate sine and cosine functions vertically and horizontally.
- Identify the vertical and horizontal translations of sine and cosine from a graph and an equation.

Vertical Translations

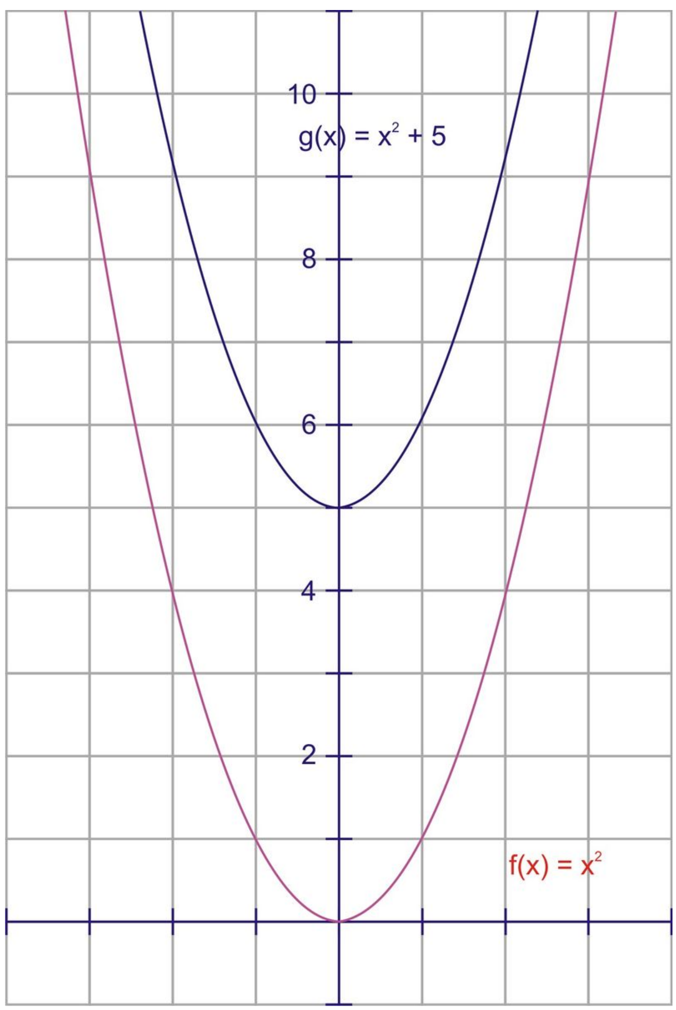
When you first learned about vertical translations in a coordinate grid, you started with simple shapes. Here is a rectangle:



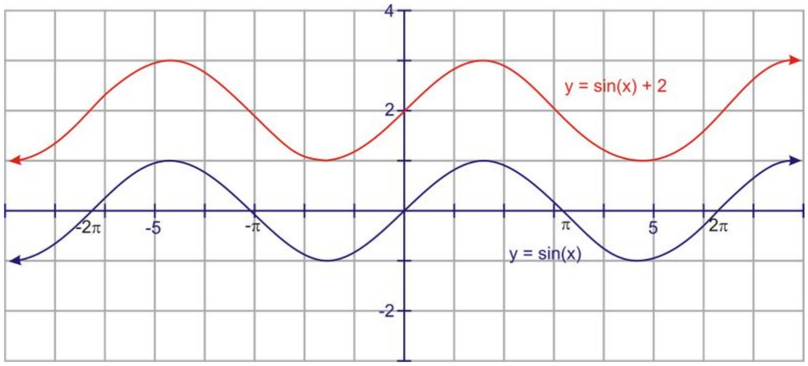
To translate this rectangle vertically, move all points and lines up by a specified number of units. We do this by adjusting the $y-$coordinate of the points. So to translate this rectangle 5 units up, add 5 to every $y-$coordinate.



This process worked the same way for functions. Since the value of a function corresponds to the $y-$value on its graph, to move a function up 5 units, we would increase the value of *the function* by 5. Therefore, to translate $y = x^2$ up five units, you would increase the $y-$value by 5. Because y is equal to x^2, then the equation $y = x^2 + 5$, will show this translation.



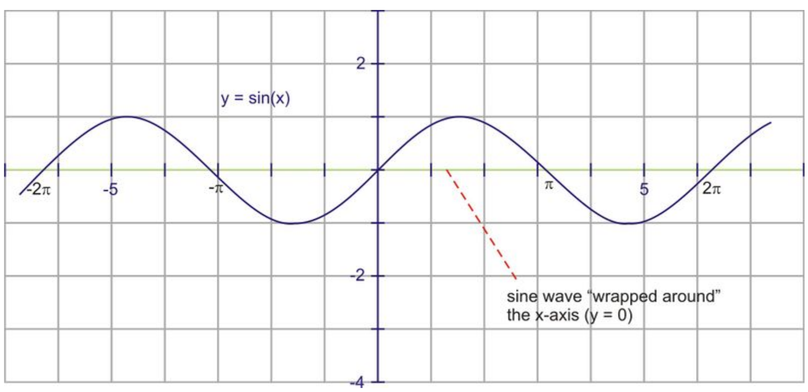
Hence, for any graph, adding a constant to the equation will move it up, and subtracting a constant will move it down. From this, we can conclude that the graphs of $y = \sin x$ and $y = \cos x$ will follow the same rules. That is, the graph of $y = \sin(x) + 2$ will be the same as $y = \sin x$, only it will be translated, or shifted, 2 units up.



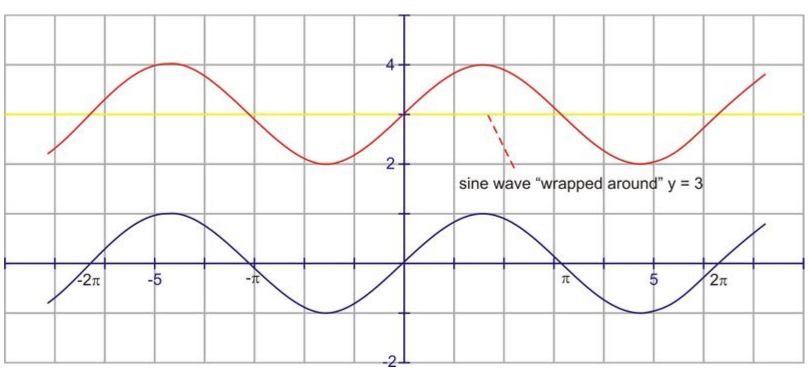
To avoid confusion, this translation is usually written *in front of* the function: $y = 2 + \sin x$.

Various texts use different notation, but we will use D as the constant for vertical translations. This would lead to the following equations: $y = D \pm \sin x$ and $y = D \pm \cos x$ where D is the vertical translation. *D can be positive or negative*.

Another way to think of this is to view sine or cosine curves "wrapped" around a horizontal line. For $y = \sin x$ and $y = \cos x$, the graphs are wrapped around the $x-$axis, or the horizontal line, $y = 0$.



For $y = 3 + \sin x$, we know the curve is translated up 3 units. In this context, think of the sine curve as being "wrapped" around the line, $y = 3$.



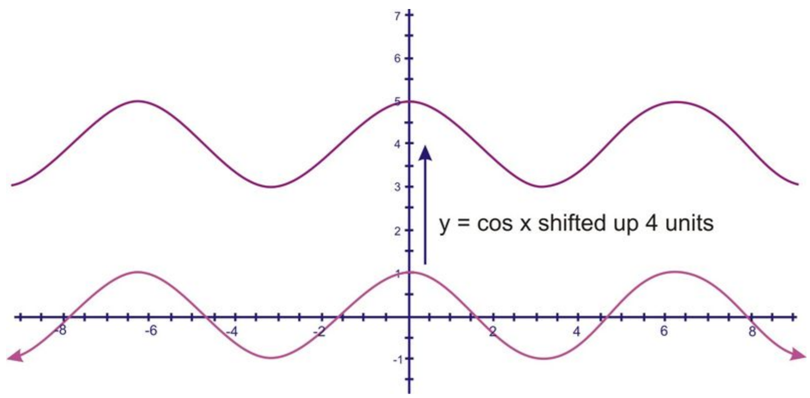
Either method works for the translation of a sine or cosine curve. Pick the thought process that works best for you.

Example 1: Find the minimum and maximum of $y = -6 + \cos x$

Solution: This is a cosine wave that has been shifted down 6 units, or is now wrapped around the line $y = -6$. Because the graph still rises and falls one unit in either direction, the cosine curve will extend one unit above the "wrapping line" and one unit below it. The minimum is -7 and the maximum is -5.

Example 2: Graph $y = 4 + \cos x$.

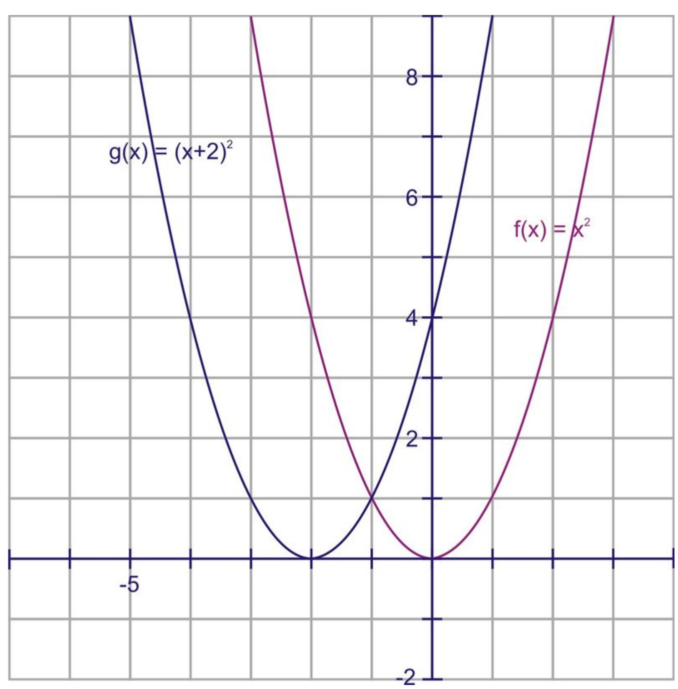
Solution: This will be the basic cosine curve, shifted up 4 units.



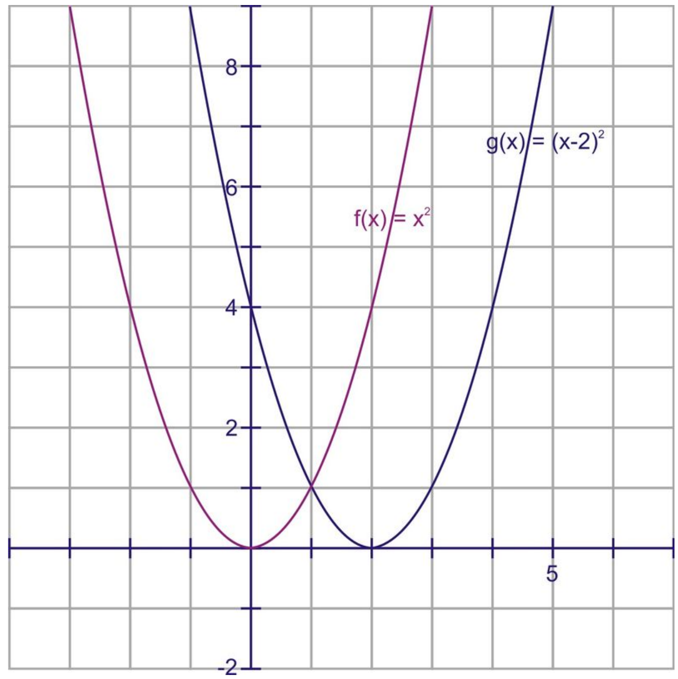
Horizontal Translations or Phase Shifts

Horizontal translations are a little more complicated. If we return to the example of the parabola, $y = x^2$, what change would you make to the equation to have it move to the right or left? Many students guess that if you move the graph vertically by adding to the $y-$value, then we should add to the $x-$value in order to translate horizontally. This is correct, but the graph behaves in the opposite way than what you may think.

Here is the graph of $y = (x+2)^2$.



Notice that *adding* 2 to the $x-$value shifted the graph 2 units *to the left*, or in the *negative* direction.



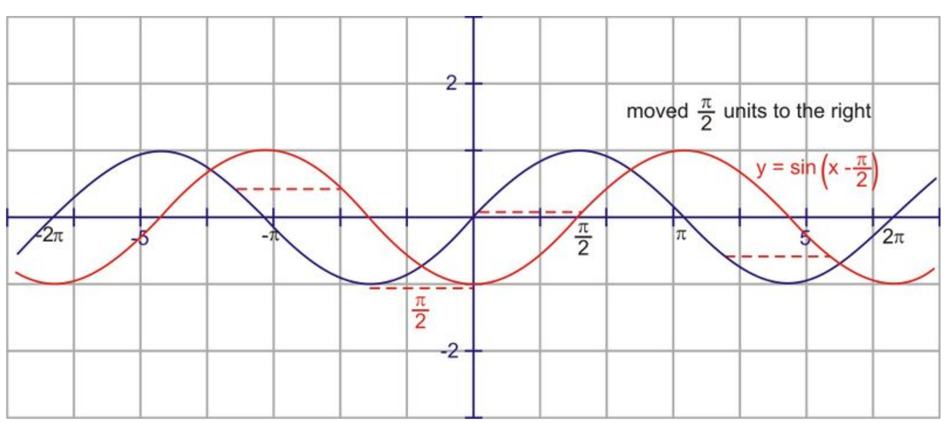
To compare, the graph $y = (x-2)^2$ moves the graph 2 units *to the right* or in the *positive* direction.

We will use the letter C to represent the horizontal shift value. Therefore, **subtracting** C from the $x-$value will shift the graph to the **right** and **adding** C will shift the graph C units to the **left**.

Adding to our previous equations, we now have $y = D \pm \sin(x \pm C)$ and $y = D \pm \cos(x \pm C)$ where D is the vertical translation and C is the *opposite sign* of the horizontal shift.

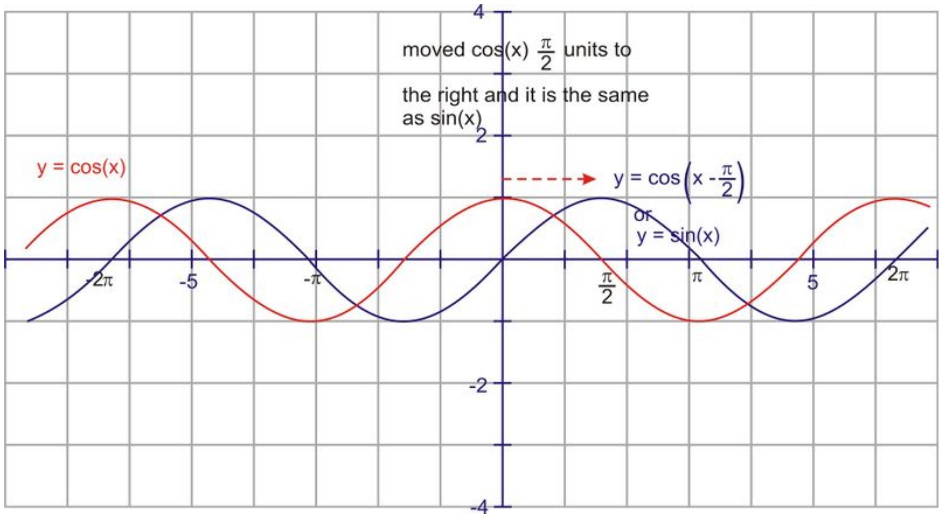
Example 3: Sketch $y = \sin\left(x - \frac{\pi}{2}\right)$

Solution: This is a sine wave that has been translated $\frac{\pi}{2}$ units to the *right*.

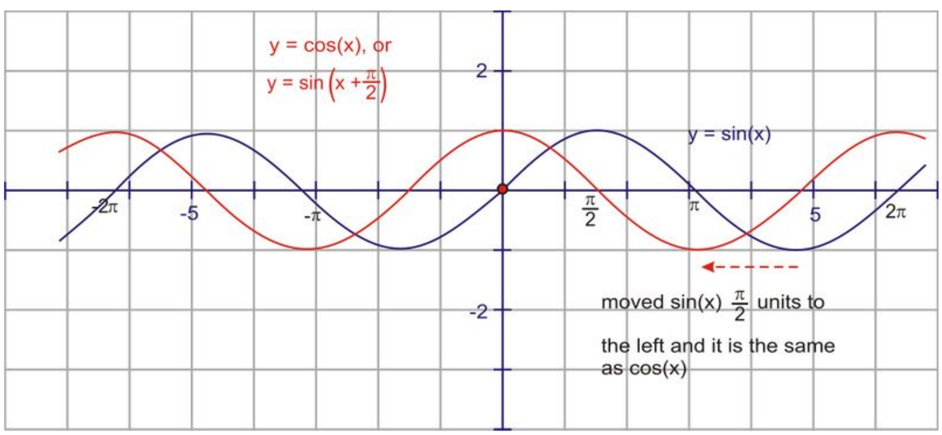


Horizontal translations are also referred to as **phase shifts**. Two waves that are identical, but have been moved horizontally are said to be "out of phase" with each other. Remember that cosine and sine are really the same waves with this phase variation.

$y = \sin x$ can be thought of as a cosine wave shifted horizontally to the right by $\frac{\pi}{2}$ radians.

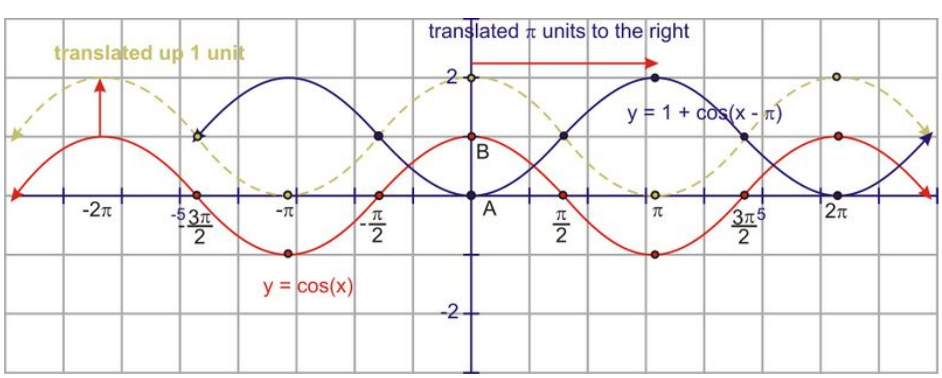


Alternatively, we could also think of cosine as a sine wave that has been shifted $\frac{\pi}{2}$ radians to the left.



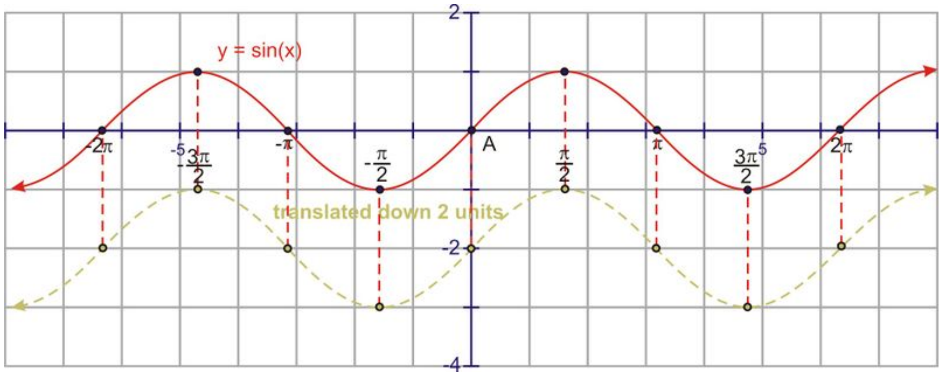
Example 4: Draw a sketch of $y = 1 + \cos(x - \pi)$

Solution: This is a cosine curve that has been translated up 1 unit and π units to the right. It may help you to use the quadrant angles to draw these sketches. Plot the points of $y = \cos x$ at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (as well as the negatives), and then translate those points before drawing the translated curve. The blue curve below is the final answer.

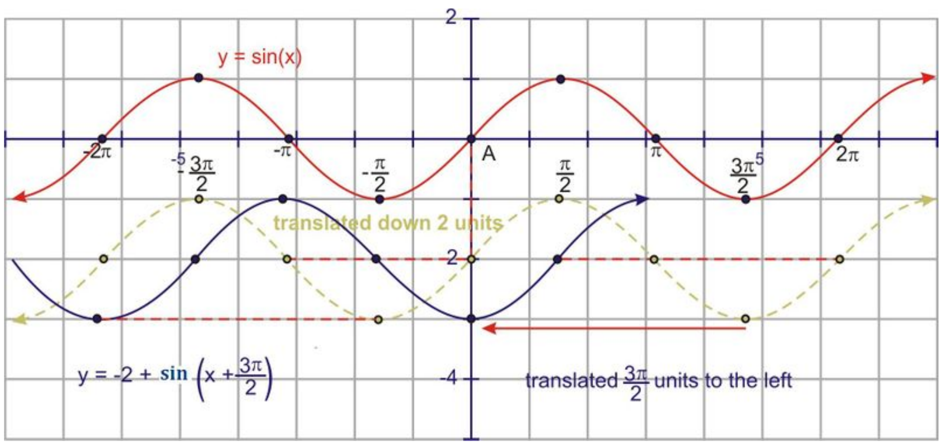


Example 5: Graph $y = -2 + \sin\left(x + \frac{3\pi}{2}\right)$

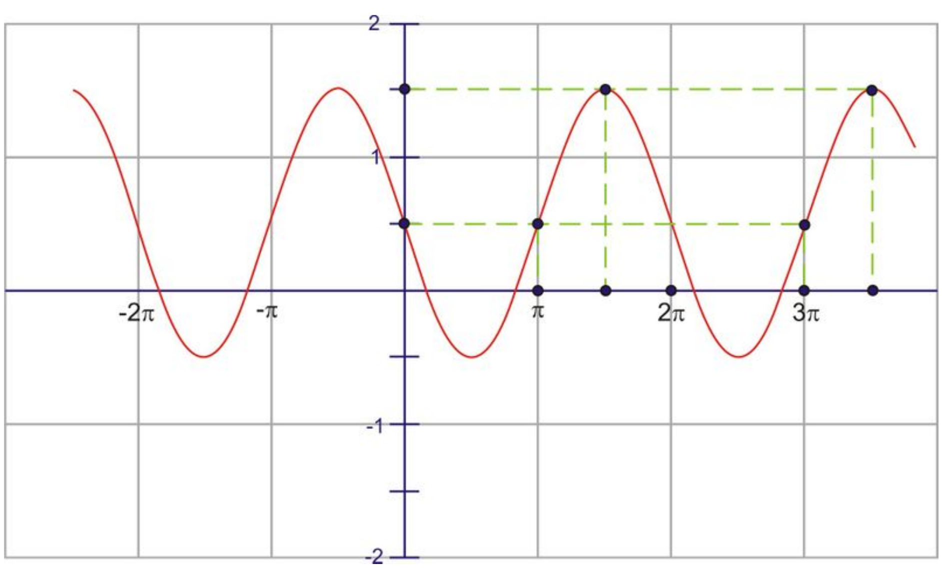
Solution: This is a sine curve that has been translated 2 units down and moved $\frac{3\pi}{2}$ radians to the left. Again, start with the quadrant angles on $y = \sin x$ and translate them down 2 units.



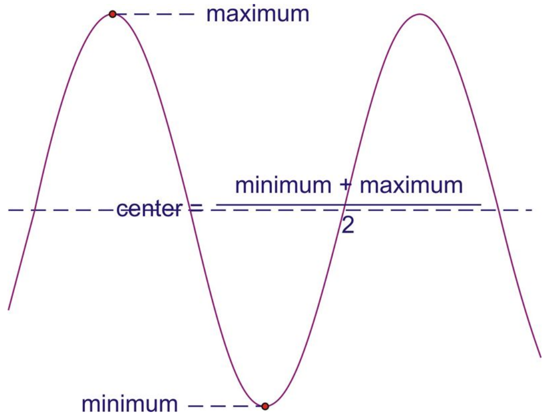
Then, take that result and shift it $\frac{3\pi}{2}$ to the left. The blue graph is the final answer.



Example 6: Write the equation of the following sinusoid



Solution: Notice that you have been given some points to help identify the curve properly. Remember that sine and cosine are essentially the same wave so you can choose to model the sinusoid with either one. Think of the function as a cosine curve because the maximum value of a cosine function is on the $y-$axis, which makes cosine easier to visualize. From the points on the curve, the first maximum point to the right of the $y-$axis occurs at halfway between π and 2π, or $\frac{3\pi}{2}$. Because the next maximum occurs 2π units to the right of that, or at $\frac{7\pi}{2}$, there is no change in the period of this function. This means that the cosine curve has been translated $\frac{3\pi}{2}$ units to the right, or $y = \cos\left(x - \frac{3\pi}{2}\right)$. The vertical translation value can be found by locating the center of the wave. If it is not obvious from the graph, you can find the center by averaging the minimum and maximum values.



This center is the wrapping line of the translated function and is therefore the same as D. In this example, the maximum value is 1.5 and the minimum is -0.5. So,

$$\frac{1.5 + (-0.5)}{2} = \frac{1}{2}$$

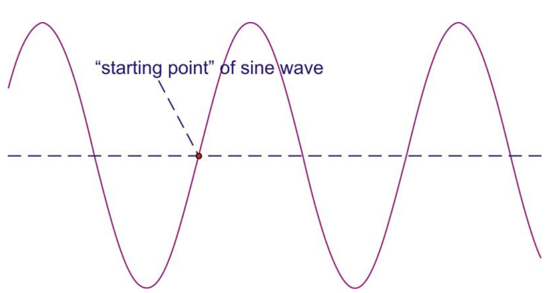
Placing these two values into our equation, $y = D \pm \cos(x \pm C)$, gives:

$$y = \frac{1}{2} + \cos\left(x - \frac{3\pi}{2}\right)$$

Because the cosine graph is periodic, there are an infinite number of possible answers for the horizontal translation. If we keep going in either direction to the next maximum and translate the wave back that far, we will obtain the same graph. Some other possible answers are:

$$y = \frac{1}{2} + \cos\left(x + \frac{\pi}{2}\right), y = \frac{1}{2} + \cos\left(x - \frac{5\pi}{2}\right), \text{and } y = \frac{1}{2} + \cos\left(x - \frac{7\pi}{2}\right).$$

Because sine and cosine are essentially the same function, we could also have modeled the curve with a sine function. Instead of looking for a maximum peak though, for sine we need to find the middle of an increasing part of the wave to consider as a starting point because sine starts at zero.



The coordinates of this point may not always be obvious from the graph. It this case, the drawing shows that the point just to the right of the $y-$axis is $\left(\pi, \frac{1}{2}\right)$. So the horizontal, or C value would be π. The vertical shift, amplitude, and frequency are all the same as the were for the cosine wave because it is the same graph. So the equation would become $y = \frac{1}{2} + \sin(x - \pi)$.

Once again, there are an infinite number of other possible answers if you extend away from the C value multiples of 2π in either direction, such as $y = \frac{1}{2} + \sin(x - 3\pi)$ or $y = \frac{1}{2} + \sin(x - \pi)$.

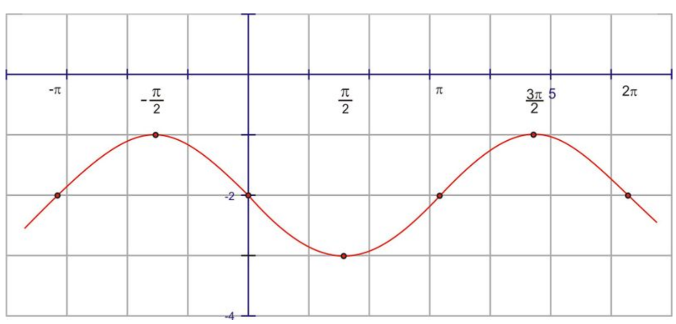
Points to Consider

- Amplitude is the "stretching" of a sine or cosine curve. Where do you think that would go in the equation?
- Do you think the other four trig functions are translated, vertically and horizontally, in the same way as sine and cosine?
- Why is there an infinitely many number of equations that can represent a sine or cosine curve?

Review Questions

For problems 1-5, find the equation that matches each description.

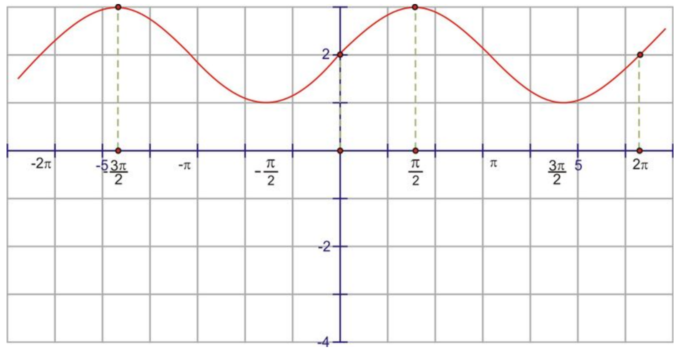
1. ____the minimum value is 0 –A. $y = \sin\left(x - \frac{\pi}{2}\right)$
2. ____the maximum value is 3 –B. $y = 1 + \sin x$
3. ____the $y-$intercept is -2 –C. $y = \cos(x - \pi)$
4. ____the $y-$intercept is -1 –D. $y = -1 + \sin\left(x - \frac{3\pi}{2}\right)$
5. ____the same graph as $y = \cos(x)$ –E. $y = 2 + \cos x$
6. Express the equation of the following graph as both a sine and a cosine function. Several points have been plotted at the quadrant angles to help.



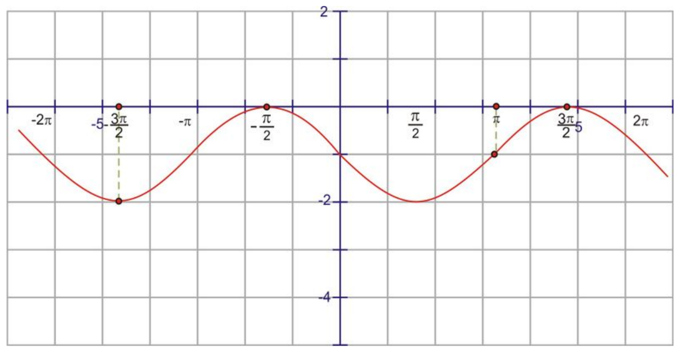
For problems 7-10, match the graph with the correct equation.

7. ____ $y = 1 + \sin\left(x - \frac{\pi}{2}\right)$
8. ____ $y = -1 + \cos\left(x + \frac{3\pi}{2}\right)$
9. ____ $y = 1 + \cos\left(x - \frac{\pi}{2}\right)$
10. ____ $y = -1 + \sin(x - \pi)$

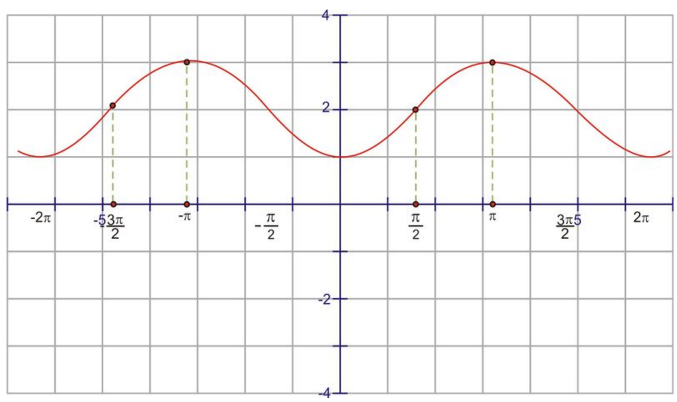
- A.



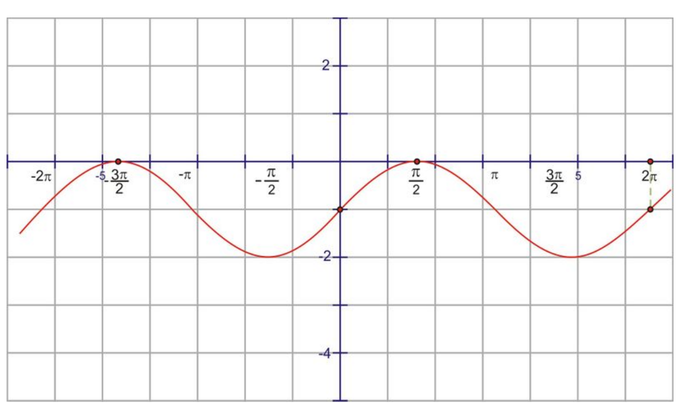
- B.



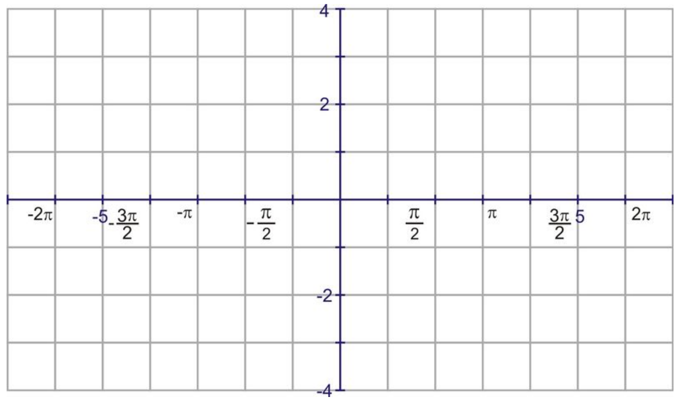
- C.



- D.



11. Sketch the graph of $y = 1 + \sin\left(x - \frac{\pi}{4}\right)$ on the axes below.


4
2
-2
-4
-2π
-5
$-\frac{3\pi}{2}$
-π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
5
2π

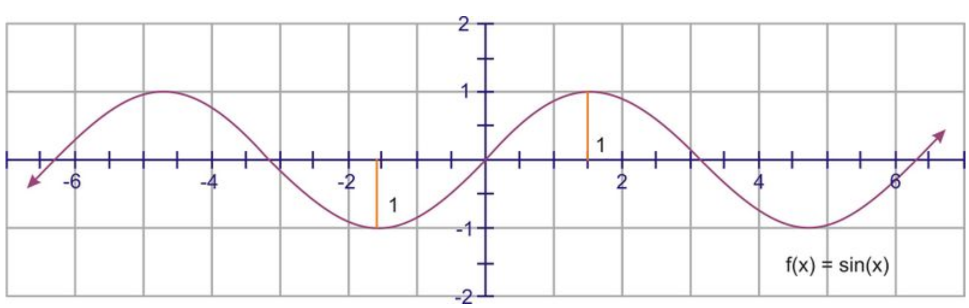
2.5 Amplitude, Period and Frequency

Learning Objectives

- Calculate the amplitude and period of a sine or cosine curve.
- Calculate the frequency of a sine or cosine wave.
- Graph transformations of sine and cosine waves involving changes in amplitude and period (frequency).

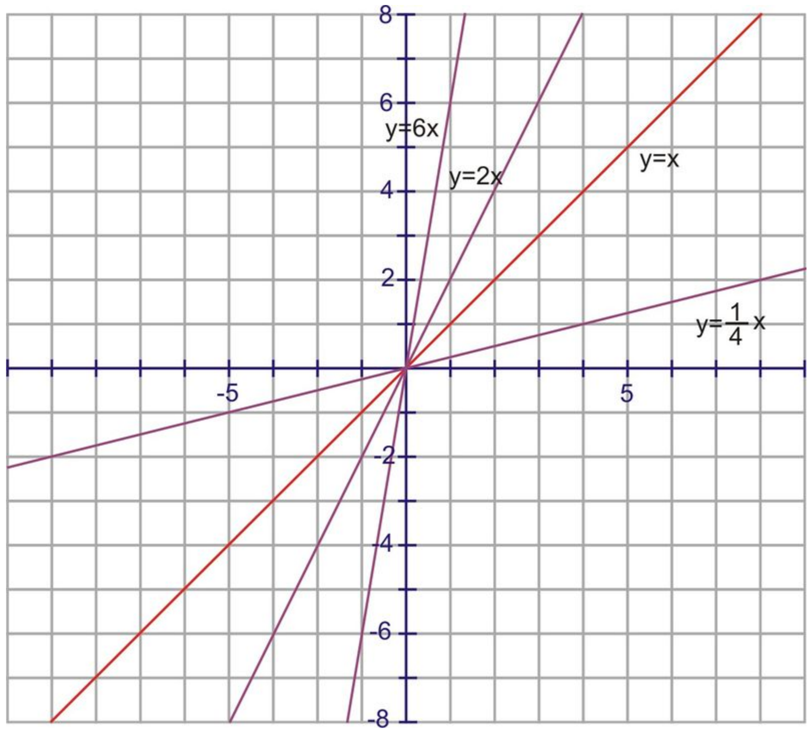
Amplitude

The **amplitude** of a wave is basically a measure of its height. Because that height is constantly changing, amplitude is defined as the *farthest* distance the wave gets from its center. In a graph of $f(x) = \sin x$, the wave is centered on the $x-$axis and the farthest away it gets (in either direction) from the axis is 1 unit.



So the amplitude of $f(x) = \sin x$ (and $f(x) = \cos x$) is 1.

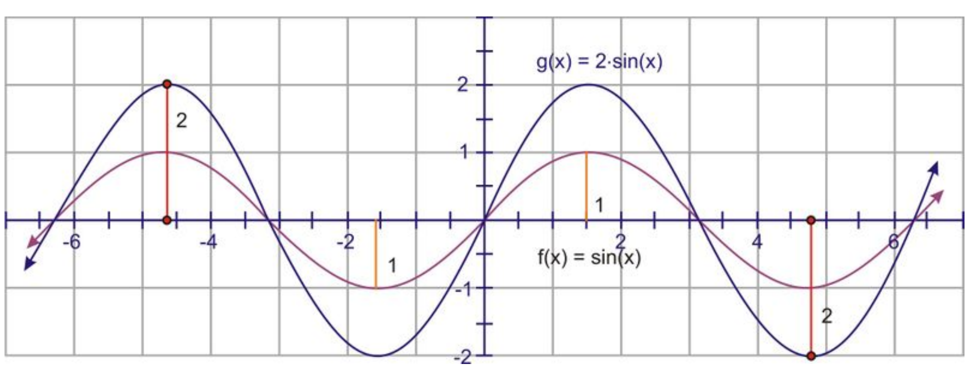
Recall how to transform a linear function, like $y = x$. By placing a constant in front of the x value, you may remember that the slope of the graph affects the steepness of the line.



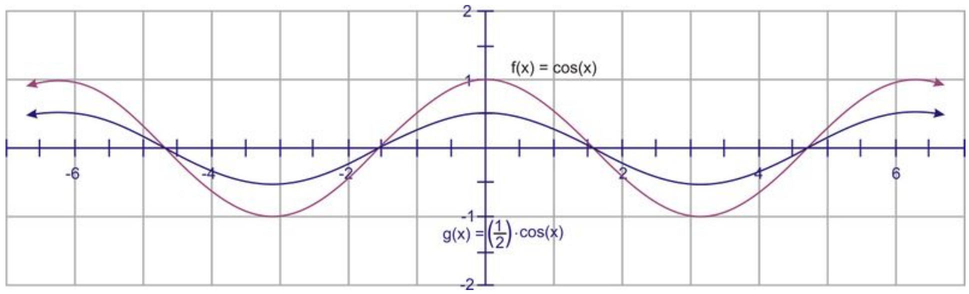
The same is true of a parabolic function, such as $y = x^2$. By placing a constant in front of the x^2, the graph would be either wider or narrower. So, a function such as $y = \frac{1}{8}x^2$, has the same parabolic shape but it has been "smooshed," or looks wider, so that it increases or decreases at a lower rate than the graph of $y = x^2$.

No matter the basic function; linear, parabolic, or trigonometric, the same principle holds. To dilate (flatten or steepen, wide or narrow) the function, multiply the function by a constant. Constants greater than 1 will stretch the graph vertically and those less than 1 will shrink it vertically.

Look at the graphs of $y = \sin x$ and $y = 2\sin x$.

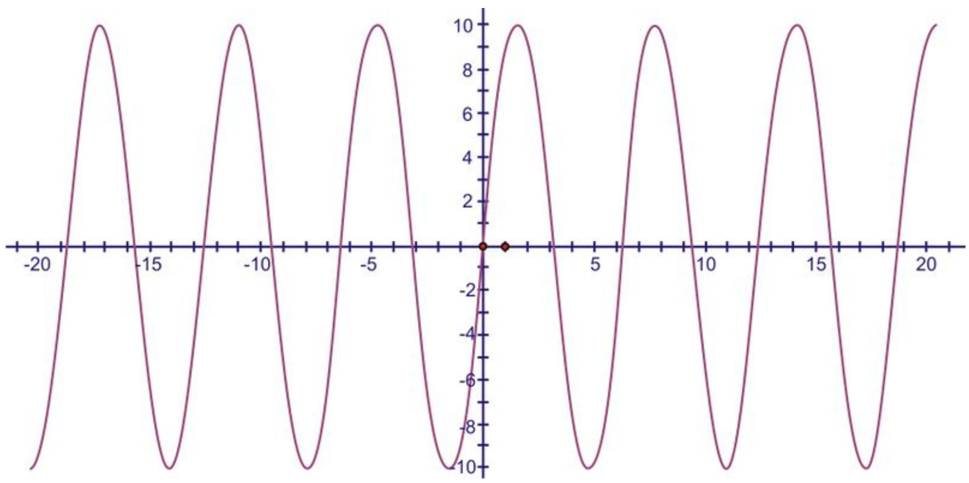


Notice that the amplitude of $y = 2\sin x$ is now 2. An investigation of some of the points will show that each $y-$value is twice as large as those for $y = \sin x$. Multiplying values less than 1 will decrease the amplitude of the wave as in this case of the graph of $y = \frac{1}{2}\cos x$:



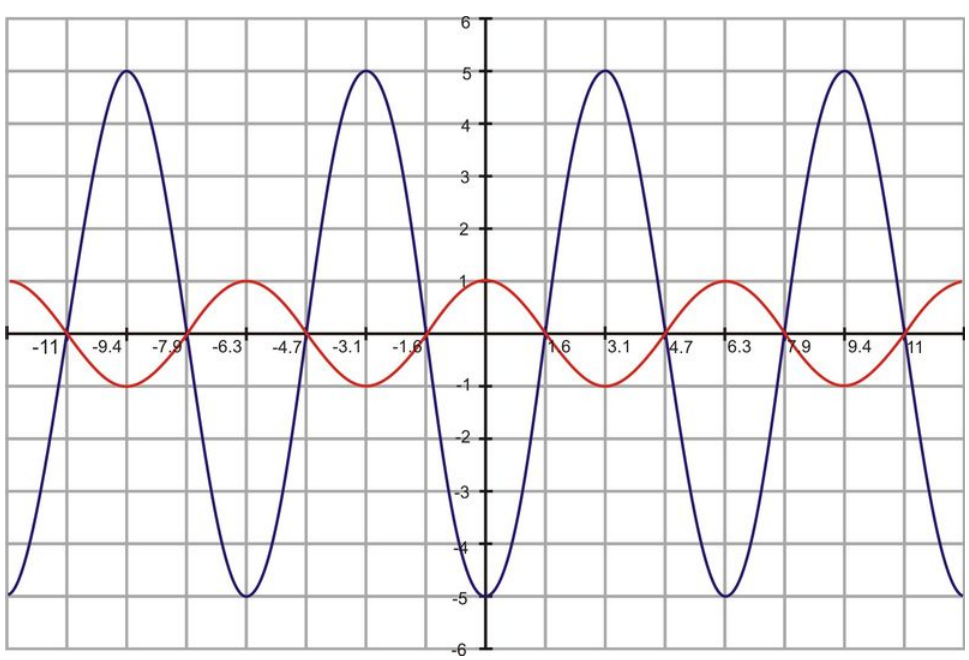
Example 1: Determine the amplitude of $f(x) = 10\sin x$.

Solution: The 10 indicates that the amplitude, or height, is 10. Therefore, the function rises and falls between 10 and -10.



Example 2: Graph $g(x) = -5\cos x$

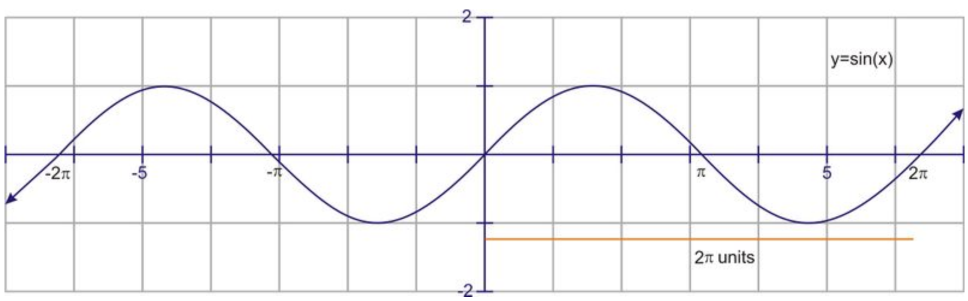
Solution: Even though the 5 is negative, the amplitude is still positive 5. The amplitude is always the absolute value of the constant A. However, the negative changes the appearance of the graph. Just like a parabola, the sine (or cosine) is flipped upside-down. Compare the blue graph, $g(x) = -5\cos x$, to the red parent graph, $f(x) = \cos x$.



So, in general, the constant that creates this stretching or shrinking is the amplitude of the sinusoid. Continuing with our equations from the previous section, we now have $y = D \pm A\sin(x \pm C)$ or $y = D \pm A\cos(x \pm C)$. Remember, if $0 < |A| < 1$, then the graph is shrunk and if $|A| > 1$, then the graph is stretched. And, if A is negative, then the graph is flipped.

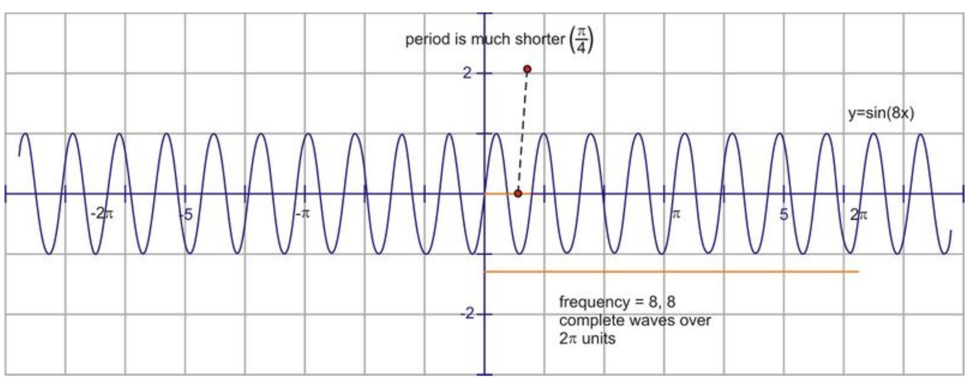
Period and Frequency

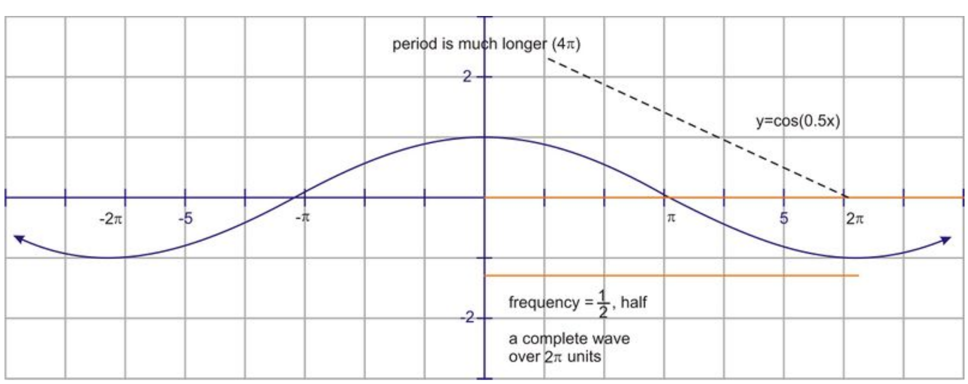
The **period** of a trigonometric function is the horizontal distance traversed before the $y-$values begin to repeat. For both graphs, $y = \sin x$ and $y = \cos x$, the period is 2π. As we learned earlier in the chapter, after completing one rotation of the unit circle, these values are the same.



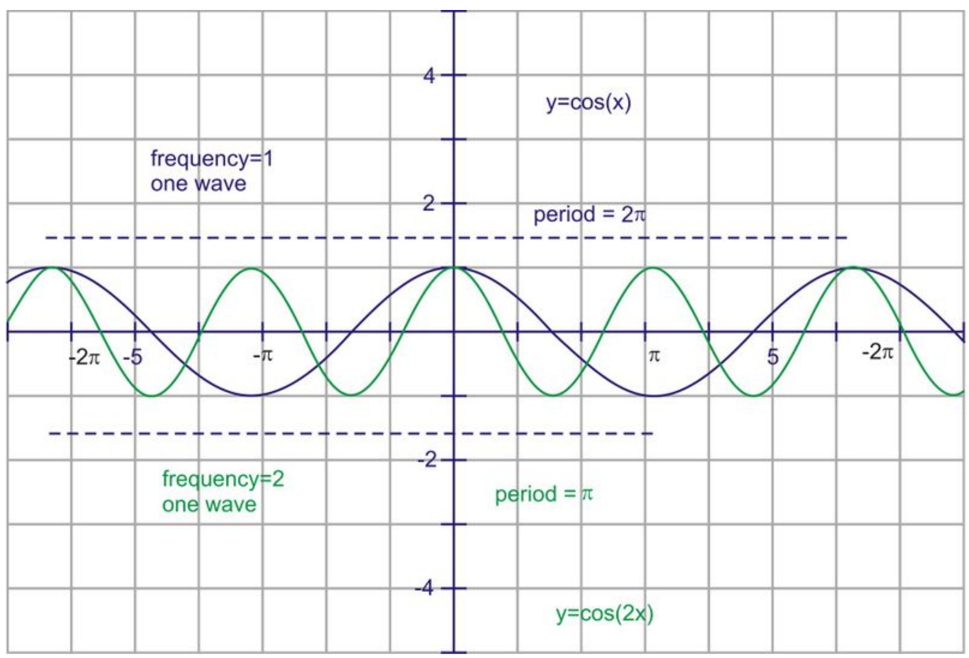
Frequency is a measurement that is closely related to period. In science, the frequency of a sound or light wave is the number of complete waves for a given time period (like seconds). In trigonometry, because all of these periodic functions are based on the unit circle, we usually measure frequency as the number of complete waves every 2π units. Because $y = \sin x$ and $y = \cos x$ cover exactly one complete wave over this interval, their frequency is 1.

Period and frequency are inversely related. That is, the higher the frequency (more waves over 2π units), the lower the period (shorter distance on the $x-$axis for each complete cycle).





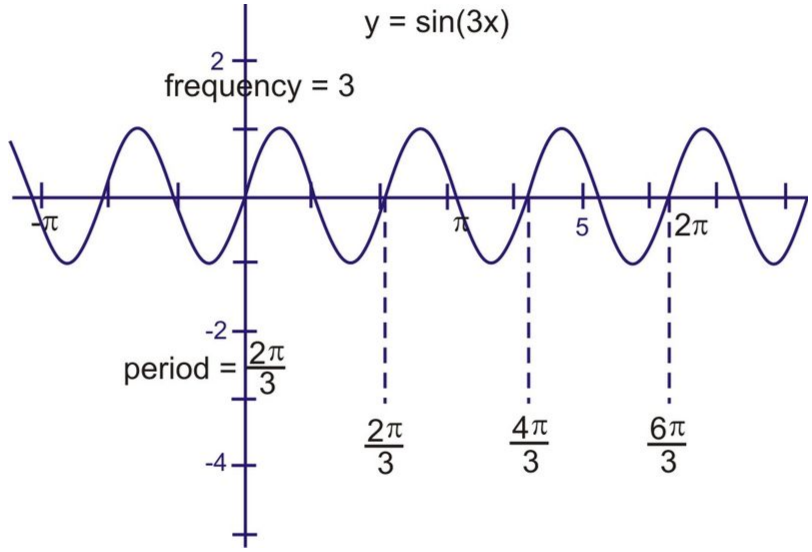
After observing the transformations that result from multiplying a number *in front of* the sinusoid, it seems natural to look at what happens if we multiply a constant *inside* the argument of the function, or in other words, by the x value. In general, the equation would be $y = \sin Bx$ or $y = \cos Bx$. For example, look at the graphs of $y = \cos 2x$ and $y = \cos x$.



Notice that the number of waves for $y = \cos 2x$ has increased, in the same interval as $y = \cos x$. There are now **2 waves** over the interval from 0 to 2π. Consider that you are doubling each of the x values because the function is $2x$. When π is plugged in, for example, the function becomes 2π. So the portion of the graph that normally corresponds to 2π units on the $x-$axis, now corresponds to *half* that distance—so the graph has been "scrunched" horizontally. The frequency of this graph is therefore 2, or the same as the constant we multiplied by in the argument. The period (the length for each complete wave) is π.

Example 3: What is the frequency and period of $y = \sin 3x$?

Solution: If we follow the pattern from the previous example, multiplying the angle by 3 should result in the sine wave completing a cycle **three times** as often as $y = \sin x$. So, there will be three complete waves if we graph it from 0 to 2π. The frequency is therefore 3. Similarly, if there are 3 complete waves in 2π units, one wave will be a third of that distance, or $\frac{2\pi}{3}$ radians. Here is the graph:



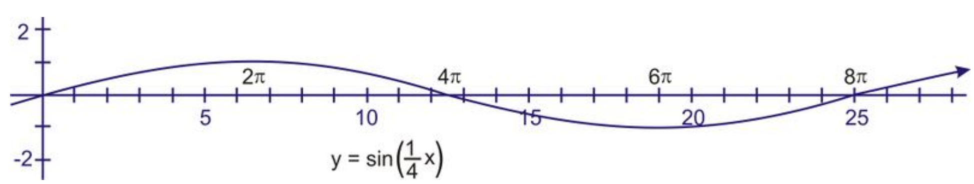
This number that is multiplied by x, called B, will create a horizontal dilation. The larger the value of B, the more compressed the waves will be horizontally. To stretch out the graph horizontally, we would need to *decrease* the frequency, or multiply by a number that is less than 1. Remember that this dilation factor is *inversely* related to the period of the graph.

Adding, one last time to our equations from before, we now have: $y = D \pm A\sin(B(x \pm C))$ or $y = D \pm A\cos(B(x \pm C))$, where B is the frequency, the period is equal to $\frac{2\pi}{B}$, and everything else is as defined before.

Example 4: What is the frequency and period of $y = \cos\frac{1}{4}x$?

Solution: Using the generalization above, the frequency must be $\frac{1}{4}$ and therefore the period is $\frac{2\pi}{\frac{1}{4}}$, which simplifies to: $\frac{2\pi}{\frac{1}{4}} = \frac{\frac{2\pi}{1}}{\frac{1}{4}} \cdot \frac{\frac{4}{1}}{\frac{4}{1}} = \frac{8\pi}{1} = 8\pi$

Thinking of it as a transformation, the graph is stretched horizontally. We would only see $\frac{1}{4}$ of the curve if we graphed the function from 0 to 2π. To see a complete wave, therefore, we would have to go four times as far, or all the way from 0 to 8π.

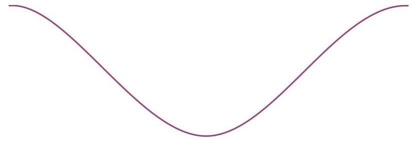


Combining Amplitude and Period

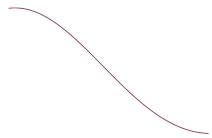
Here are a few examples with both amplitude and period.

Example 5: Find the period, amplitude and frequency of $y = 2\cos\frac{1}{2}x$ and sketch a graph from 0 to 2π.

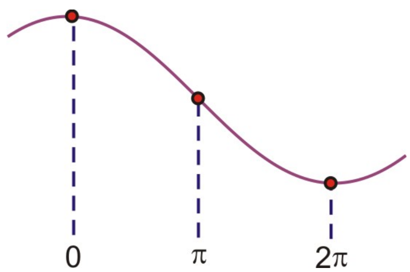
Solution: This is a cosine graph that has been stretched both vertically and horizontally. It will now reach up to 2 and down to -2. The frequency is $\frac{1}{2}$ and to see a complete period we would need to graph the interval $[0, 4\pi]$. Since we are only going out to 2π, we will only see half of a wave. A complete cosine wave looks like this:



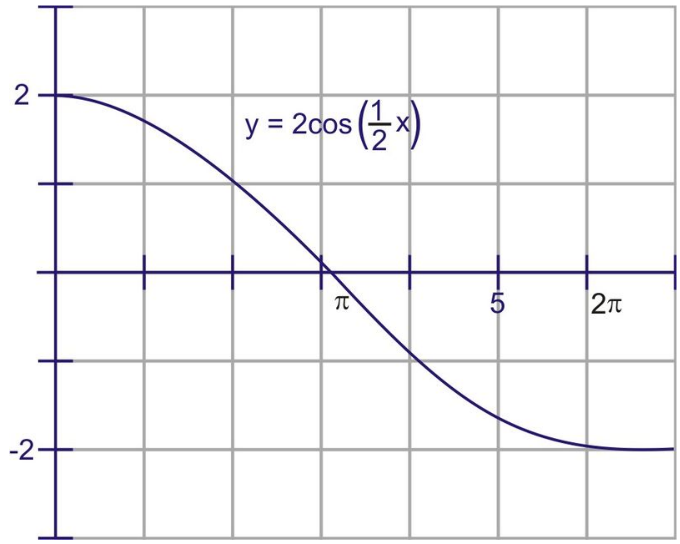
So, half of it is this:



This means that this half needs to be stretched out so it finishes at 2π, which means that at π the graph should cross the $x-$axis:

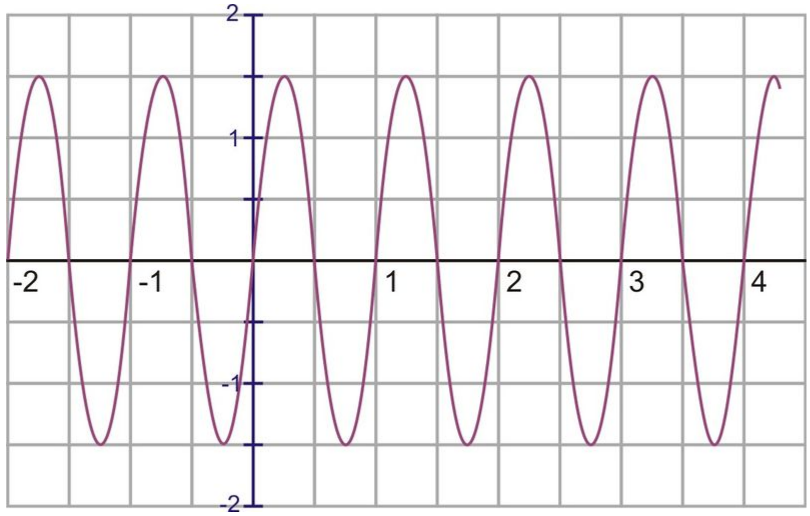


The final sketch would look like this:



$\text{amplitude} = 2, \text{frequency} = \frac{1}{2}, \text{period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Example 6: Identify the period, amplitude, frequency, and equation of the following sinusoid:



Solution: The amplitude is 1.5. Notice that the units on the $x-$axis are not labeled in terms of π. This appears to be a sine wave because the $y-$intercept is 0.

One wave appears to complete in 1 unit (*not* 1π *units!*), so the period is 1. If one wave is completed in 1 unit, how many waves will be in 2π units? In previous examples, you were given the frequency and asked to find the period using the following relationship:

$$p = \frac{2\pi}{B}$$

Where B is the frequency and p is the period. With just a little bit of algebra, we can transform this formula and solve it for B:

$$p = \frac{2\pi}{B} \to Bp = 2\pi \to B = \frac{2\pi}{p}$$

Therefore, the frequency is:

$$B = \frac{2\pi}{1} = 2\pi$$

If we were to graph this out to 2π we would see 2π (or a little more than 6) complete waves.

Replacing these values in the equation gives: $f(x) = 1.5\sin 2\pi x$.

Points to Consider

- Are the graphs of the other four trigonometric functions affected in the same way as sine and cosine by amplitude and period?
- We saw what happens to a graph when A is negative. What happens when B is negative?

Review Questions

1. Using the graphs from section 3, identify the period and frequency of $y = \sec x, y = \cot x$ and $y = \csc x$.
2. Identify the minimum and maximum values of these functions.

   a. $y = \cos x$
   b. $y = 2\sin x$
   c. $y = -\sin x$
   d. $y = \tan x$
   e. $y = \frac{1}{2}\cos 2x$
   f. $y = -3\sin 4x$

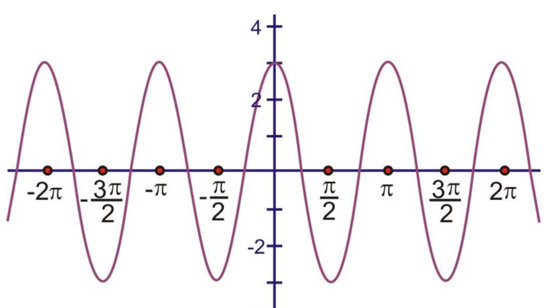
3. How many real solutions are there for the equation $4\sin x = \sin x$ over the interval $0 \leq x \leq 2\pi$?

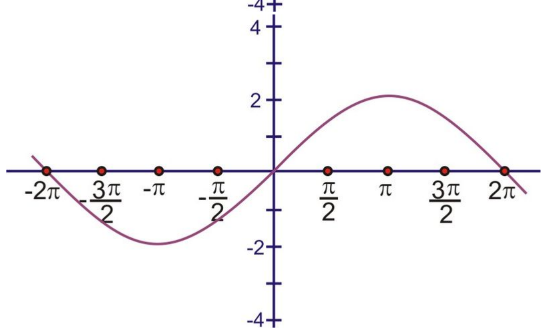
   a. 0
   b. 1
   c. 2
   d. 3

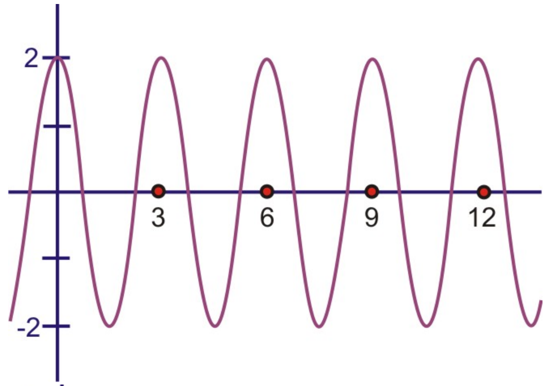
4. For each equation, identify the period, amplitude, and frequency.

   a. $y = \cos 2x$
   b. $y = 3\sin x$
   c. $y = 2\sin \pi x$
   d. $y = 2\cos 3x$
   e. $y = \frac{1}{2}\cos \frac{1}{2}x$
   f. $y = 3\sin \frac{1}{2}x$

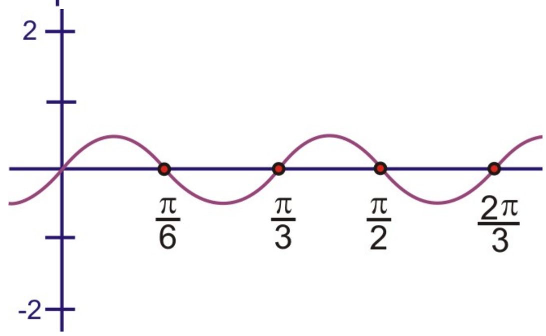
5. For each of the following graphs; 1) identify the period, amplitude, and frequency and 2) write the equation.

a. 

b. 



c.



d.

6. For each equation, draw a sketch from 0 to 2π.
   a. $y = 3\sin 2x$
   b. $y = 2.5\cos \pi x$
   c. $y = 4\sin \frac{1}{2}x$

2.6 General Sinusoidal Graphs

Learning Objectives

- Given any sinusoid in the form: $y = D \pm A\cos(B(x \pm C))$ or $y = D \pm A\sin(B(x \pm C))$ identify the transformations performed by A, B, C, and D.
- Graph any sinusoid given an equation in the form $y = D \pm A\cos(B(x \pm C))$ or $y = D \pm A\sin(B(x \pm C))$.
- Identify the equation of any sinusoid given a graph and critical values.

The Generalized Equations

In the previous two sections, you learned how to translate and dilate sine and cosine waves both horizontally and vertically. Combining all the information learned, the general equations are: $y = D \pm A\cos(B(x \pm C))$ or $y = D \pm A\sin(B(x \pm C))$, where A is the amplitude, B is the frequency, C is the horizontal translation, and D is the vertical translation.

Recall the relationship between period, p, and frequency, B.

$$p = \frac{2\pi}{B} \text{ and } B = \frac{2\pi}{p}$$

With this knowledge, we should be able to sketch any sine or cosine function as well as write an equation given its graph.

Drawing Sketches/Identifying Transformations from the Equation

Example 1: Given the function: $f(x) = 1 + 2\sin(2(x + \pi))$

a. Identify the period, amplitude, and frequency.

b. Explain any vertical or horizontal translations present in the equation.

c. Sketch the graph from -2π to 2π.

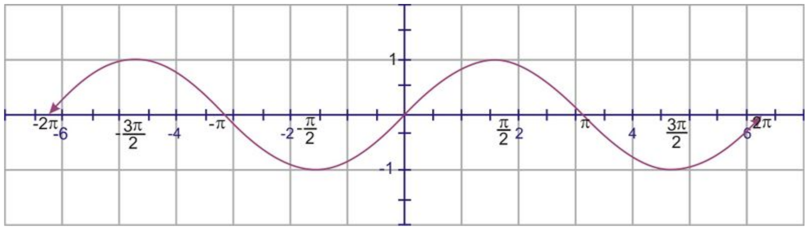
Solution: a. From the equation, the amplitude is 2 and the frequency is also 2. To find the period we use:

$$p = \frac{2\pi}{B} \rightarrow p = \frac{2\pi}{2} = \pi$$

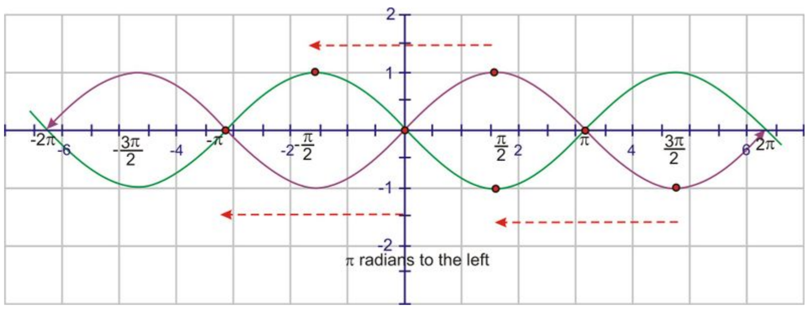
So, there are two complete waves from $[0, 2\pi]$ and each individual wave requires π radians to complete.

b. $D = 1$ and $C = -\pi$, so this graph has been translated 1 unit up, and π units to the left.

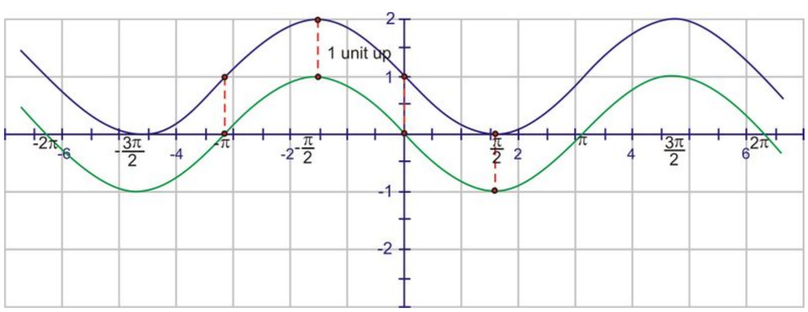
c. To sketch the graph, start with the graph of $y = \sin(x)$



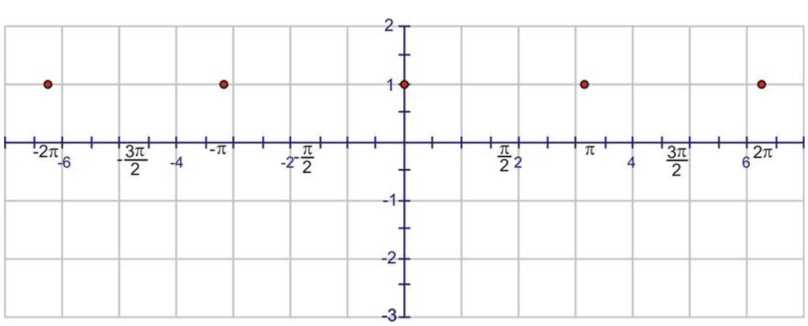
Translate the graph π units to the left (the *C* value).



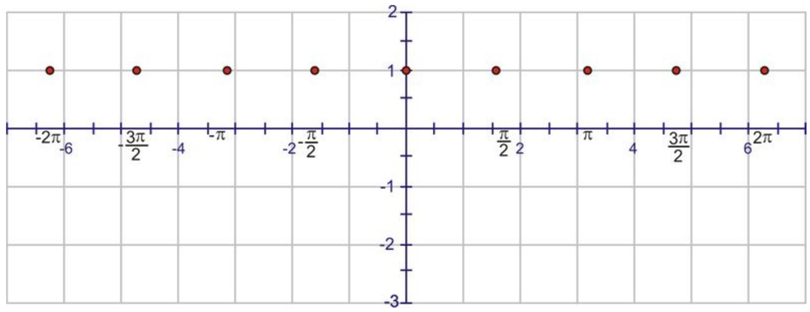
Next, move the graph 1 unit up (*D* value)



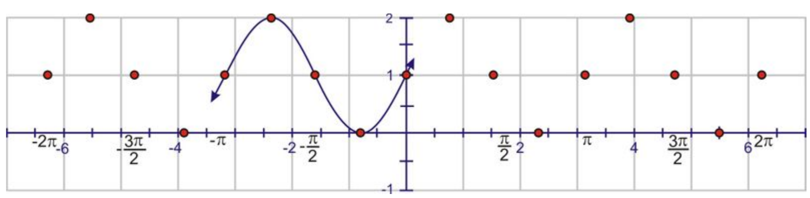
Now we can add the dilations. Remember that the "starting point" of the wave is $-\pi$ because of the horizontal translation. A normal sine wave takes 2π units to complete a cycle, but this wave completes one cycle in π units. The first wave will complete at 0, then we will see a second wave from 0 to π and a third from π to 2π. Start by placing points at these values:



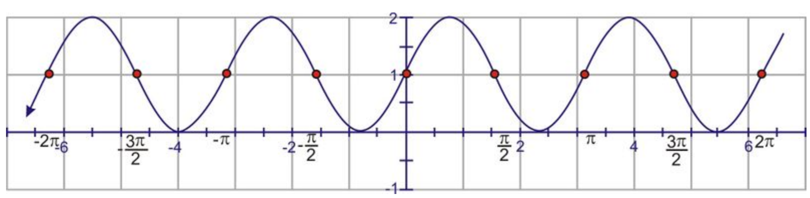
Using symmetry, each interval needs to cross the line $y = 1$ through the center of the wave.



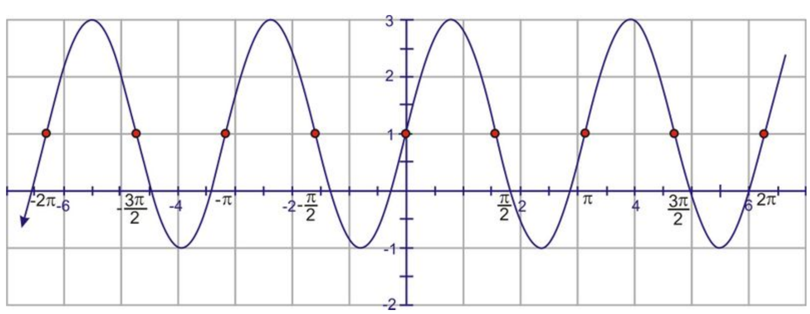
One sine wave contains a "mountain" and a "valley". The mountain "peak" and the valley low point must occur halfway between the points above.



Extend the curve through the domain.



Finally, extend the minimum and maximum points to match the amplitude of 2.



Example 2: Given the function: $f(x) = 3 + 3\cos\left(\frac{1}{2}(x - \frac{\pi}{2})\right)$

a. Identify the period, amplitude, and frequency.

b. Explain any vertical or horizontal translations present in the equation.

c. Sketch the graph from -2π to 2π.

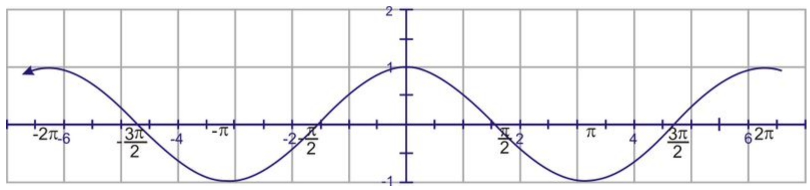
Solution: a. From the equation, the amplitude is 3 and the frequency is $\frac{1}{2}$. To find the period we use:

$$period = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

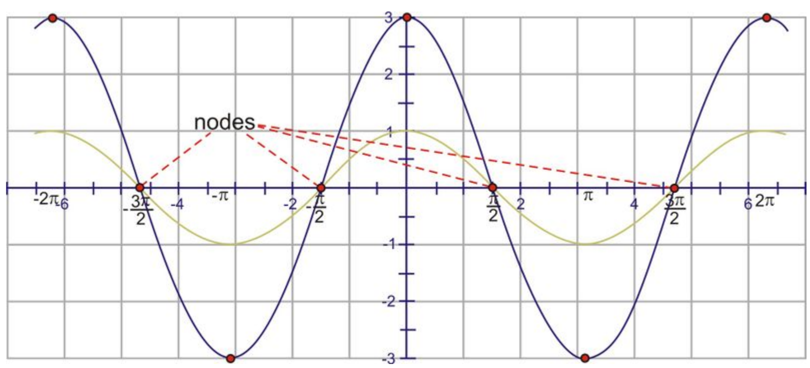
So, there is only one half of a cosine curve from 0 to 2π and each individual wave requires 4π radians to complete.

b. $D = 3$ and $C = \frac{\pi}{2}$, so this graph has been translated 3 units up, and $\frac{\pi}{2}$ units to the right.

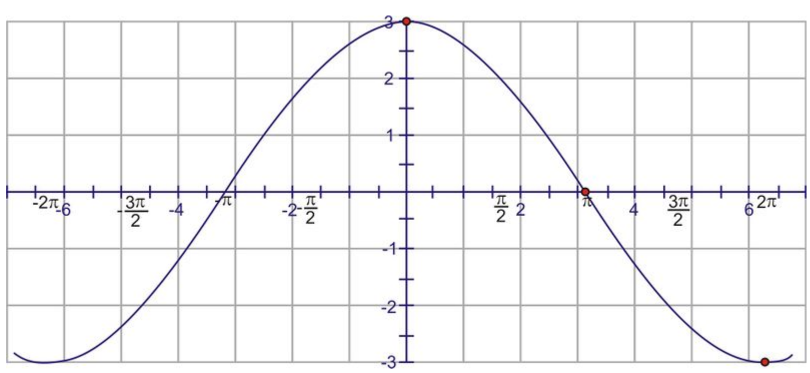
c. To sketch the graph, start with the graph of $y = \cos(x)$



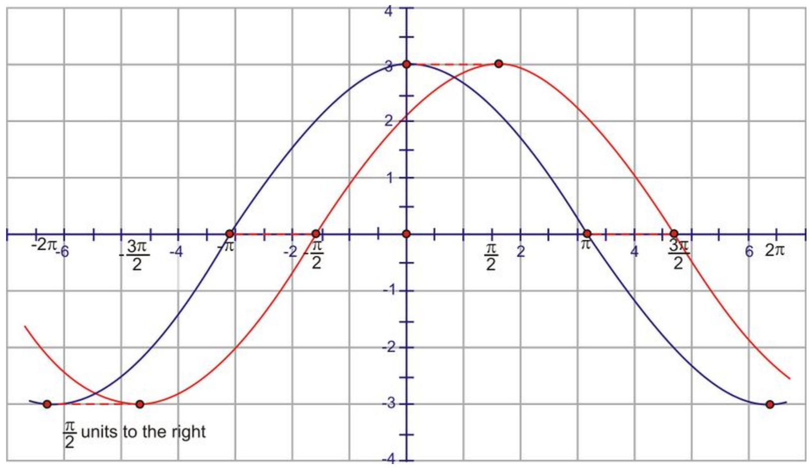
Adjust the amplitude so the cosine wave reaches up to 3 and down to negative three. This affects the maximum points, but the points on the $x-$axis remain the same. These points are sometimes called **nodes.**



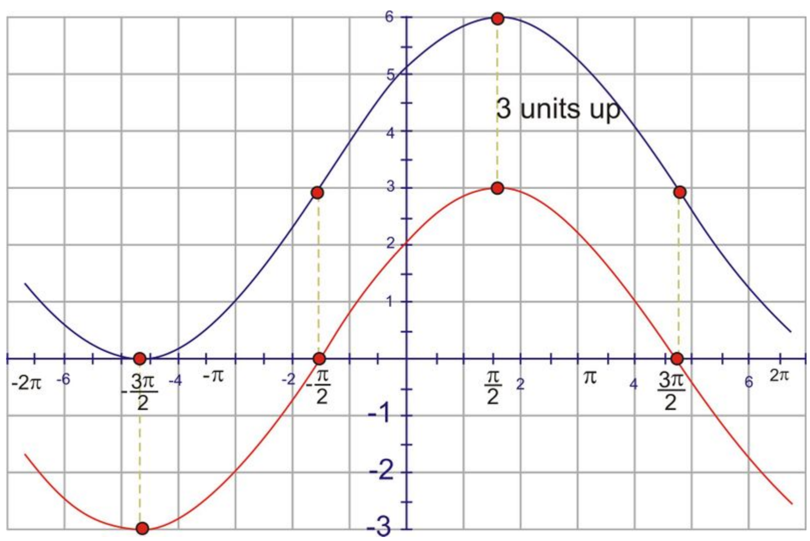
According to the period, we should see one of these shapes every 4π units. Because the interval specified is $[-2\pi, 2\pi]$ and the cosine curve "starts" at the $y-$axis, at (0, 3) and at 2π the value is -3. Conversely, at -2π, the function is also -3.



Now, shift the graph $\frac{\pi}{2}$ units to the right.



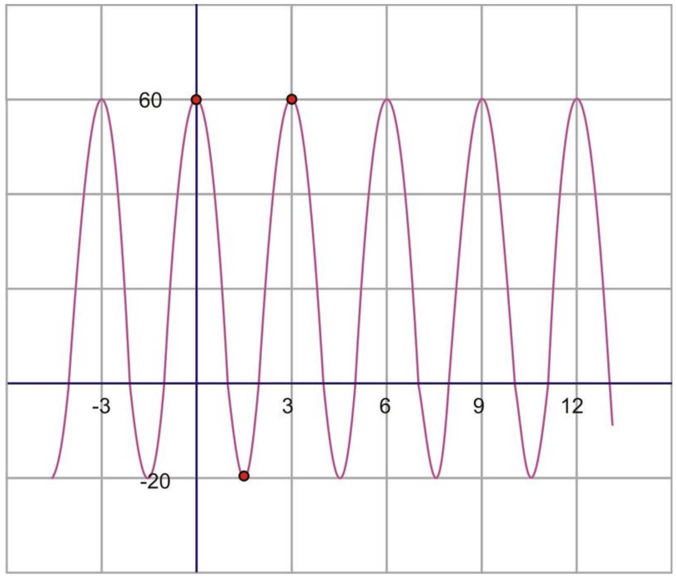
Finally, we need to adjust for the vertical shift by moving it up 3 units.



Writing the Equation from a Sketch

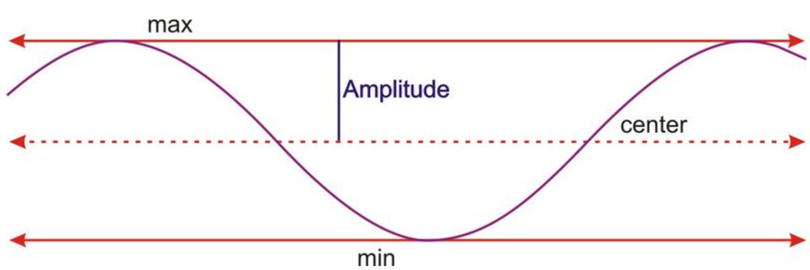
In order to write the equation from a graph, you need to be provided with enough information to find the four constants.

Example 3: Find the equation of the sinusoid graphed here.



Solution: First of all, remember that either sine or cosine could be used to model these graphs. However, it is usually easier to use cosine because the horizontal shift is easier to locate in most cases. Therefore, the model that we will be using is $y = D \pm A\cos(B(x \pm C))$.

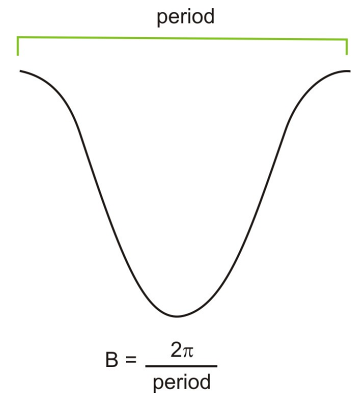
First, if we think of the graph as a cosine function, it has a horizontal translation of zero. The maximum point is also the $y-$intercept of the graph, so there is no need to shift the graph horizontally and therefore, $C = 0$. The amplitude is the height from the center of the wave. If you can't find the center of the wave by sight, you can calculate it. The center should be halfway between the highest and the lowest points, which is really the **average** of the maximum and minimum. This value will actually be the vertical shift, or D value.



$$D = center = \frac{60 + -20}{2} = \frac{40}{2} = 20$$

The amplitude is the height from the center line, or vertical shift, to either the minimum or the maximum. So, $A = 60 - 20 = 40$.

The last value to find is the frequency. In order to do so, we must first find the period. The period is the distance required for one complete wave. To find this value, look at the horizontal distance between two consecutive maximum points.



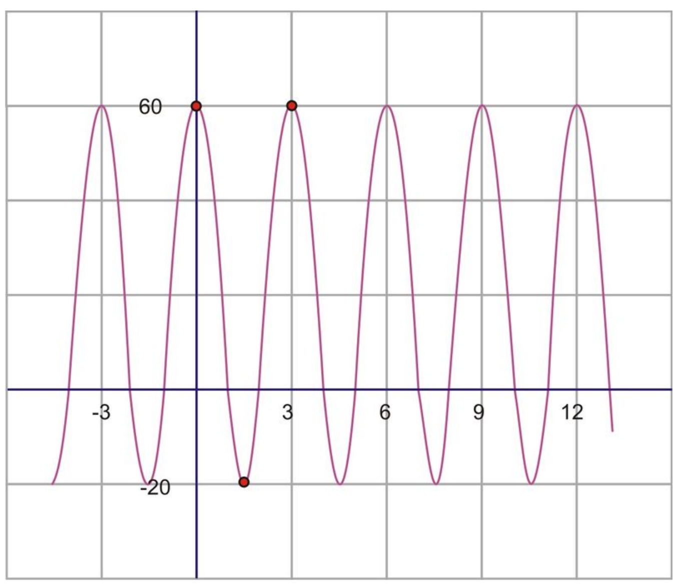
On our graph, from maximum to maximum is 3.

Therefore, the period is 3, so the frequency is $B = \frac{2\pi}{3}$.

We have now calculated each of the four parameters necessary to write the equation. Replacing them in the equation gives:

$$y = 20 + 40 \cos \frac{2\pi}{3} x$$

If we had chosen to model this curve with a sine function instead, the amplitude, period and frequency, as well as the vertical shift would all be the same. The only difference would be the horizontal shift. The sine wave starts in the middle of an upward sloped section of the curve as shown by the red circle.



This point intersects with the vertical translation line and is a third of the distance back to -3. So, in this case, the sine wave has been translated 1 unit *to the left*. The equation using a sine function instead would have been: $y = 20 + 40 \sin \left(\frac{2\pi}{3}(x+1) \right)$

Points to Consider

- When using either sine or cosine to model a graph, why is only the phase shift different?

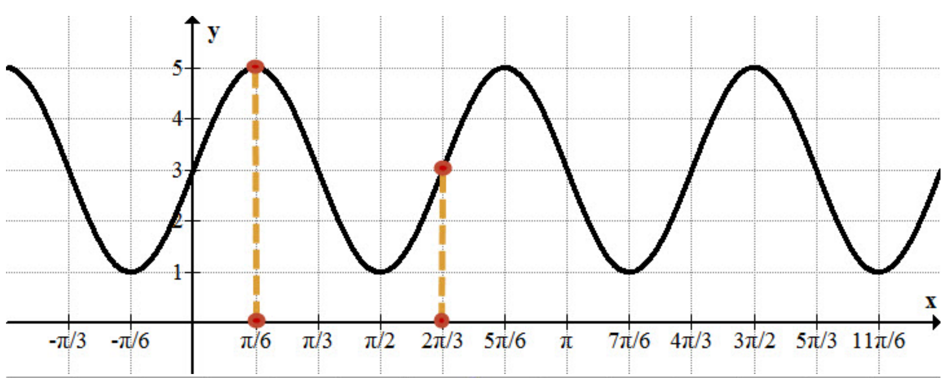
- How would you write $y = \sin x$ in the form $y = D \pm A\sin(B(x \pm C))$? What are A, B, C, and D?
- Is it possible to solve $y = D \pm A\sin(B(x \pm C))$ for x?

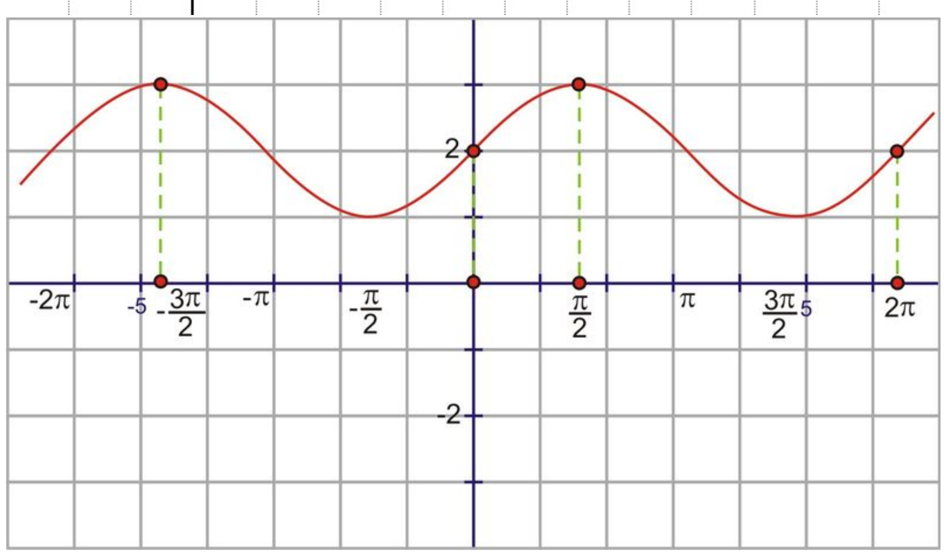
Review Questions

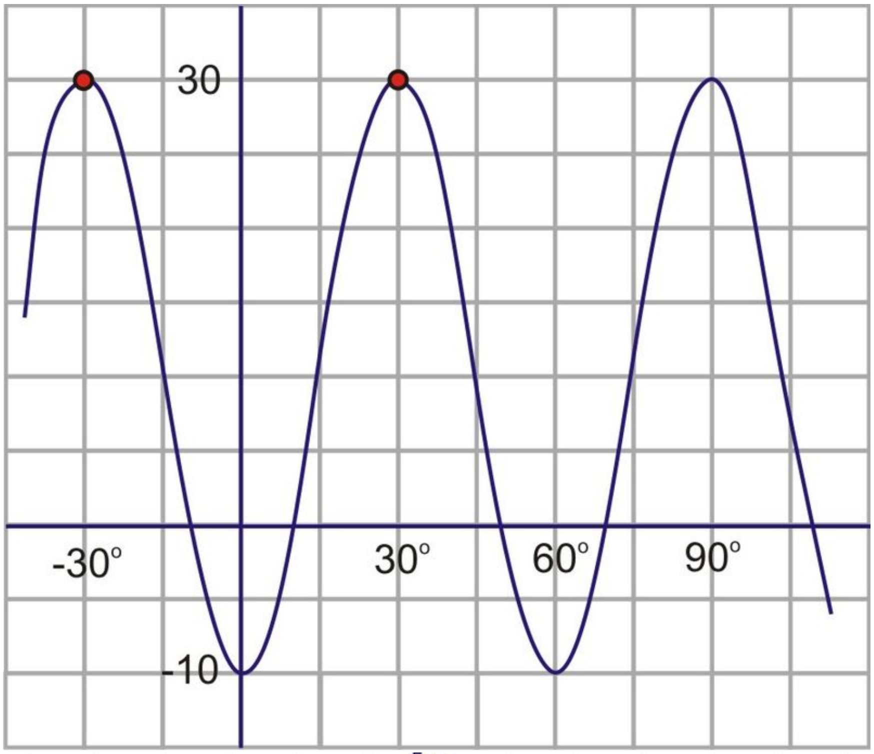
For problems 1-5, identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift.

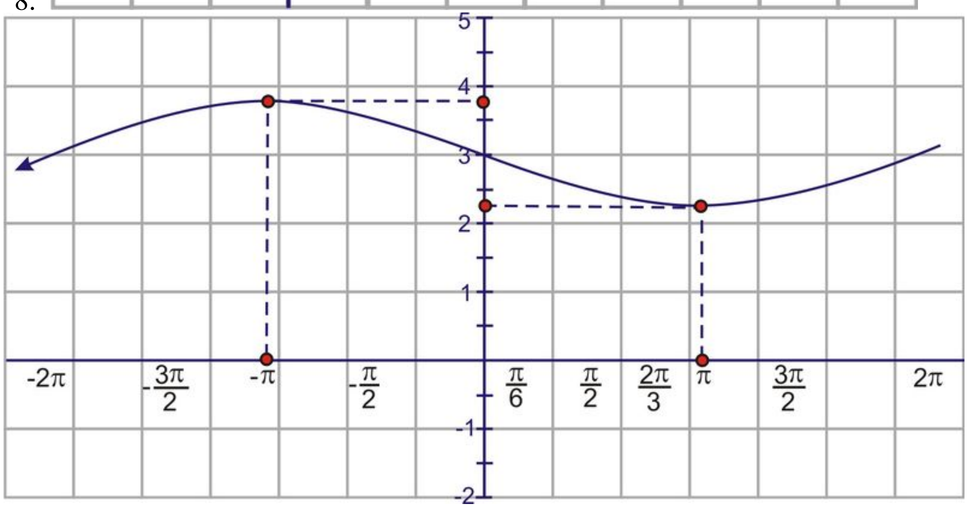
1. $y = 2 + 3\sin(2(x-1))$
2. $y = -1 + \sin\left(\pi(x + \frac{\pi}{3})\right)$
3. $y = \cos(40(x-120)) + 5$
4. $y = -\cos\left(\frac{1}{2}(x + \frac{5\pi}{4})\right)$
5. $y = 2\cos(-x) + 3$

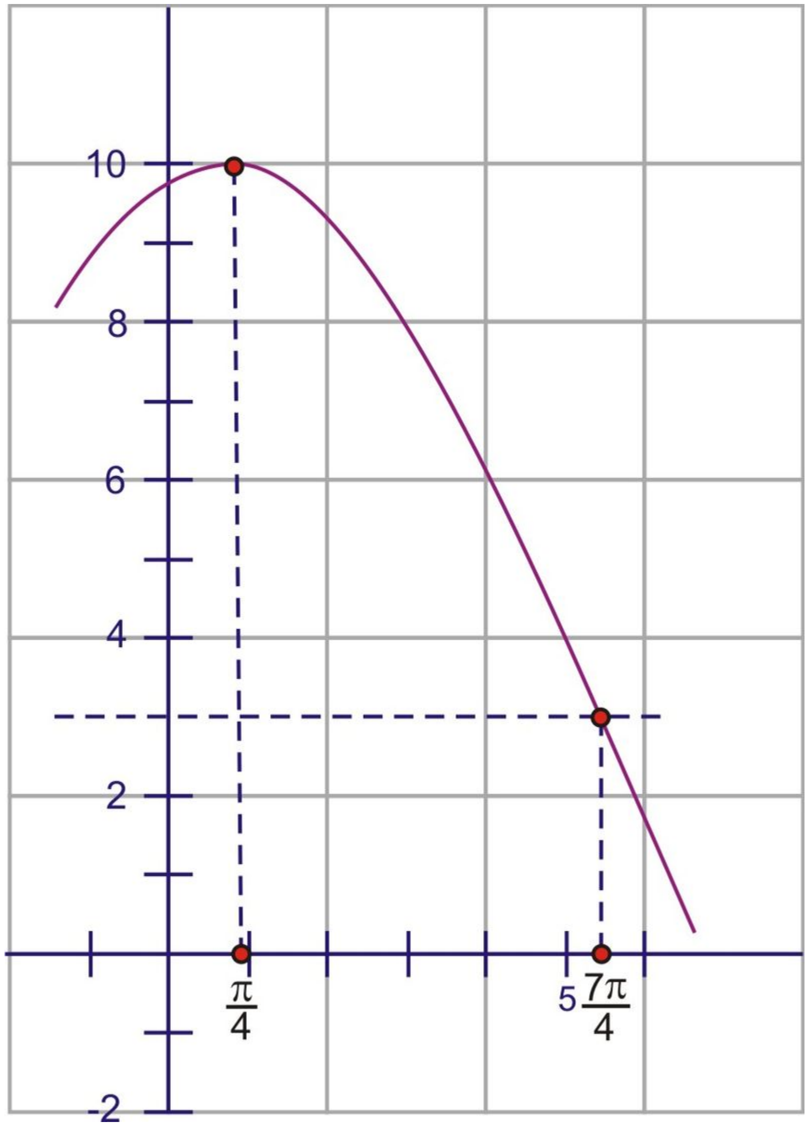
For problems 6-10, write the equation of each graph. Recall that cosine might be an easier model, but you may write your answer in terms of cosine or sine.

6. 

7. 

8. 

9. 

10. 

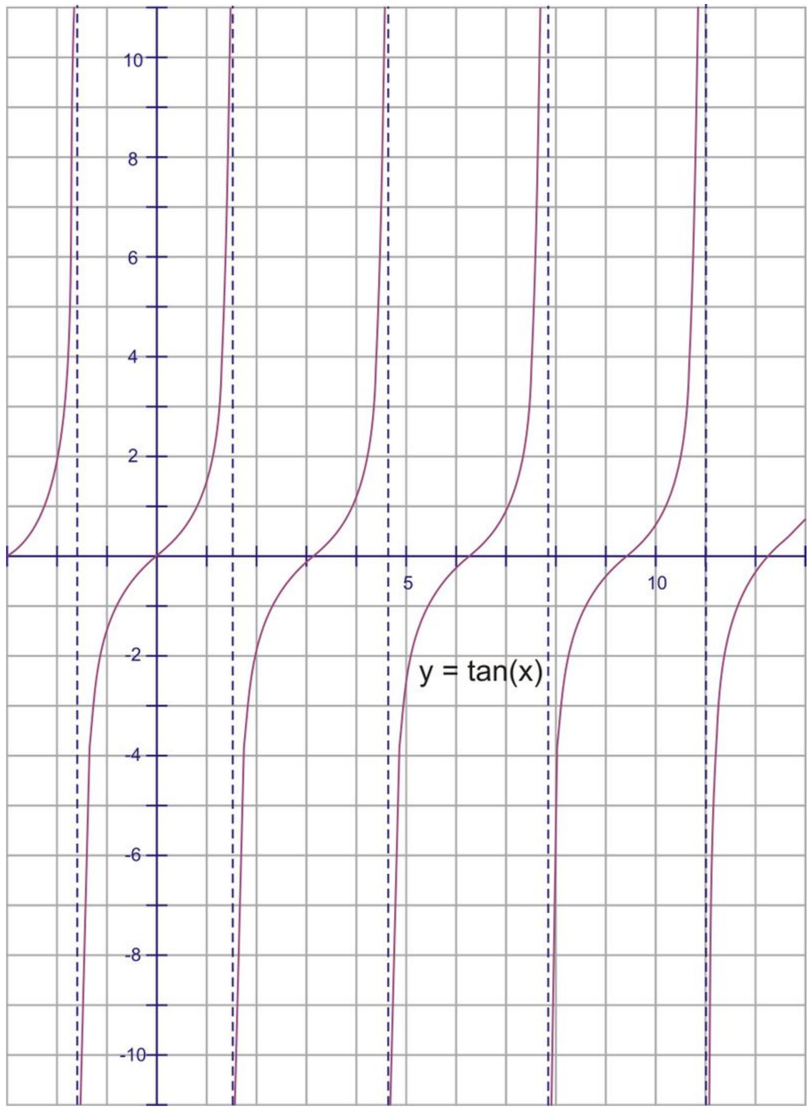
2.7 Graphing Tangent, Cotangent, Secant, and Cosecant

Learning Objectives

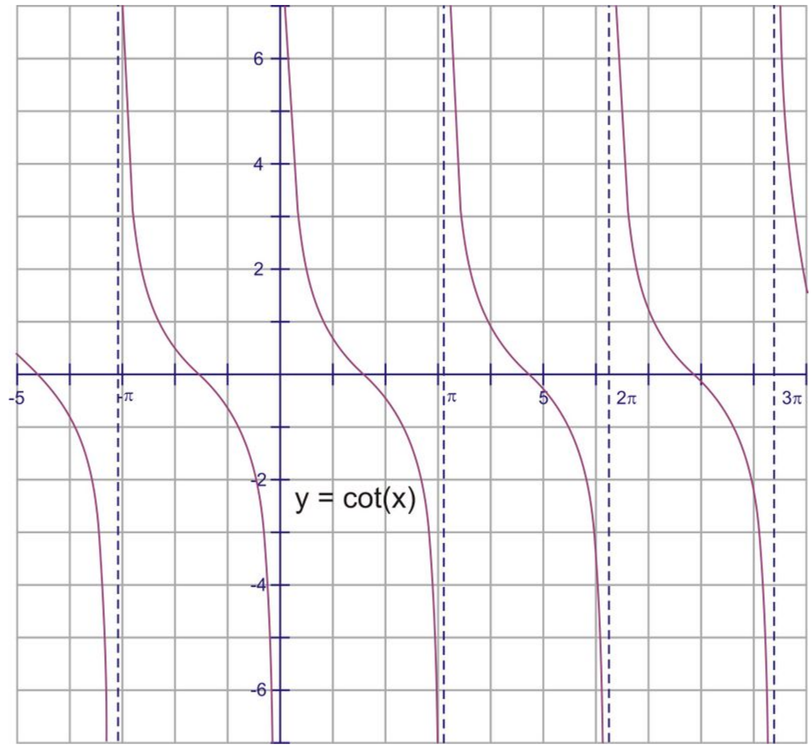
- Apply transformations to the remaining four trigonometric functions.
- Identify the equation, given a basic graph.

Tangent and Cotangent

From Section 2.3, the graph of tangent looks like the picture below, where the period is π and vertical asymptotes are at $2\pi n \pm \frac{\pi}{2}$ and $2\pi n \pm \frac{3\pi}{2}$, where n is any integer. Notice that the period is only π and the function repeats after every asymptote. The $x-$intercepts are $\ldots, -\pi, 0, \pi, 2\pi, \ldots$ The general equation of a tangent function is just like sine and cosine, $f(x) = D \pm A\tan(B(x \pm C))$, where A, B, C, and D represent the same transformations as they did before.



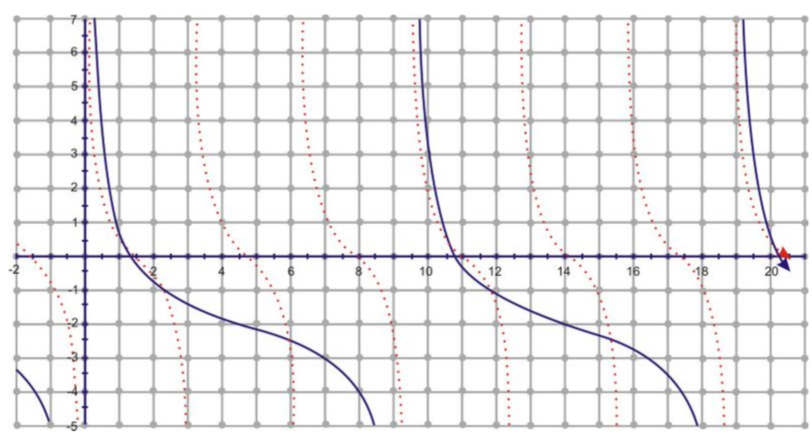
Cotangent also has a period of π, but the asymptotes and $x-$intercepts are reversed. What this means is that the vertical asymptotes are now at 0 and $\pm n\pi$, and the $x-$intercepts are at $2\pi n \pm \frac{\pi}{2}$ and $2\pi n \pm \frac{3\pi}{2}$, where n is an integer. The general equation of a cotangent function is just like sine and cosine, $f(x) = D \pm A\cot(B(x \pm C))$, where A, B, C, and D represent the same transformations as they did before.



One important difference: the period of sine and cosine is defined as $\frac{2\pi}{B}$. The period of tangent and cotangent is only π, so the period would be $\frac{\pi}{B}$.

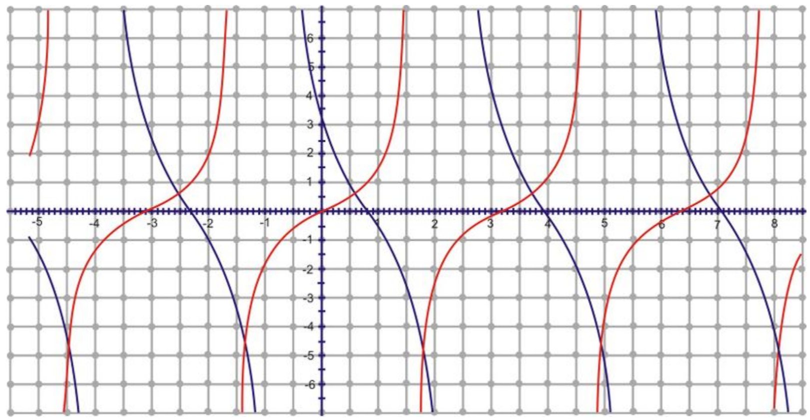
Example 1: Sketch the graph of $g(x) = -2 + \cot\frac{1}{3}x$ over the interval $[0, 6\pi]$.

Solution: Starting with $y = \cot x$, $g(x)$ would be shifted down two and frequency is $\frac{1}{3}$, which means the period would be 3π, instead of π. So, in our interval of $[0, 6\pi]$ there would be two complete repetitions. The red graph is $y = \cot x$.



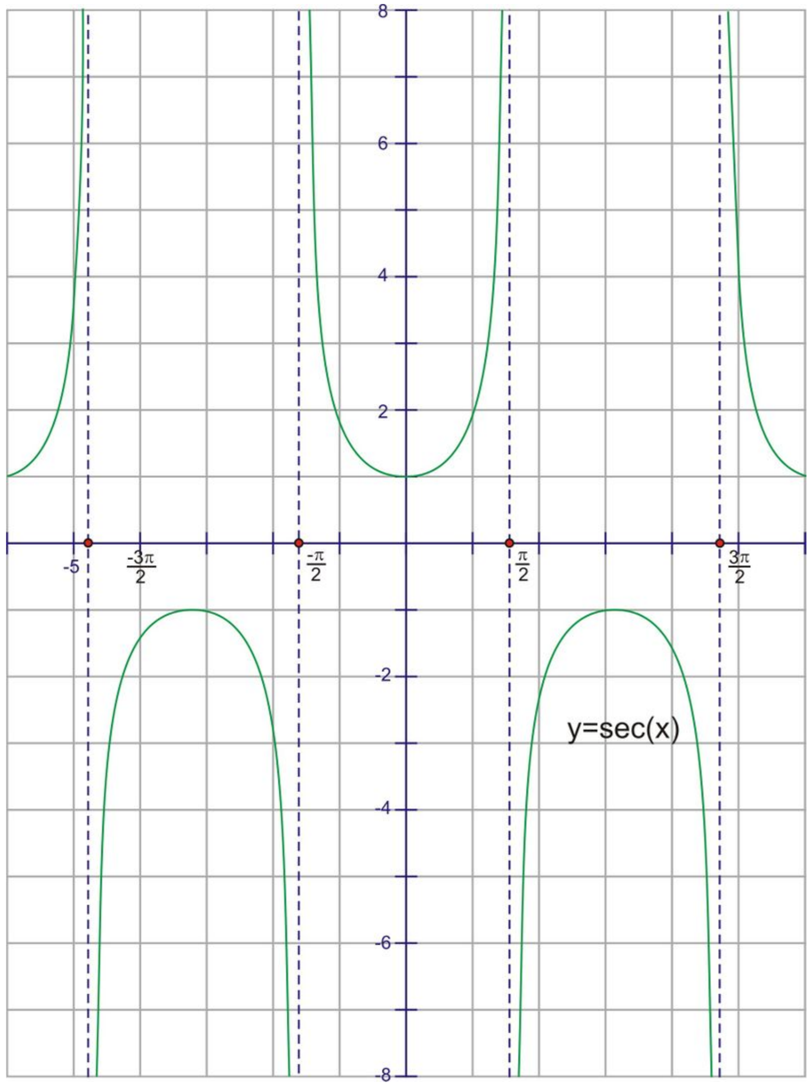
Example 2: Sketch the graph of $y = -3\tan\left(x - \frac{\pi}{4}\right)$ over the interval $[-\pi, 2\pi]$.

Solution: If you compare this graph to $y = \tan x$, it will be stretched and flipped. It will also have a phase shift of $\frac{\pi}{4}$ to the right. The red graph is $y = \tan x$.

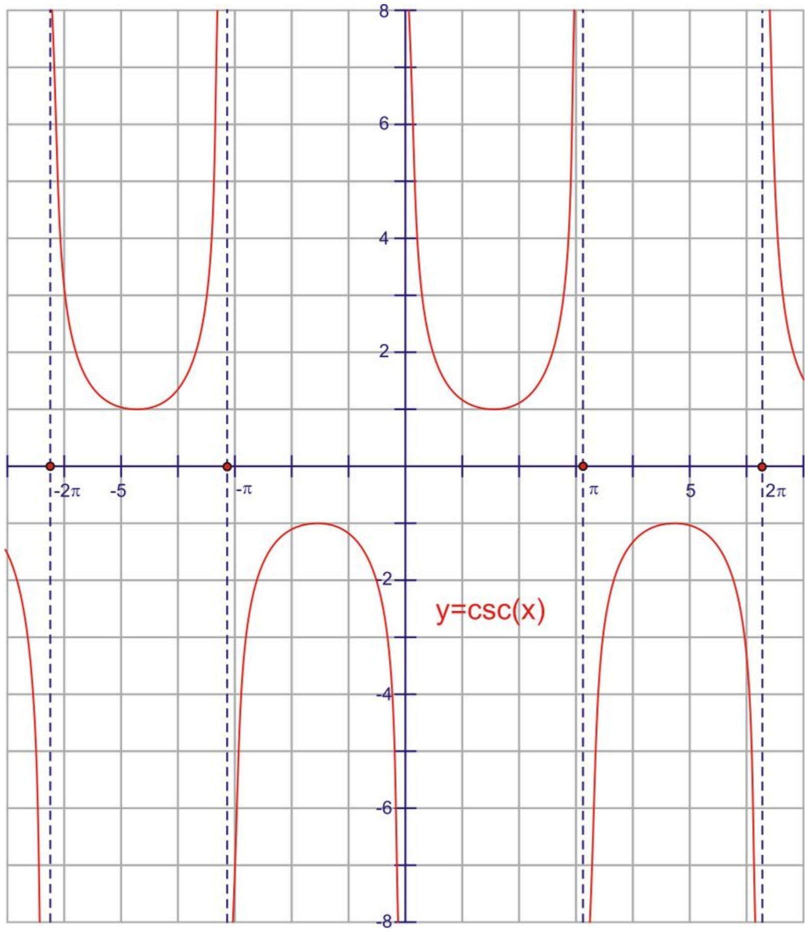


Secant and Cosecant

Because secant is the reciprocal of cosine, it will have the same period, 2π. Notice that an entire period encompasses an upward $\bigcup$ and downward $\bigcap$ and the asymptote between them. There are no $x-$intercepts and only one $y-$intercept at $(0,1)$. The vertical asymptotes are everywhere cosine is zero, so $\pi n \pm \frac{\pi}{2}$ and $\pi n \pm \frac{3\pi}{2}$, where n is any integer. The general equation of a secant function is just like the others, $f(x) = D \pm A\sec(B(x \pm C))$, where A, B, C, and D represent the same transformations as they did before.



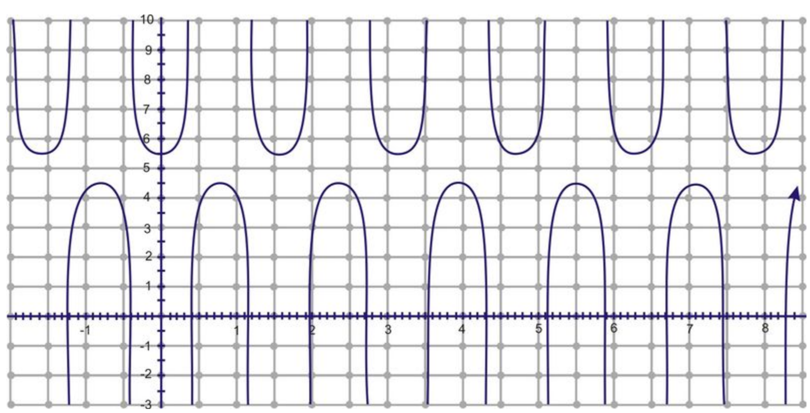
The cosecant is the reciprocal of sine and it has the same period, 2π. Notice that an entire period encompasses an upward $\bigcup$ and downward $\bigcap$ and the asymptote between them, just like secant. There are no $x-$intercepts and no $y-$intercepts. The vertical asymptotes are everywhere sine is zero, so $\pm n\pi$, where n is any integer. The general equation of a cosecant function is just like the others, $f(x) = D \pm A\csc(B(x \pm C))$, where A, B, C, and D represent the same transformations as they did before.



Recall that the period of sine and cosine is defined as $\frac{2\pi}{B}$. The period of secant and cosecant will also be defined this way.

Example 3: Sketch a graph of $h(x) = 5 + \frac{1}{2}\sec 4x$ over the interval $[0, 2\pi]$.

Solution: If you compare this example to $f(x) = \sec x$, it will be translated 5 units up, with an amplitude of $\frac{1}{2}$ and a frequency of 4. This means in our interval of 0 to 2π, there will be 4 secant curves.



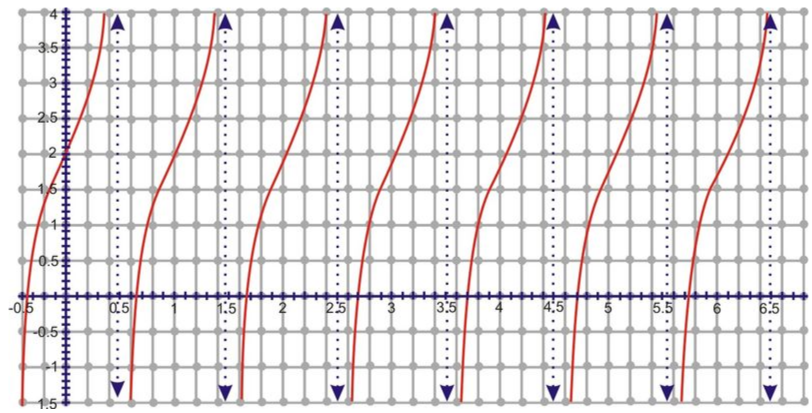
Graphing Calculator Note

For the two examples above, it might seem difficult to graph these on a graphing calculator. Most graphing calculators do not have sec, csc, or cot buttons. However, we do know that these three functions are reciprocals of cosine, sine, and tangent, respectively. So, you must enter them into the calculator in this way. For example, the equation $f(x) = 2+3\csc\left(\frac{3}{4}(x-2)\right)$ would be entered like $2+\frac{3}{\sin\left(\frac{3}{4}(x-2)\right)}$ in the $y=$ menu.

Find the Equation from a Graph

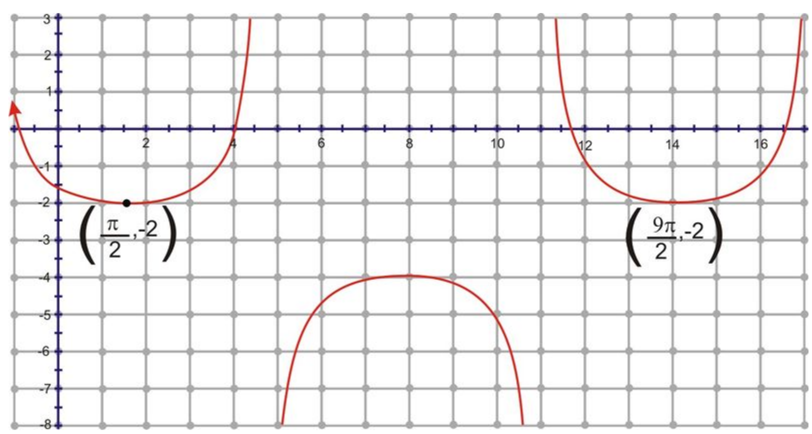
For tangent, cotangent, secant, and cosecant it can be difficult to determine the equation from a graph, so to simplify this section amplitude changes will not be included.

Example 4: Find the equation of the graph below.



Solution: From the graph, we can see this is tangent. Usually tangent intercepts the origin, but here it intercepts at $(0,2)$. Therefore, we know that there is no horizontal shift and the vertical shift is up 2. Because we have eliminated amplitude from this section, the only thing left to find is the period. Normally, the period of tangent is π, but as you can see from the graph, there are three curves from $[0,\pi]$. So, the frequency is 3. The equation is $y = 2+\tan 3x$.

Example 5: Find the equation for the graph below.



Solution: First of all, this could be either a secant or cosecant function. Let's say this is a secant function. Secant usually intersects the $y-$axis at $(0,1)$ at a minimum. Now, that corresponding minimum is $\left(\frac{\pi}{2},-2\right)$. Because there is no amplitude change, we can say that the vertical shift is the difference between the two $y-$values, -3. It looks like there is a phase shift and a period change. From minimum to minimum is one period, which is $\frac{9\pi}{2}-\frac{\pi}{2}=\frac{8\pi}{2}=4\pi$ and $B=\frac{2\pi}{4\pi}=\frac{1}{2}$. Lastly, we need to find the horizontal shift. Since secant usually intersects the $y-$axis at $(0,1)$ at a minimum, and now the corresponding minimum is $\left(\frac{\pi}{2},-2\right)$, we can say that the horizontal shift is the difference between the two $x-$values, $\frac{\pi}{2}$. Therefore, our equation is $f(x) = -3+\sec\left(\frac{1}{2}\left(x-\frac{\pi}{2}\right)\right)$.

Points to Consider

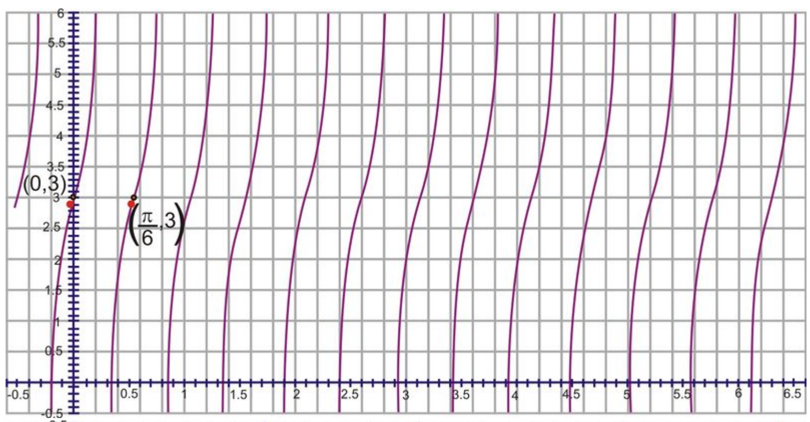
- How can you shift or change tangent to make it look like cotangent?
- Are secant and cosecant "out of phase" like sine and cosine?
- Why do tangent and cotangent have a different period than the other four trig functions?

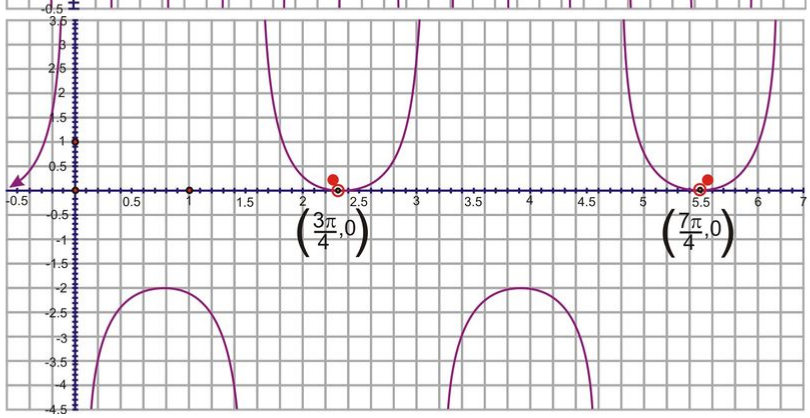
Review Questions

For questions 1-6, graph the following functions. Determine the amplitude, period, frequency, phase shift and vertical translation.

1. $y = -1 + \frac{1}{3}\cot 2x$
2. $g(x) = 5\csc\left(\frac{1}{4}(x+\pi)\right)$
3. $f(x) = 4 + \tan(0.5(x-\pi))$
4. $y = -2 + \frac{1}{2}\sec(4(x-1))$
5. $y = -2\tan 2x$
6. $h(x) = -\cot\frac{1}{3}x + 1$
7. We know that sine and cosine (and secant and cosecant) would be the same graph with a shift of $\frac{\pi}{2}$. How can we manipulate the graph of $y = \cot x$ to match up with $y = \tan x$?

For problems 8 and 9, determine the equation of the trig functions below. All amplitudes are 1.

8. 

9. 

Chapter Summary

In this chapter we learned about another way to measure degrees, called radians. Radians are based off of the unit circle, which is a circle with a radius of one. Because all circles are similar, it doesn't matter which one we use to measure radians, the ratios will always reduce to be the same. Therefore, we can now graph the six trigonometric functions on the $x-y$ plane. All the trigonometric functions are periodic and, wherever the trig value is undefined the graph has a vertical asymptote. Finally, we learned that trig functions can be transformed, much like a quadratic or cubic function. Every trig function has amplitude, phase shift, vertical shift and a period, or frequency.

Vocabulary

Amplitude
A dilation on the $y-$value of a trigonometric function. A is multiplied by the function, to make it stretch or shorten.

Angular Velocity
The velocity of rotation, measured in radians.

Arc Length
The length of a portion of the circumference of a circle. The formula is $\theta \cdot 2\pi r$, where θ is the corresponding central angle, in radians.

Circular Function
The collective term for a function that can be defined by the unit circle.

Critical Angle(s)
Any angle that is a multiple of 30° or 45°.

Dilation
A transformation that changes the size of an object or function.

Frequency
The number of times the graph repeats in 2π or π for tangent and cotangent.

Period
The distance it takes a graph to complete one phase.

Phase Shift
The shift, or translation, in the $x-$direction of a trigonometric function. Also called a horizontal translation.

Radian
An alternate way to measure degrees, defined by the arc length on a circle that is equal to the radius. $360^\circ = 2\pi$ radians.

Sector
The area of a portion of a circle. The formula is $\theta \cdot \pi r^2$, where θ is the central angle, measured in radians.

Transformation
Any change made to an object or graph. Transformations can either be dilations or translations.

Translation
Either a vertical or horizontal movement of an object or function.

Vertical Shift
The vertical translation of a function.

Review Questions

1. Convert 160° to radians.
2. Convert $\frac{11\pi}{12}$ to degrees.
3. Find the exact value of $\cos\frac{3\pi}{4}$.
4. Find all possible answers in radians, between $0 < \theta < 2\pi : \tan\theta = \sqrt{3}$
5. This is an image of the state flag of Colorado

FIGURE 2.5

It turns out that the diameter of the gold circle is $\frac{1}{3}$ the total height of the flag (the same width as the white stripe) and the outer diameter of the red circle is $\frac{2}{3}$ of the total height of the flag. The angle formed by the missing portion of the red band is $\frac{\pi}{4}$ radians. In a flag that is 66 inches tall, what is the area of the red portion of the flag to the nearest square inch?
6. Suppose the radius of the dial of an electric meter on a house is 7 cm.

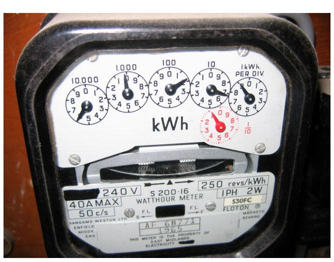
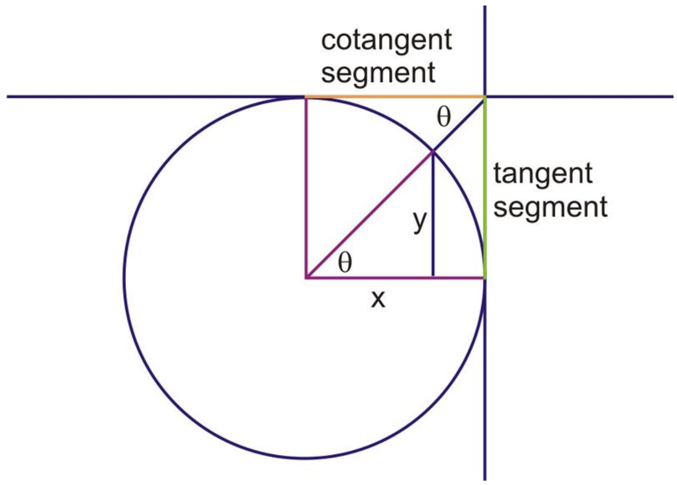


FIGURE 2.6

a. How fast is a point on the outside edge of the dial moving if it completes a revolution in 9 seconds?
b. Find the angular velocity of a point on the dial.

7. In the figure below, there is a quadrilateral formed by four line segments: the two radii of the circle (in pink), the orange segment (marked as "cotangent"), and the green segment (marked as "tangent"). The tangent (green) segment has been constructed as tangent to the circle (forming a 90-degree angle with the radius).

How do you know that the number of units that is the length of the cotangent segment is equal to $\frac{x}{y}$? You may assume that the radii shown (pink) are 1 unit.

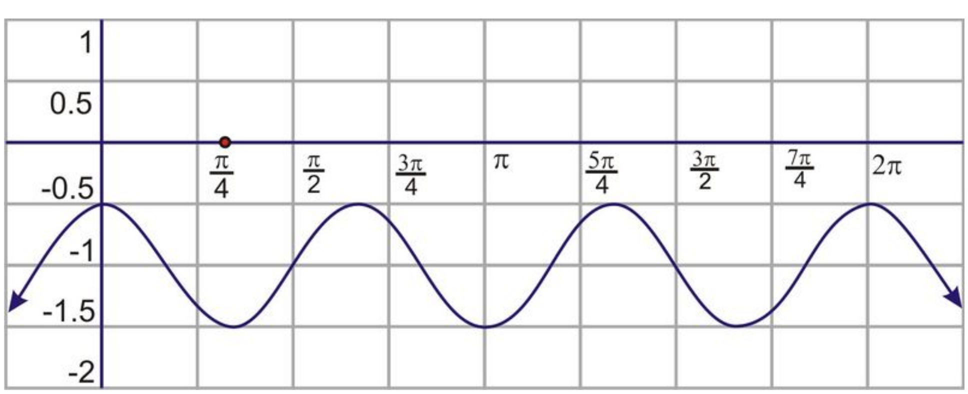


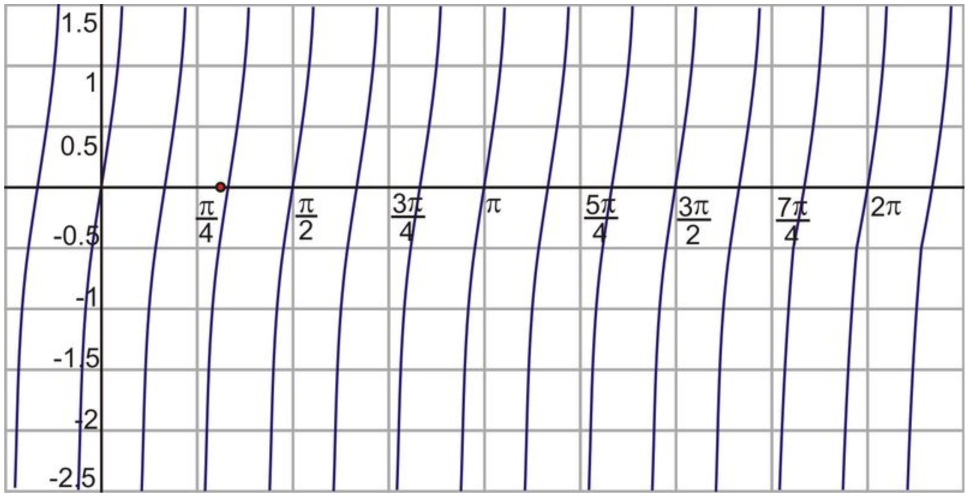
8. Graph $y = \sin x$ and $y = \cos x$ on the same set of axes over the interval $[0, 2\pi]$. Where do they intersect?

For questions 9-12, determine the amplitude, period, frequency, vertical shift, and phase shift. Then, graph each function over the interval $[0, 2\pi]$.

9. $y = -2 + 4\sin 5x$
10. $f(x) = \frac{1}{4}\cos\left(\frac{1}{2}(x - \frac{\pi}{3})\right)$
11. $g(x) = 4 + \tan\left(2(x + \frac{\pi}{2})\right)$
12. $h(x) = 3 - 6\cos(\pi x)$

For questions 13 and 14, find the equation of the graph below. Only sine and cosine functions will have an amplitude other than 1.

13. 

14. 

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9700 .

2.8 References

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CHAPTER 3 Trigonometric Identities and Equations

Chapter Outline

3.1 FUNDAMENTAL IDENTITIES
3.2 PROVING IDENTITIES
3.3 SOLVING TRIGONOMETRIC EQUATIONS
3.4 SUM AND DIFFERENCE IDENTITIES
3.5 DOUBLE ANGLE IDENTITIES
3.6 HALF-ANGLE IDENTITIES
3.7 PRODUCTS, SUMS, LINEAR COMBINATIONS, AND APPLICATIONS

3.1 Fundamental Identities

Introduction

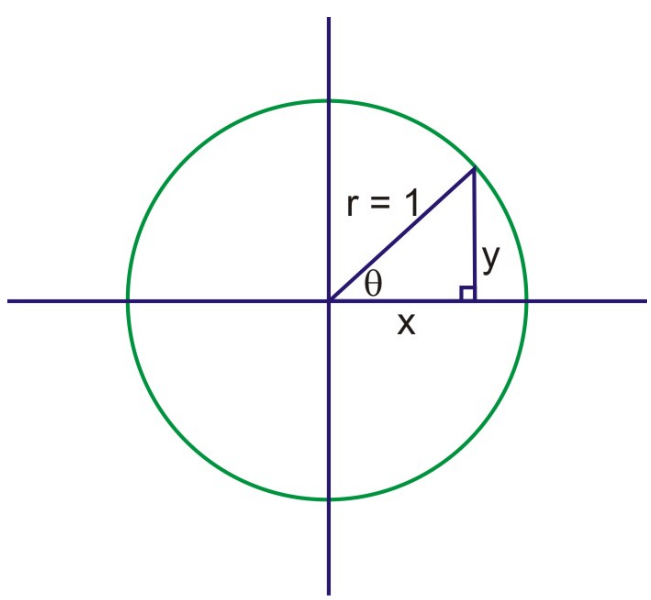
We now enter into the proof portion of trigonometry. Starting with the basic definitions of sine, cosine, and tangent, identities (or fundamental trigonometric equations) emerge. Students will learn how to prove certain identities, using other identities and definitions. Finally, students will be able solve trigonometric equations for theta, also using identities and definitions.

Learning Objectives

- use the fundamental identities to prove other identities.
- apply the fundamental identities to values of θ and show that they are true.

Quotient Identity

In Chapter 1, the three fundamental trigonometric functions sine, cosine and tangent were introduced. All three functions can be defined in terms of a right triangle or the unit circle.



$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} = \frac{y}{1} = y$$
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} = \frac{x}{1} = x$$
$$\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

The Quotient Identity is $\tan\theta = \frac{\sin\theta}{\cos\theta}$. We see that this is true because tangent is equal to $\frac{y}{x}$, in the unit circle. We know that y is equal to the sine values of θ and x is equal to the cosine values of θ. Substituting $\sin\theta$ for y and $\cos\theta$ for x and we have a new identity.

Example 1: Use $\theta = 45^\circ$ to show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ holds true.

Solution: Plugging in 45°, we have: $\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ}$. Then, substitute each function with its actual value and simplify both sides.

$\frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = 1$ and we know that $\tan 45^\circ = 1$, so this is true.

Example 2: Show that $\tan 90^\circ$ is undefined using the Quotient Identity.

Solution: $\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$, because you cannot divide by zero, the tangent at 90° is undefined.

Reciprocal Identities

Chapter 1 also introduced us to the idea that the three fundamental reciprocal trigonometric functions are cosecant (csc), secant (sec) and cotangent (cot) and are defined as:

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

If we apply the Quotient Identity to the reciprocal of tangent, an additional quotient is created:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

Example 3: Prove $\tan\theta = \sin\theta\sec\theta$

Solution: First, you should change everything into sine and cosine. Feel free to work from either side, as long as the end result from both sides ends up being the same.

$$\begin{aligned}\tan\theta &= \sin\theta\ \sec\theta \\ &= \sin\theta \cdot \frac{1}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Here, we end up with the Quotient Identity, which we know is true. Therefore, this identity is also true and we are finished.

Example 4: Given $\sin\theta = -\frac{\sqrt{6}}{5}$ and θ is in the fourth quadrant, find $\sec\theta$.

Solution: Secant is the reciprocal of cosine, so we need to find the adjacent side. We are given the opposite side, $\sqrt{6}$ and the hypotenuse, 5. Because θ is in the fourth quadrant, cosine will be positive. From the Pythagorean Theorem, the third side is:

$$\begin{aligned}\left(\sqrt{6}\right)^2 + b^2 &= 5^2\\ 6 + b^2 &= 25\\ b^2 &= 19\\ b &= \sqrt{19}\end{aligned}$$

From this we can now find $\cos\theta = \frac{\sqrt{19}}{5}$. Since secant is the reciprocal of cosine, $\sec\theta = \frac{5}{\sqrt{19}}$, or $\frac{5\sqrt{19}}{19}$.

Pythagorean Identity

Using the fundamental trig functions, sine and cosine and some basic algebra can reveal some interesting trigonometric relationships. Note when a trig function such as $\sin\theta$ is multiplied by itself, the mathematical convention is to write it as $\sin^2\theta$. ($\sin\theta^2$ can be interpreted as the sine of the square of the angle, and is therefore avoided.)

$\sin^2\theta = \frac{y^2}{r^2}$ and $\cos^2\theta = \frac{x^2}{r^2}$ or $\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2+y^2}{r^r}$

Using the Pythagorean Theorem for the triangle above: $x^2 + y^2 = r^2$

Then, divide both sides by r^2, $\frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1$. So, because $\frac{x^2+y^2}{r^2} = 1, \sin^2\theta + \cos^2\theta$ also equals 1. This is known as the Trigonometric Pythagorean Theorem or the Pythagorean Identity and is written $\sin^2\theta + \cos^2\theta = 1$. Alternative forms of the Theorem are: $1 + \cot^2\theta = \csc^2\theta$ and $\tan^2\theta + 1 = \sec^2\theta$. The second form is found by taking the original form and dividing each of the terms by $\sin^2\theta$, while the third form is found by dividing all the terms of the first by $\cos^2\theta$.

Example 5: Use 30° to show that $\sin^2\theta + \cos^2\theta = 1$ holds true.

Solution: Plug in 30° and find the values of $\sin 30^\circ$ and $\cos 30^\circ$.

$$\begin{gathered}\sin^2 30^\circ + \cos^2 30^\circ\\ \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\\ \frac{1}{4} + \frac{3}{4} = 1\end{gathered}$$

Even and Odd Identities

Functions are even or odd depending on how the end behavior of the graphical representation looks. For example, $y = x^2$ is considered an even function because the ends of the parabola both point in the same direction and the parabola is symmetric about the $y-$axis. $y = x^3$ is considered an odd function for the opposite reason. The ends of a cubic function point in opposite directions and therefore the parabola is not symmetric about the $y-$axis. What about the trig functions? They do not have exponents to give us the even or odd clue (when the degree is even, a function is even, when the degree is odd, a function is odd).

Even Function	Odd Function
$y=(-x)^2=x^2$	$y=(-x)^3=-x^3$

Let's consider sine. Start with $\sin(-x)$. Will it equal $\sin x$ or $-\sin x$? Plug in a couple of values to see.

$$\sin(-30^\circ)=\sin 330^\circ=-\frac{1}{2}=-\sin 30^\circ$$
$$\sin(-135^\circ)=\sin 225^\circ=-\frac{\sqrt{2}}{2}=-\sin 135^\circ$$

From this we see that sine is **odd**. Therefore, $\sin(-x)=-\sin x$, for any value of x. For cosine, we will plug in a couple of values to determine if it's even or odd.

$$\cos(-30^\circ)=\cos 330^\circ=\frac{\sqrt{3}}{2}=\cos 30^\circ$$
$$\cos(-135^\circ)=\cos 225^\circ=-\frac{\sqrt{2}}{2}=\cos 135^\circ$$

This tells us that the cosine is **even**. Therefore, $\cos(-x)=\cos x$, for any value of x. The other four trigonometric functions are as follows:

$$\tan(-x)=-\tan x$$
$$\csc(-x)=-\csc x$$
$$\sec(-x)=\sec x$$
$$\cot(-x)=-\cot x$$

Notice that cosecant is odd like sine and secant is even like cosine.

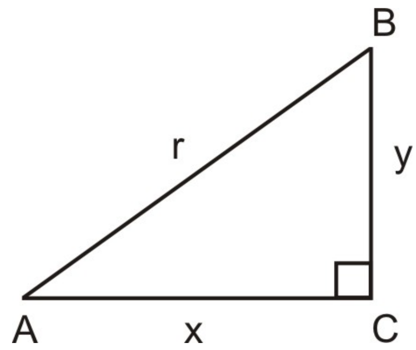
Example 6: If $\cos(-x)=\frac{3}{4}$ and $\tan(-x)=-\frac{\sqrt{7}}{3}$, find $\sin x$.

Solution: We know that sine is odd. Cosine is even, so $\cos x=\frac{3}{4}$. Tangent is odd, so $\tan x=\frac{\sqrt{7}}{3}$. Therefore, sine is positive and $\sin x=\frac{\sqrt{7}}{4}$.

Cofunction Identities

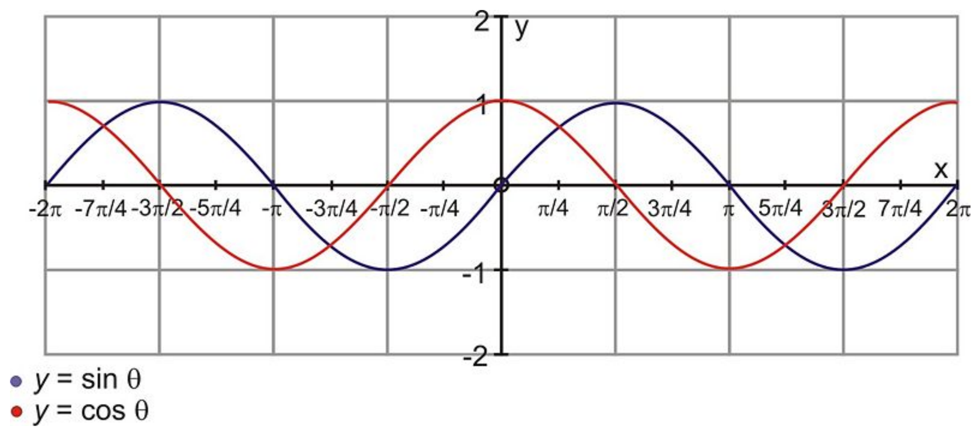
Recall that two angles are complementary if their sum is 90°. In every triangle, the sum of the interior angles is 180° and the right angle has a measure of 90°. Therefore, the two remaining acute angles of the triangle have a sum equal to 90°, and are complementary. Let's explore this concept to identify the relationship between a function of one angle and the function of its complement in any right triangle, or the cofunction identities. A cofunction is a pair of trigonometric functions that are equal when the variable in one function is the complement in the other.

In $\triangle ABC$, $\angle C$ is a right angle, $\angle A$ and $\angle B$ are complementary.



Chapter 1 introduced the cofunction identities (section 1.8) and because $\angle A$ and $\angle B$ are complementary, it was found that $\sin A = \cos B, \cos A = \sin B, \tan A = \cot B, \cot A = \tan B, \csc A = \sec B$ and $\sec A = \csc B$. For each of the above $\angle A = \frac{\pi}{2} - \angle B$. To generalize, $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ and $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta, \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ and $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$.

The following graph represents two complete cycles of $y = \sin x$ and $y = \cos\theta$.



Notice that a phase shift of $\frac{\pi}{2}$ on $y = \cos x$, would make these graphs exactly the same. These cofunction identities hold true for all real numbers for which both sides of the equation are defined.

Example 7: Use the cofunction identities to evaluate each of the following expressions:

a. If $\tan\left(\frac{\pi}{2} - \theta\right) = -4.26$ determine $\cot\theta$

b. If $\sin\theta = 0.91$ determine $\cos\left(\frac{\pi}{2} - \theta\right)$.

Solution:

a. $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ therefore $\cot\theta = -4.26$

b. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ therefore $\cos\left(\frac{\pi}{2} - \theta\right) = 0.91$

Example 8: Show $\sin\left(\frac{\pi}{2} - x\right) = \cos(-x)$ is true.

Solution: Using the identities we have derived in this section, $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, and we know that cosine is an even function so $\cos(-x) = \cos x$. Therefore, each side is equal to $\cos x$ and thus equal to each other.

Points to Consider

- Why do you think secant is even like cosine?
- How could you show that tangent is odd?

Review Questions

1. Use the Quotient Identity to show that the tan 270° is undefined.
2. If $\cos\left(\frac{\pi}{2}-x\right)=\frac{4}{5}$, find $\sin(-x)$.
3. If $\tan(-x)=-\frac{5}{12}$ and $\sin x=-\frac{5}{13}$, find $\cos x$.
4. Simplify $\sec x\cos\left(\frac{\pi}{2}-x\right)$.
5. Verify $\sin^2\theta+\cos^2\theta=1$ using:
   a. the sides $5, 12$, and 13 of a right triangle, in the first quadrant
   b. the ratios from a $30-60-90$ triangle
6. Prove $1+\tan^2\theta=\sec^2\theta$ using the Pythagorean Identity
7. If $\csc z=\frac{17}{8}$ and $\cos z=-\frac{15}{17}$, find $\cot z$.
8. Factor:
   a. $\sin^2\theta-\cos^2\theta$
   b. $\sin^2\theta+6\sin\theta+8$
9. Simplify $\frac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta}$ using the trig identities
10. Rewrite $\frac{\cos x}{\sec x-1}$ so that it is only in terms of cosine. Simplify completely.
11. Prove that tangent is an odd function.

3.2 Proving Identities

Learning Objectives

- Prove identities using several techniques.

Working with Trigonometric Identities

During the course, you will see complex trigonometric expressions. Often, complex trigonometric expressions can be equivalent to less complex expressions. The process for showing two trigonometric expressions to be equivalent (regardless of the value of the angle) is known as validating or proving trigonometric identities.

There are several options a student can use when proving a trigonometric identity.

Option One: Often one of the steps for proving identities is to change each term into their sine and cosine equivalents:

Example 1: Prove the identity: $\csc\theta \times \tan\theta = \sec\theta$

Solution: Reducing each side separately. It might be helpful to put a line down, through the equals sign. Because we are proving this identity, we don't know if the two sides are equal, so wait until the end to include the equality.

$$\begin{array}{c|c} \csc x \times \tan x & \sec x \\ \frac{1}{\sin x} \times \frac{\sin x}{\cos x} & \frac{1}{\cos x} \\ \frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x} & \frac{1}{\cos x} \\ \frac{1}{\cos x} & \frac{1}{\cos x} \end{array}$$

At the end we ended up with the same thing, so we know that this is a valid identity.

Notice when working with identities, unlike equations, conversions and mathematical operations are performed only on one side of the identity. In more complex identities sometimes both sides of the identity are simplified or expanded. The thought process for establishing identities is to view each side of the identity separately, and at the end to show that both sides do in fact transform into identical mathematical statements.

Option Two: Use the Trigonometric Pythagorean Theorem and other Fundamental Identities.

Example 2: Prove the identity: $(1-\cos^2 x)(1+\cot^2 x) = 1$

Solution: Use the Pythagorean Identity and its alternate form. Manipulate $\sin^2\theta + \cos^2\theta = 1$ to be $\sin^2\theta = 1 - \cos^2\theta$. Also substitute $\csc^2 x$ for $1+\cot^2 x$, then cross-cancel.

$$\begin{array}{c|c} (1-\cos^2 x)(1+\cot^2 x) & 1 \\ \sin^2 x \cdot \csc^2 x & 1 \\ \sin^2 x \cdot \frac{1}{\sin^2 x} & 1 \\ 1 & 1 \end{array}$$

Option Three: When working with identities where there are fractions- combine using algebraic techniques for adding expressions with unlike denominators:

Example 3: Prove the identity: $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\csc\theta$.

Solution: Combine the two fractions on the left side of the equation by finding the common denominator: $(1+\cos\theta)\times\sin\theta$, and the change the right side into terms of sine.

$$\begin{array}{c|l} \frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} & 2\csc\theta \\ \frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} & 2\csc\theta \\ \frac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} & 2\csc\theta \end{array}$$

Now, we need to apply another algebraic technique, FOIL. (FOIL is a memory device that describes the process for multiplying two binomials, meaning multiplying the First two terms, the Outer two terms, the Inner two terms, and then the Last two terms, and then summing the four products.) Always leave the denominator factored, because you might be able to cancel something out at the end.

$$\begin{array}{c|l} \frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \end{array}$$

Using the second option, substitute $\sin^2\theta+\cos^2\theta=1$ and simplify.

$$\begin{array}{c|l} \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\ \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\ \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\ \frac{2}{\sin\theta} & \frac{2}{\sin\theta} \end{array}$$

Option Four: If possible, factor trigonometric expressions. Actually procedure four was used in the above example: $\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}=2\csc\theta$ can be *factored* to $\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}=2\csc\theta$ and in this situation, the factors cancel each other.

Example 4: Prove the identity: $\frac{1+\tan\theta}{(1+\cot\theta)}=\tan\theta$.

Solution: Change $\cot\theta$ to $\frac{1}{\tan\theta}$ and find a common denominator.

$$\frac{1+\tan\theta}{\left(1+\frac{1}{\tan\theta}\right)}=\tan\theta$$

$$\frac{1+\tan\theta}{\left(\frac{\tan\theta}{\tan\theta}+\frac{1}{\tan\theta}\right)}=\tan\theta \qquad \text{or} \qquad \frac{1+\tan\theta}{\frac{\tan\theta+1}{\tan\theta}}=\tan\theta$$

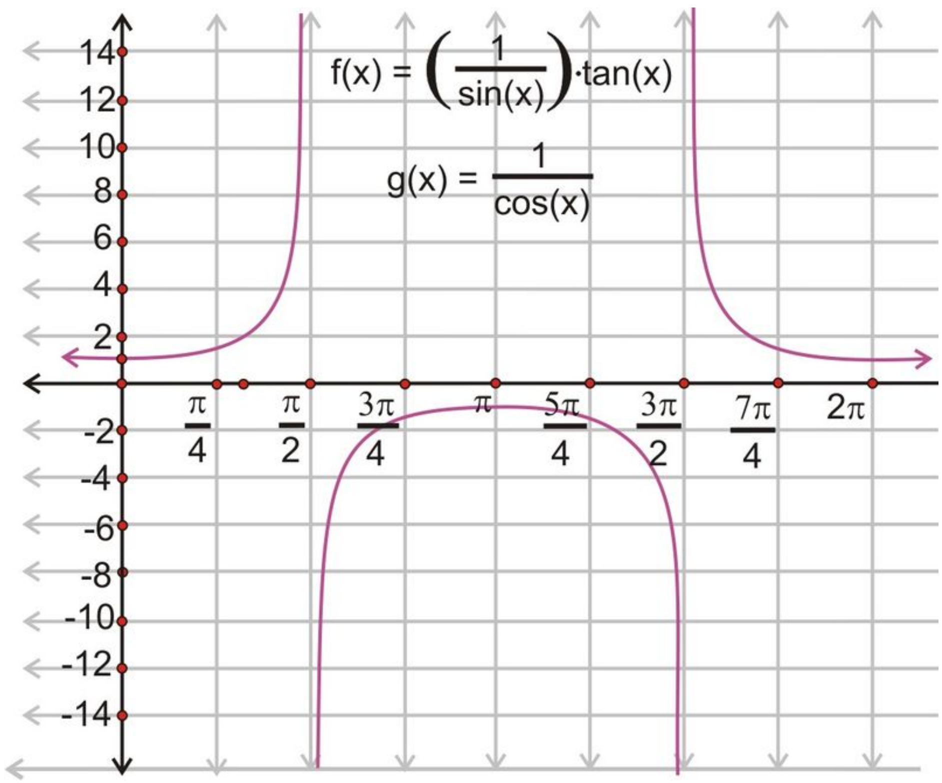
Now invert the denominator and multiply.

$$\frac{\tan\theta(1+\tan\theta)}{\tan\theta+1}=\tan\theta$$

$$\tan\theta=\tan\theta$$

Technology Note

A graphing calculator can help provide the correctness of an identity. For example looking at: $\csc x \times \tan x = \sec x$, first graph $y = \csc x \times \tan x$, and then graph $y = \sec x$. Examining the viewing screen for each demonstrates that the results produce the same graph.



To summarize, when verifying a trigonometric identity, use the following tips:

1. Work on one side of the identity- usually the more complicated looking side.
2. Try rewriting all given expressions in terms of sine and cosine.
3. If there are fractions involved, combine them.
4. After combining fractions, if the resulting fraction can be reduced, reduce it.
5. The goal is to make one side look exactly like the other—so as you change one side of the identity, look at the other side for a potential hint to what to do next. If you are stumped, work with the other side. Don't limit yourself to working only on the left side, a problem might require you to work on the right.

Points to Consider

- Are there other techniques that you could use to prove identities?
- What else, besides what is listed in this section, do you think would be useful in proving identities?

Review Questions

Prove the following identities true:

1. $\sin x \tan x + \cos x = \sec x$
2. $\cos x - \cos x \sin^2 x = \cos^3 x$
3. $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$
4. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
5. $\frac{1}{1+\cos a} + \frac{1}{1-\cos a} = 2 + 2\cot^2 a$
6. $\cos^4 b - \sin^4 b = 1 - 2\sin^2 b$
7. $\frac{\sin y+\cos y}{\sin y} - \frac{\cos y-\sin y}{\cos y} = \sec y \csc y$
8. $(\sec x - \tan x)^2 = \frac{1-\sin x}{1+\sin x}$
9. Show that $2\sin x \cos x = \sin 2x$ is true using $\frac{5\pi}{6}$.
10. Use the trig identities to prove $\sec x \cot x = \csc x$

3.3 Solving Trigonometric Equations

Learning Objectives

- Use the fundamental identities to solve trigonometric equations.
- Express trigonometric expressions in simplest form.
- Solve trigonometric equations by factoring.
- Solve trigonometric equations by using the Quadratic Formula.

By now we have seen trigonometric functions represented in many ways: Ratios between the side lengths of right triangles, as functions of coordinates as one travels along the unit circle and as abstract functions with graphs. Now it is time to make use of the properties of the trigonometric functions to gain knowledge of the connections between the functions themselves. The patterns of these connections can be applied to simplify trigonometric expressions and to solve trigonometric equations.

Simplifying Trigonometric Expressions

Example 1: Simplify the following expressions using the basic trigonometric identities:

a. $\frac{1+\tan^2 x}{\csc^2 x}$

b. $\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x}$

c. $\cos x - \cos^3 x$

Solution:

a.

$$\begin{aligned}
\frac{1+\tan^2 x}{\csc^2 x} & \ldots (1+\tan^2 x = \sec^2 x) \text{Pythagorean Identity} \\
\frac{\sec^2 x}{\csc^2 x} & \ldots (\sec^2 x = \frac{1}{\cos^2 x} \text{ and } \csc^2 x = \frac{1}{\sin^2 x}) \text{Reciprocal Identity} \\
\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} & = \left(\frac{1}{\cos^2 x}\right) \div \left(\frac{1}{\sin^2 x}\right) \\
\left(\frac{1}{\cos^2 x}\right) \cdot \left(\frac{\sin^2 x}{1}\right) & = \frac{\sin^2 x}{\cos^2 x} \\
& = \tan^2 x \rightarrow \text{Quotient Identity}
\end{aligned}$$

b.

$$\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x} \ldots (\sin^2 x + \cos^2 x = 1) \text{Pythagorean Identity}$$
$$\frac{1 + \tan^2 x}{\sec x} \ldots (1 + \tan^2 x = \sec^2 x) \text{Pythagorean Identity}$$
$$\frac{\sec^2 x}{\sec x} = \sec x$$

c.

$$\cos x - \cos^3 x$$
$$\cos x(1 - \cos^2 x) \qquad \ldots \text{Factor out } \cos x \text{ and } \sin^2 x = 1 - \cos^2 x$$
$$\cos x(\sin^2 x)$$

In the above examples, the given expressions were simplified by applying the patterns of the basic trigonometric identities. We can also apply the fundamental identities to trigonometric equations to solve for x. When solving trig equations, restrictions on x (or θ) must be provided, or else there would be infinitely many possible answers (because of the periodicity of trig functions).

Solving Trigonometric Equations

Example 2: Without the use of technology, find all solutions $\tan^2(x) = 3$, such that $0 \le x \le 2\pi$.

Solution:

$$\tan^2 x = 3$$
$$\sqrt{\tan^2 x} = \sqrt{3}$$
$$\tan x = \pm\sqrt{3}$$

This means that there are four answers for x, because tangent is positive in the first and third quadrants and negative in the second and fourth. Combine that with the values that we know would generate $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}, x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

Example 3: Solve $2\cos x \sin x - \cos x = 0$ for all values of x between $[0, 2\pi]$.

Solution:

$$\cos x \ (2\sin x - 1) = 0 \rightarrow \text{set each factor equal to zero and solve them separately}$$
$$\downarrow \qquad \searrow$$
$$\cos x = 0 \qquad 2\sin x = 1$$
$$x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2} \qquad \sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}$$

In the above examples, exact values were obtained for the solutions of the equations. These solutions were within the domain that was specified.

Example 4: Solve $2\sin^2 x - \cos x - 1 = 0$ for all values of x.

Solution: The equation now has two functions –sine and cosine. Study the equation carefully and decide in which function to rewrite the equation. $\sin^2 x$ can be expressed in terms of cosine by manipulating the Pythagorean Identity, $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
2\sin^2 x - \cos x - 1 &= 0 \\
2(1-\cos^2 x) - \cos x - 1 &= 0 \\
2 - 2\cos^2 x - \cos x - 1 &= 0 \\
-2\cos^2 x - \cos x + 1 &= 0 \\
2\cos^2 x + \cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1) &= 0 \\
\swarrow \qquad\qquad &\searrow \\
2\cos x - 1 = 0 \quad \text{or} &\quad \cos x + 1 = 0 \\
\cos x = \frac{1}{2} \qquad &\quad \cos x = -1 \\
x = \frac{\pi}{3} + 2\pi k, k\varepsilon Z \qquad &\quad x = \pi + 2\pi k, k\varepsilon Z \\
x = \frac{5\pi}{3} + 2\pi k, k\varepsilon Z &
\end{aligned}$$

Solving Trigonometric Equations Using Factoring

Algebraic skills like factoring and substitution that are used to solve various equations are very useful when solving trigonometric equations. As with algebraic expressions, one must be careful to avoid dividing by zero during these maneuvers.

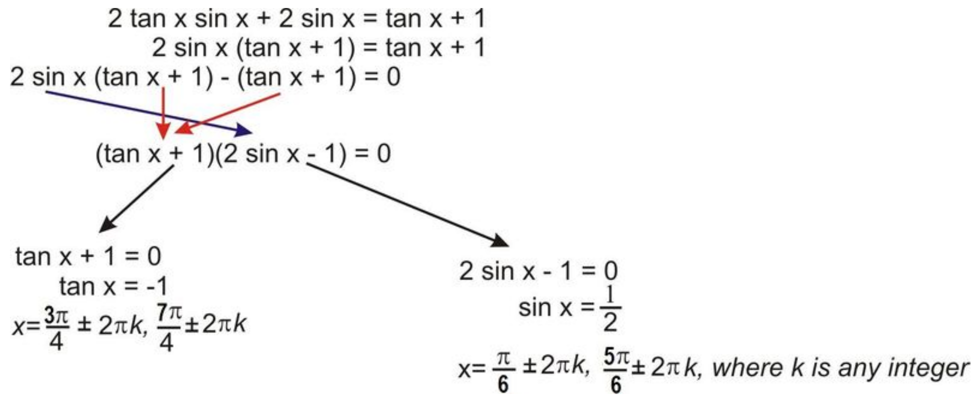
Example 5: Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 < x \leq 2\pi$.

Solution:

$$\begin{aligned}
2\sin^2 x - 3\sin x + 1 &= 0 \quad \text{Factor this like a quadratic equation} \\
(2\sin x - 1)(\sin x - 1) &= 0 \\
\downarrow \qquad\qquad &\searrow \\
2\sin x - 1 = 0 \quad \text{or} &\quad \sin x - 1 = 0 \\
2\sin x = 1 \qquad &\quad \sin x = 1 \\
\sin x = \frac{1}{2} \qquad &\quad x = \frac{\pi}{2} \\
x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} &
\end{aligned}$$

Example 6: Solve $2\tan x \sin x + 2\sin x = \tan x + 1$ for all values of x.

Solution:



Pull out $\sin x$

There is a common factor of $(\tan x + 1)$

Think of the $-(\tan x+1)$ as $(-1)(\tan x+1)$, which is why there is a -1 behind the $2\sin x$.

Example 7: Solve $2\sin^2 x+3\sin x-2=0$ for all x, $[0,\pi]$.

Solution:

$$2\sin^2 x+3\sin x-2=0 \rightarrow \text{Factor like a quadratic}$$
$$(2\sin x-1)(\sin x+2)=0$$
$$\swarrow \qquad \searrow$$
$$2\sin x-1=0 \qquad \sin x+2=0$$
$$\sin x=\frac{1}{2} \qquad \sin x=-2$$
$$x=\frac{\pi}{6} \text{ and } x=\frac{5\pi}{6} \quad \text{There is no solution because the range of } \sin x \text{ is } [-1,1].$$

Some trigonometric equations have no solutions. This means that there is no replacement for the variable that will result in a true expression.

Example 8: Solve $4\sin^3 x+2\sin^2 x-2\sin x-1=0$ for x in the interval $[0,2\pi]$.

Solution: Even though this does not look like a factoring problem, it is. We are going to use factoring by grouping, from Algebra II. First group together the first two terms and the last two terms. Then find the greatest common factor for each pair.

$$\underbrace{4\sin^3 x+2\sin^2 x} \qquad \underbrace{-2\sin x-1}=0$$
$$2\sin^2 x(2\sin x+1)-1(2\sin x+1)$$

Notice we have gone from four terms to two. These new two terms have a common factor of $2\sin x+1$. We can pull this common factor out and reduce our number of terms from two to one, comprised of two factors.

$$2\sin^2 x(2\sin x+1)-1(2\sin x+1)=0$$
$$\searrow \qquad \swarrow$$
$$(2\sin x+1)(2\sin^2 x \; - \; 1)=0$$

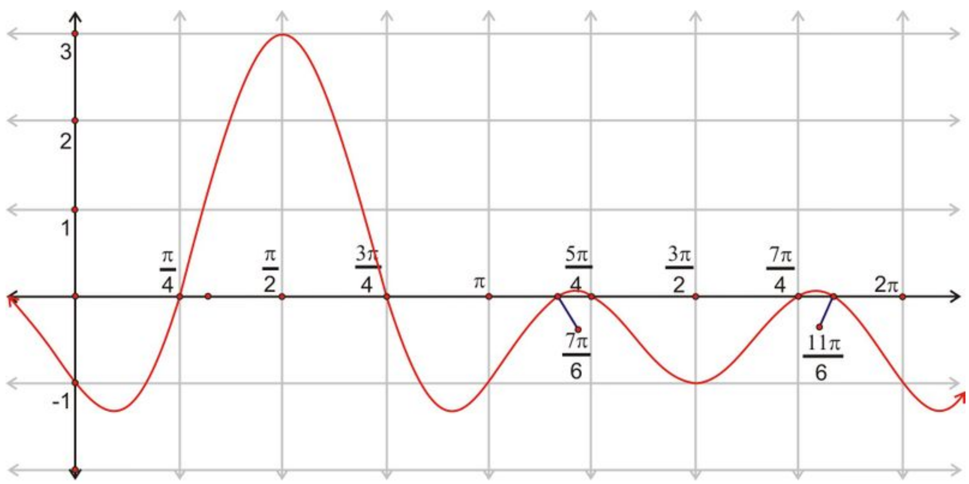
We can take this one step further because $2\sin^2 x - 1$ can factor again.

$$(2\sin x + 1)\left(\sqrt{2}\sin x - 1\right)\left(\sqrt{2}\sin x + 1\right) = 0$$

Set each factor equal to zero and solve.

$$\begin{aligned} 2\sin x + 1 &= 0 \\ 2\sin x &= -1 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned} \quad \text{or} \quad \begin{aligned} \sqrt{2}\sin x + 1 &= 0 \\ \sqrt{2}\sin x &= -1 \\ \sin x &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ x &= \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned} \quad \text{or} \quad \begin{aligned} \sqrt{2}\sin x - 1 &= 0 \\ \sqrt{2}\sin x &= 1 \\ \sin x &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ x &= \frac{\pi}{4}, \frac{3\pi}{4} \end{aligned}$$

Notice there are six solutions for x. Graphing the original function would show that the equation crosses the $x-$axis six times in the interval $[0, 2\pi]$.



Solving Trigonometric Equations Using the Quadratic Formula

When solving quadratic equations that do not factor, the quadratic formula is often used. The same can be applied when solving trigonometric equations that do not factor. The values for a is the numerical coefficient of the function's squared term, b is the numerical coefficient of the function term that is to the first power and c is a constant. The formula will result in two answers and both will have to be evaluated within the designated interval.

Example 9: Solve $3\ \cot^2 x - 3\ \cot x = 1$ for exact values of x over the interval $[0, 2\pi]$.

Solution:

$$\begin{aligned} 3\cot^2 x - 3\cot x &= 1 \\ 3\cot^2 x - 3\cot x - 1 &= 0 \end{aligned}$$

The equation will not factor. Use the quadratic formula for $\cot x$, $a = 3, b = -3, c = -1$.

$$\cot x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$\cot x = \frac{-(-3) \pm \sqrt{(-3)^2-4(3)(-1)}}{2(3)}$$
$$\cot x = \frac{3 \pm \sqrt{9+12}}{6}$$

$$\cot x = \frac{3+\sqrt{21}}{6} \quad \text{or} \quad \cot x = \frac{3-\sqrt{21}}{6}$$
$$\cot x = \frac{3+4.5826}{6} \qquad \cot x = \frac{3-4.5826}{6}$$
$$\cot x = 1.2638 \qquad \cot x = -0.2638$$
$$\tan x = \frac{1}{1.2638} \qquad \tan x = \frac{1}{-0.2638}$$
$$x = 0.6694, 3.81099 \qquad x = 1.8287, 4.9703$$

Example 10: Solve $-5\cos^2 x + 9\sin x + 3 = 0$ for values of x over the interval $[0, 2\pi]$.

Solution: Change $\cos^2 x$ to $1-\sin^2 x$ from the Pythagorean Identity.

$$\begin{aligned} -5\cos^2 x + 9\sin x + 3 &= 0 \\ -5(1-\sin^2 x) + 9\sin x + 3 &= 0 \\ -5 + 5\sin^2 x + 9\sin x + 3 &= 0 \\ 5\sin^2 x + 9\sin x - 2 &= 0 \end{aligned}$$

$$\begin{aligned} \sin x &= \frac{-9 \pm \sqrt{9^2-4(5)(-2)}}{2(5)} \\ \sin x &= \frac{-9 \pm \sqrt{81+40}}{10} \\ \sin x &= \frac{-9 \pm \sqrt{121}}{10} \\ \sin x &= \frac{-9+11}{10} \text{ and } \sin x = \frac{-9-11}{10} \\ \sin x &= \frac{1}{5} \text{ and } -2 \\ &\sin^{-1}(0.2) \text{ and } \sin^{-1}(-2) \end{aligned}$$

$x \approx .201$ *rad* and $\pi - .201 \approx 2.941$

This is the only solutions for x since -2 is not in the range of values.

To summarize, to solve a trigonometric equation, you can use the following techniques:

1. Simplify expressions with the fundamental identities.
2. Factor, pull out common factors, use factoring by grouping.
3. The Quadratic Formula.
4. Be aware of the intervals for x. Make sure your final answer is in the specified domain.

Points to Consider

- Are there other methods for solving equations that can be adapted to solving trigonometric equations?
- Will any of the trigonometric equations involve solving quadratic equations?
- Is there a way to solve a trigonometric equation that will not factor?
- Is substitution of a function with an identity a feasible approach to solving a trigonometric equation?

Review Questions

1. Solve the equation $\sin 2\theta = 0.6$ for $0 \leq \theta < 2\pi$.
2. Solve the equation $\cos^2 x = \frac{1}{16}$ over the interval $[0, 2\pi]$
3. Solve the trigonometric equation $\tan^2 x = 1$ for all values of θ such that $0 \leq \theta \leq 2\pi$
4. Solve the trigonometric equation $4\sin x\cos x + 2\cos x - 2\sin x - 1 = 0$ such that $0 \leq x < 2\pi$.
5. Solve $\sin^2 x - 2\sin x - 3 = 0$ for x over $[0, \pi]$.
6. Solve $\tan^2 x = 3\tan x$ for x over $[0, \pi]$.
7. Find all the solutions for the trigonometric equation $2\sin^2 \frac{x}{4} - 3\cos \frac{x}{4} = 0$ over the interval $[0, 2\pi)$.
8. Solve the trigonometric equation $3 - 3\sin^2 x = 8\sin x$ over the interval $[0, 2\pi]$.
9. Solve $2\sin x\tan x = \tan x + \sec x$ for all values of $x\ \varepsilon[0, 2\pi]$.
10. Solve the trigonometric equation $2\cos^2 x + 3\sin x - 3 = 0$ over the interval $[0, 2\pi]$.
11. Solve $\tan^2 x + \tan x - 2 = 0$ for values of x over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
12. Solve the trigonometric equation such that $5\cos^2\theta - 6\sin\theta = 0$ over the interval $[0, 2\pi]$.

3.4 Sum and Difference Identities

Learning Objectives

- Use and identify the sum and difference identities.
- Apply the sum and difference identities to solve trigonometric equations.
- Find the exact value of a trigonometric function for certain angles.

In this section we are going to explore $\cos(a \pm b), \sin(a \pm b)$, and $\tan(a \pm b)$. These identities have very useful expansions and can help to solve identities and equations.

Sum and Difference Formulas: Cosine

Is $\cos 15^\circ = \cos(45^\circ - 30^\circ)$? Upon appearance, yes, it is. This section explores how to find an expression that would equal $\cos(45^\circ - 30^\circ)$. To simplify this, let the two given angles be a and b where $0 < b < a < 2\pi$.

Begin with the unit circle and place the angles a and b in standard position as shown in Figure A. Point Pt1 lies on the terminal side of b, so its coordinates are $(\cos b, \sin b)$ and Point Pt2 lies on the terminal side of a so its coordinates are $(\cos a, \sin a)$. Place the $a - b$ in standard position, as shown in Figure B. The point A has coordinates $(1, 0)$ and the Pt3 is on the terminal side of the angle $a - b$, so its coordinates are $(\cos[a - b], \sin[a - b])$.

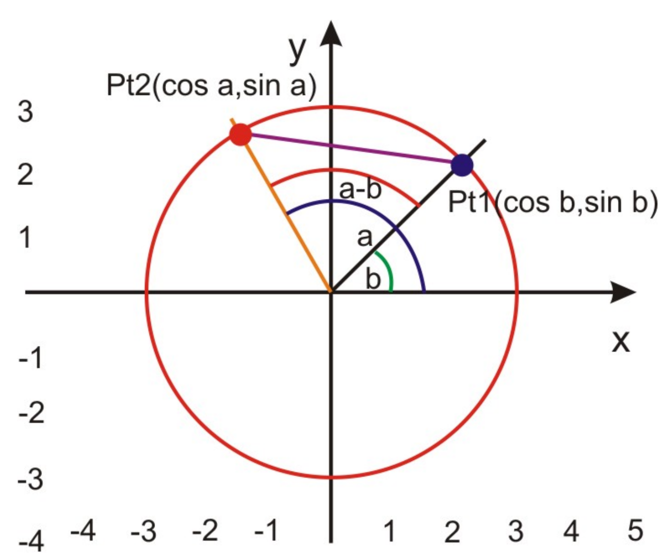


Figure A

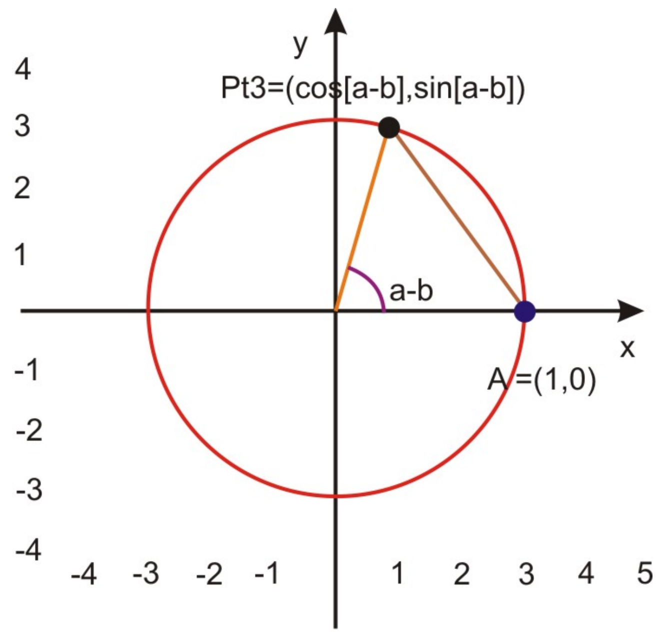


Figure B

Triangles OP_1P_2 in figure A and Triangle OAP_3 in figure B are congruent. (Two sides and the included angle, $a-b$, are equal). Therefore the unknown side of each triangle must also be equal. That is: $d\ (A,P_3) = d\ (P_1,P_2)$

Applying the distance formula to the triangles in Figures A and B and setting them equal to each other:

$$\sqrt{[\cos(a-b)-1]^2+[\sin(a-b)-0]^2} = \sqrt{(\cos a-\cos b)^2+(\sin a-\sin b)^2}$$

Square both sides to eliminate the square root.

$$[\cos(a-b)-1]^2+[\sin(a-b)-0]^2 = (\cos a-\cos b)^2+(\sin a-\sin b)^2$$

FOIL all four squared expressions and simplify.

$$\begin{aligned}
\cos^2(a-b)-2\cos(a-b)+1+\sin^2(a-b) &= \cos^2 a-2\cos a\cos b+\cos^2 b+\sin^2 a-2\sin a\sin b+\sin^2 b\\
\underbrace{\sin^2(a-b)+\cos^2(a-b)}-2\cos(a-b)+1 &= \underbrace{\sin^2 a+\cos^2 a}-2\cos a\cos b+\underbrace{\sin^2 b+\cos^2 b}-2\sin a\sin b\\
1-2\cos(a-b)+1 &= 1-2\cos a\cos b+1-2\sin a\sin b\\
2-2\cos(a-b) &= 2-2\cos a\cos b-2\sin a\sin b\\
-2\cos(a-b) &= -2\cos a\cos b-2\sin a\sin b\\
\cos(a-b) &= \cos a\cos b+\sin a\sin b
\end{aligned}$$

In $\cos(a-b) = \cos a\cos b+\sin a\sin b$, the *difference* formula for cosine, you can substitute $a-(-b) = a+b$ to obtain: $\cos(a+b) = \cos[a-(-b)]$ or $\cos a\cos(-b)+\sin a\sin(-b)$. since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$, then $\cos(a+b) = \cos a\cos b-\sin a\sin b$, which is the *sum* formula for cosine.

Using the Sum and Difference Identities of Cosine

The sum/difference formulas for cosine can be used to establish other identities:

Example 1: Find an equivalent form of $\cos\left(\frac{\pi}{2}-\theta\right)$ using the cosine difference formula.

Solution:

$$\begin{aligned}
\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta \\
\cos\left(\frac{\pi}{2}-\theta\right) &= 0\times\cos\theta+1\times\sin\theta,\ \text{substitute}\ \cos\frac{\pi}{2}=0\ \text{and}\ \sin\frac{\pi}{2}=1 \\
\cos\left(\frac{\pi}{2}-\theta\right) &= \sin\theta
\end{aligned}$$

We know that is a true identity because of our understanding of the sine and cosine curves, which are a phase shift of $\frac{\pi}{2}$ off from each other.

The cosine formulas can also be used to find exact values of cosine that we weren't able to find before, such as $15^\circ=(45^\circ-30^\circ), 75^\circ=(45^\circ+30^\circ)$, among others.

Example 2: Find the exact value of $\cos 15^\circ$

Solution: Use the difference formula where $a=45^\circ$ and $b=30^\circ$.

$$\begin{aligned}
\cos(45^\circ-30^\circ) &= \cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ \\
\cos 15^\circ &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2} \\
\cos 15^\circ &= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

Example 3: Find the exact value of $\cos 105^\circ$.

Solution: There may be more than one pair of key angles that can add up (or subtract to) 105°. Both pairs, $45^\circ+60^\circ$ and $150^\circ-45^\circ$, will yield the correct answer.

1.

$$\begin{aligned}
\cos 105^\circ &= \cos(45^\circ+60^\circ) \\
&= \cos 45^\circ\cos 60^\circ-\sin 45^\circ\sin 60^\circ,\ \text{substitute in the known values} \\
&= \frac{\sqrt{2}}{2}\times\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

2.

$$\begin{aligned}
\cos 105^\circ &= \cos(150^\circ-45^\circ) \\
&= \cos 150^\circ\cos 45^\circ+\sin 150^\circ\sin 45^\circ \\
&= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

You do not need to do the problem multiple ways, just the one that seems easiest to you.

Example 4: Find the exact value of $\cos\frac{5\pi}{12}$, in radians.

Solution: $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$, notice that $\frac{\pi}{4} = \frac{3\pi}{12}$ and $\frac{\pi}{6} = \frac{2\pi}{12}$

$$\begin{aligned}\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Sum and Difference Identities: Sine

To find $\sin(a+b)$, use Example 1, from above:

$$\begin{aligned}\sin(a+b) &= \cos\left[\frac{\pi}{2}-(a+b)\right] && \text{Set } \theta = a+b\\ &= \cos\left[\left(\frac{\pi}{2}-a\right)-b\right] && \text{Distribute the negative}\\ &= \cos\left(\frac{\pi}{2}-a\right)\cos b + \sin\left(\frac{\pi}{2}-a\right)\sin b && \text{Difference Formula for cosines}\\ &= \sin a\cos b + \cos a\sin b && \text{Co-function Identities}\end{aligned}$$

In conclusion, $\sin(a+b) = \sin a\cos b + \cos a\sin b$, which is the *sum* formula for sine.

To obtain the identity for $\sin(a-b)$:

$$\begin{aligned}\sin(a-b) &= \sin[a+(-b)]\\ &= \sin a\cos(-b) + \cos a\sin(-b) && \text{Use the sine sum formula}\\ \sin(a-b) &= \sin a\cos b - \cos a\sin b && \text{Use } \cos(-b) = \cos b, \text{ and } \sin(-b) = -\sin b\end{aligned}$$

In conclusion, $\sin(a-b) = \sin a\cos b - \cos a\sin b$, so, this is the *difference* formula for sine.

Example 5: Find the exact value of $\sin\frac{5\pi}{12}$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}\sin\frac{5\pi}{12} &= \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)\\ &= \sin\frac{3\pi}{12}\cos\frac{2\pi}{12} + \cos\frac{3\pi}{12}\sin\frac{2\pi}{12}\\ \sin\frac{5\pi}{12} &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\times\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example 6: Given $\sin\alpha = \frac{12}{13}$, where α is in Quadrant II, and $\sin\beta = \frac{3}{5}$, where β is in Quadrant I, find the exact value of $\sin(\alpha+\beta)$.

Solution: To find the exact value of $\sin(\alpha+\beta)$, here we use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. The values of $\sin\alpha$ and $\sin\beta$ are known, however the values of $\cos\alpha$ and $\cos\beta$ need to be found.

Use $\sin^2\alpha + \cos^2\alpha = 1$, to find the values of each of the missing cosine values.

For $\cos a : \sin^2\alpha + \cos^2\alpha = 1$, substituting $\sin\alpha = \frac{12}{13}$ transforms to $\left(\frac{12}{13}\right)^2 + \cos^2\alpha = \frac{144}{169} + \cos^2\alpha = 1$ or $\cos^2\alpha = \frac{25}{169}$ $\cos\alpha = \pm\frac{5}{13}$, however, since α is in Quadrant II, the cosine is negative, $\cos\alpha = -\frac{5}{13}$.

For $\cos\beta$ use $\sin^2\beta + \cos^2\beta = 1$ and substitute $\sin\beta = \frac{3}{5}$, $\left(\frac{3}{5}\right)^2 + \cos^2\beta = \frac{9}{25} + \cos^2\beta = 1$ or $\cos^2\beta = \frac{16}{25}$ and $\cos\beta = \pm\frac{4}{5}$ and since β is in Quadrant I, $\cos\beta = \frac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$$\begin{aligned}
\sin(\alpha+\beta) &= \frac{12}{13}\times\frac{4}{5} + \left(-\frac{5}{13}\right)\times\frac{3}{5} \text{ or } \frac{48}{65} - \frac{15}{65} \\
\sin(\alpha+\beta) &= \frac{33}{65}
\end{aligned}$$

Sum and Difference Identities: Tangent

To find the sum formula for tangent:

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} && \text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a\sin b} && \text{Substituting the sum formulas for sine and cosine} \\
&= \frac{\frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b - \sin a\sin b}{\cos a\cos b}} && \text{Divide both the numerator and the denominator by } \cos a\cos b \\
&= \frac{\frac{\sin a\cos b}{\cos a\cos b} + \frac{\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b} - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Reduce each of the fractions} \\
&= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Substitute } \frac{\sin\theta}{\cos\theta} = \tan\theta \\
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a\tan b} && \text{Sum formula for tangent}
\end{aligned}$$

In conclusion, $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$. Substituting $-b$ for b in the above results in the difference formula for tangent:

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$

Example 7: Find the exact value of $\tan 285^\circ$.

Solution: Use the difference formula for tangent, with $285^\circ = 330^\circ - 45^\circ$

$$\begin{aligned}
\tan(330^\circ - 45^\circ) &= \frac{\tan 330^\circ - \tan 45^\circ}{1 + \tan 330^\circ \tan 45^\circ} \\
&= \frac{-\frac{\sqrt{3}}{3} - 1}{1 - \frac{\sqrt{3}}{3} \cdot 1} = \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \\
&= \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \\
&= \frac{-9 - 6\sqrt{3} - 3}{9 - 3} \\
&= \frac{-12 - 6\sqrt{3}}{6} \\
&= -2 - \sqrt{3}
\end{aligned}$$

To verify this on the calculator, $\tan 285^\circ = -3.732$ and $-2 - \sqrt{3} = -3.732$.

Using the Sum and Difference Identities to Verify Other Identities

Example 8: Verify the identity $\frac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

$$\begin{aligned}
\cot x \cot y + 1 &= \frac{\cos(x-y)}{\sin x \sin y} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} && \text{Expand using the cosine difference formula.} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + 1 \\
\cot x \cot y + 1 &= \cot x \cot y + 1 && \text{cotangent equals cosine over sine}
\end{aligned}$$

Example 9: Show $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

Solution: First, expand left hand side using the sum and difference formulas:

$$\begin{aligned}
\cos(a+b)\cos(a-b) &= (\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b) \\
&= \cos^2 a \cos^2 b - \sin^2 a \sin^2 b \rightarrow \text{FOIL, middle terms cancel out}
\end{aligned}$$

Substitute $(1 - \sin^2 b)$ for $\cos^2 b$ and $(1 - \cos^2 a)$ for $\sin^2 a$ and simplify.

$$\cos^2 a(1 - \sin^2 b) - \sin^2 b(1 - \cos^2 a)$$
$$\cos^2 a - \cos^2 a \sin^2 b - \sin^2 b + \cos^2 a \sin^2 b$$
$$\cos^2 a - \sin^2 b$$

Solving Equations with the Sum and Difference Formulas

Just like the section before, we can incorporate all of the sum and difference formulas into equations and solve for values of x. In general, you will apply the formula *before* solving for the variable. Typically, the goal will be to isolate $\sin x, \cos x$, or $\tan x$ and then apply the inverse. Remember, that you may have to use the identities in addition to the formulas seen in this section to solve an equation.

Example 10: Solve $3\sin(x-\pi)=3$ in the interval $[0,2\pi)$.

Solution: First, get $\sin(x-\pi)$ by itself, by dividing both sides by 3.

$$\begin{aligned}\frac{3\sin(x-\pi)}{3}&=\frac{3}{3}\\ \sin(x-\pi)&=1\end{aligned}$$

Now, expand the left side using the sine difference formula.

$$\begin{aligned}\sin x\cos\pi-\cos x\sin\pi&=1\\ \sin x(-1)-\cos x(0)&=1\\ -\sin x&=1\\ \sin x&=-1\end{aligned}$$

The $\sin x=-1$ when x is $\frac{3\pi}{2}$.

Example 11: Find all the solutions for $2\cos^2\left(x+\frac{\pi}{2}\right)=1$ in the interval $[0,2\pi)$.

Solution: Get the $\cos^2\left(x+\frac{\pi}{2}\right)$ by itself and then take the square root.

$$\begin{aligned}2\cos^2\left(x+\frac{\pi}{2}\right)&=1\\ \cos^2\left(x+\frac{\pi}{2}\right)&=\frac{1}{2}\\ \cos\left(x+\frac{\pi}{2}\right)&=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\end{aligned}$$

Now, use the cosine sum formula to expand and solve.

$$\begin{aligned}\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}&=\frac{\sqrt{2}}{2}\\ \cos x(0)-\sin x(1)&=\frac{\sqrt{2}}{2}\\ -\sin x&=\frac{\sqrt{2}}{2}\\ \sin x&=-\frac{\sqrt{2}}{2}\end{aligned}$$

The $\sin x=-\frac{\sqrt{2}}{2}$ is in Quadrants III and IV, so $x=\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Points to Consider

- What are the angles that have 15° and 75° as reference angles?

- Are the only angles that we can find the exact sine, cosine, or tangent values for, multiples of $\frac{\pi}{12}$? (Recall that $\frac{\pi}{2}$ would be $6 \cdot \frac{\pi}{12}$, making it a multiple of $\frac{\pi}{12}$)

Review Questions

1. Find the exact value for:
   a. $\cos \frac{5\pi}{12}$
   b. $\cos \frac{7\pi}{12}$
   c. $\sin 345^\circ$
   d. $\tan 75^\circ$
   e. $\cos 345^\circ$
   f. $\sin \frac{17\pi}{12}$
2. If $\sin y = \frac{12}{13}$, y is in quad II, and $\sin z = \frac{3}{5}$, z is in quad I find $\cos(y - z)$
3. If $\sin y = -\frac{5}{13}$, y is in quad III, and $\sin z = \frac{4}{5}$, z is in quad II find $\sin(y + z)$
4. Simplify:
   a. $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$
   b. $\sin 25^\circ \cos 5^\circ + \cos 25^\circ \sin 5^\circ$
5. Prove the identity: $\frac{\cos(m-n)}{\sin m \cos n} = \cot m + \tan n$
6. Simplify $\cos(\pi + \theta) = -\cos\theta$
7. Verify the identity: $\sin(a + b)\sin(a - b) = \cos^2 b - \cos^2 a$
8. Simplify $\tan(\pi + \theta)$
9. Verify that $\sin \frac{\pi}{2} = 1$, using the sine sum formula.
10. Reduce the following to a single term: $\cos(x + y)\cos y + \sin(x + y)\sin y$.
11. Prove $\frac{\cos(c+d)}{\cos(c-d)} = \frac{1-\tan c \tan d}{1+\tan c \tan d}$
12. Find all solutions to $2\cos^2\left(x + \frac{\pi}{2}\right) = 1$, when x is between $[0, 2\pi)$.
13. Solve for all values of x between $[0, 2\pi)$ for $2\tan^2\left(x + \frac{\pi}{6}\right) + 1 = 7$.
14. Find all solutions to $\sin\left(x + \frac{\pi}{6}\right) = \sin\left(x - \frac{\pi}{4}\right)$, when x is between $[0, 2\pi)$.

3.5 Double Angle Identities

Learning Objectives

- Use the double angle identities to solve other identities.
- Use the double angle identities to solve equations.

Deriving the Double Angle Identities

One of the formulas for calculating the sum of two angles is:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}\sin(\alpha+\alpha) &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha\\ \sin 2\alpha &= 2\sin\alpha\cos\alpha\end{aligned}$$

This is the double angle formula for the sine function. The same procedure can be used in the sum formula for cosine, start with the sum angle formula:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}\cos(\alpha+\alpha) &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha\\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\end{aligned}$$

This is one of the double angle formulas for the cosine function. Two more formulas can be derived by using the Pythagorean Identity, $\sin^2\alpha + \cos^2\alpha = 1$.

$\sin^2\alpha = 1-\cos^2\alpha$ and likewise $\cos^2\alpha = 1-\sin^2\alpha$

$$\begin{aligned}&\text{Using } \sin^2\alpha = 1-\cos^2\alpha:\\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \cos^2\alpha - (1-\cos^2\alpha)\\ &= \cos^2\alpha - 1 + \cos^2\alpha\\ &= 2\cos^2\alpha - 1\end{aligned}$$

$$\begin{aligned}&\text{Using } \cos^2\alpha = 1-\sin^2\alpha:\\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= (1-\sin^2\alpha) - \sin^2\alpha\\ &= 1 - \sin^2\alpha - \sin^2\alpha\\ &= 1 - 2\sin^2\alpha\end{aligned}$$

Therefore, the double angle formulas for $\cos 2a$ are:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

Finally, we can calculate the double angle formula for tangent, using the tangent sum formula:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

If α and β are both the same angle in the above formula, then

$$\tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Applying the Double Angle Identities

Example 1: If $\sin a = \frac{5}{13}$ and a is in Quadrant II, find $\sin 2a$, $\cos 2a$, and $\tan 2a$.

Solution: To use $\sin 2a = 2\sin a\cos a$, the value of $\cos a$ must be found first.

$$= \cos^2 a + \sin^2 a = 1$$
$$= \cos^2 a + \left(\frac{5}{13}\right)^2 = 1$$
$$= \cos^2 a + \frac{25}{169} = 1$$
$$= \cos^2 a = \frac{144}{169}, \cos a = \pm\frac{12}{13}$$

.

However since a is in Quadrant II, $\cos a$ is negative or $\cos a = -\frac{12}{13}$.

$$\sin 2a = 2\sin a\cos a = 2\left(\frac{5}{13}\right) \times \left(-\frac{12}{13}\right) = \sin 2a = -\frac{120}{169}$$

For $\cos 2a$, use $\cos(2a) = \cos^2 a - \sin^2 a$

$$\cos(2a) = \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \text{ or } \frac{144-25}{169}$$
$$\cos(2a) = \frac{119}{169}$$

For $\tan 2a$, use $\tan 2a = \frac{2\tan a}{1-\tan^2 a}$. From above, $\tan a = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$.

$$\tan(2a) = \frac{2 \cdot \frac{-5}{12}}{1-\left(\frac{-5}{12}\right)^2} = \frac{\frac{-5}{6}}{1-\frac{25}{144}} = \frac{\frac{-5}{6}}{\frac{119}{144}} = -\frac{5}{6} \cdot \frac{144}{119} = -\frac{120}{119}$$

Example 2: Find $\cos 4\theta$.

Solution: Think of $\cos 4\theta$ as $\cos(2\theta + 2\theta)$.

$$\cos 4\theta = \cos(2\theta + 2\theta) = \cos 2\theta \cos 2\theta - \sin 2\theta \sin 2\theta = \cos^2 2\theta - \sin^2 2\theta$$

Now, use the double angle formulas for both sine and cosine. For cosine, you can pick which formula you would like to use. In general, because we are proving a cosine identity, stay with cosine.

$$\begin{aligned}
&= (2\cos^2\theta - 1)^2 - (2\sin\theta\cos\theta)^2 \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\sin^2\theta\cos^2\theta \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4(1-\cos^2\theta)\cos^2\theta \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\cos^2\theta + 4\cos^4\theta \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

Example 3: If $\cot x = \frac{4}{3}$ and x is an acute angle, find the exact value of $\tan 2x$.

Solution: Cotangent and tangent are reciprocal functions, $\tan x = \frac{1}{\cot x}$ and $\tan x = \frac{3}{4}$.

$$\begin{aligned}
\tan 2x &= \frac{2\tan x}{1-\tan^2 x} \\
&= \frac{2 \cdot \frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} \\
&= \frac{\frac{3}{2}}{1-\frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} \\
&= \frac{3}{2} \cdot \frac{16}{7} = \frac{24}{7}
\end{aligned}$$

Example 4: Given $\sin(2x) = \frac{2}{3}$ and x is in Quadrant I, find the value of $\sin x$.

Solution: Using the double angle formula, $\sin 2x = 2\sin x \cos x$. Because we do not know $\cos x$, we need to solve for $\cos x$ in the Pythagorean Identity, $\cos x = \sqrt{1-\sin^2 x}$. Substitute this into our formula and solve for $\sin x$.

$$\begin{aligned}
\sin 2x &= 2\sin x \cos x \\
\frac{2}{3} &= 2\sin x \sqrt{1-\sin^2 x} \\
\left(\frac{2}{3}\right)^2 &= \left(2\sin x \sqrt{1-\sin^2 x}\right)^2 \\
\frac{4}{9} &= 4\sin^2 x(1-\sin^2 x) \\
\frac{4}{9} &= 4\sin^2 x - 4\sin^4 x
\end{aligned}$$

At this point we need to get rid of the fraction, so multiply both sides by the reciprocal.

$$\begin{aligned}
\frac{9}{4}\left(\frac{4}{9} = 4\sin^2 x - 4\sin^4 x\right) \\
1 = 9\sin^2 x - 9\sin^4 x \\
0 = 9\sin^4 x - 9\sin^2 x + 1
\end{aligned}$$

Now, this is in the form of a quadratic equation, even though it is a quartic. Set $a = \sin^2 x$, making the equation $9a^2 - 9a + 1 = 0$. Once we have solved for a, then we can substitute $\sin^2 x$ back in and solve for x. In the Quadratic Formula, $a = 9, b = -9, c = 1$.

$$\frac{9 \pm \sqrt{(-9)^2 - 4(9)(1)}}{2(9)} = \frac{9 \pm \sqrt{81-36}}{18} = \frac{9 \pm \sqrt{45}}{18} = \frac{9 \pm 3\sqrt{5}}{18} = \frac{3 \pm \sqrt{5}}{6}$$

So, $a = \frac{3+\sqrt{5}}{6} \approx 0.873$ or $\frac{3-\sqrt{5}}{6} \approx .1273$. This means that $\sin^2 x \approx 0.873$ or $.1273$ so $\sin x \approx 0.934$ or $\sin x \approx .357$.

Example 5: Prove $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$

Solution: Substitute in the double angle formulas. Use $\cos 2\theta = 1 - 2\sin^2\theta$, since it will produce only one term in the numerator.

$$\begin{aligned}
\tan\theta &= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} \\
&= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta
\end{aligned}$$

Solving Equations with Double Angle Identities

Much like the previous sections, these problems all involve similar steps to solve for the variable. Isolate the trigonometric function, using any of the identities and formulas you have accumulated thus far.

Example 6: Find all solutions to the equation $\sin 2x = \cos x$ in the interval $[0, 2\pi]$

Solution: Apply the double angle formula $\sin 2x = 2\sin x\cos x$

$$
\begin{aligned}
2\sin x\cos x &= \cos x \\
2\sin x\cos x-\cos x &= \cos x-\cos x \\
2\sin x\cos x-\cos x &= 0 \\
\cos x(2\sin x-1) &= 0 \text{ Factor out } \cos x \\
\text{Then } \cos x &= 0 \text{ or } 2\sin x-1=0 \\
\cos x &= 0 \text{ or } 2\sin x-1+1=0+1 \\
\frac{2}{2}\sin x &= \frac{1}{2} \\
\sin x &= \frac{1}{2}
\end{aligned}
$$

The values for $\cos x = 0$ in the interval $[0, 2\pi]$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ and the values for $\sin x = \frac{1}{2}$ in the interval $[0, 2\pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Thus, there are four solutions.

Example 7: Solve the trigonometric equation $\sin 2x = \sin x$ such that $(-\pi \le x < \pi)$

Solution: Using the sine double angle formula:

$$
\begin{aligned}
\sin 2x &= \sin x \\
2\sin x\cos x &= \sin x \\
2\sin x\cos x-\sin x &= 0 \\
\sin x(2\cos x-1) &= 0 \\
&\downarrow \qquad \searrow \\
& \qquad 2\cos x-1=0 \\
& \qquad 2\cos x=1 \\
\sin x &= 0 \\
x &= 0, -\pi \qquad \cos x=\frac{1}{2} \\
& \qquad x=\frac{\pi}{3}, -\frac{\pi}{3}
\end{aligned}
$$

Example 8: Find the exact value of $\cos 2x$ given $\cos x = -\frac{13}{14}$ if x is in the second quadrant.

Solution: Use the double-angle formula with cosine only.

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 2\left(-\frac{13}{14}\right)^2 - 1$$
$$\cos 2x = 2\left(\frac{169}{196}\right) - 1$$
$$\cos 2x = \left(\frac{338}{196}\right) - 1$$
$$\cos 2x = \frac{338}{196} - \frac{196}{196}$$
$$\cos 2x = \frac{142}{196} = \frac{71}{98}$$

Example 9: Solve the trigonometric equation $4\sin\theta\cos\theta = \sqrt{3}$ over the interval $[0, 2\pi)$.

Solution: Pull out a 2 from the left-hand side and this is the formula for $\sin 2x$.

$$4\sin\theta\cos\theta = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = 2\sin 2\theta$$
$$2\sin 2\theta = \sqrt{3}$$
$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

The solutions for 2θ are $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$, dividing each of these by 2, we get the solutions for θ, which are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{8\pi}{6}$.

Points to Consider

- Are there similar formulas that can be derived for other angles?
- Can technology be used to either solve these trigonometric equations or to confirm the solutions?

Review Questions

1. If $\sin x = \frac{4}{5}$ and x is in Quad II, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$
2. Find the exact value of $\cos^2 15° - \sin^2 15°$
3. Verify the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
4. Verify the identity: $\sin 2t - \tan t = \tan t \cos 2t$
5. If $\sin x = -\frac{9}{41}$ and x is in Quad III, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$
6. Find all solutions to $\sin 2x + \sin x = 0$ if $0 \le x < 2\pi$
7. Find all solutions to $\cos^2 x - \cos 2x = 0$ if $0 \le x < 2\pi$
8. If $\tan x = \frac{3}{4}$ and $0° < x < 90°$, use the double angle formulas to determine each of the following:
   a. $\tan 2x$
   b. $\sin 2x$

c. $\cos 2x$

9. Use the double angle formulas to prove that the following equations are identities.
    a. $2\csc 2x = \csc^2 x \tan x$
    b. $\cos^4\theta - \sin^4\theta = \cos 2\theta$
    c. $\frac{\sin 2x}{1+\cos 2x} = \tan x$
10. Solve the trigonometric equation $\cos 2x - 1 = \sin^2 x$ such that $[0, 2\pi)$
11. Solve the trigonometric equation $\cos 2x = \cos x$ such that $0 \le x < \pi$
12. Prove $2\csc 2x \tan x = \sec^2 x$.
13. Solve $\sin 2x - \cos 2x = 1$ for x in the interval $[0, 2\pi)$.
14. Solve the trigonometric equation $\sin^2 x - 2 = \cos 2x$ such that $0 \le x < 2\pi$

3.6 Half-Angle Identities

Learning Objectives

- Apply the half angle identities to expressions, equations and other identities.
- Use the half angle identities to find the exact value of trigonometric functions for certain angles.

Just as there are double angle identities, there are also half angle identities. For example: $\sin \frac{1}{2}a$ can be found in terms of the angle "a". Recall that $\frac{1}{2}a$ and $\frac{a}{2}$ are the same thing and will be used interchangeably throughout this section.

Deriving the Half Angle Formulas

In the previous lesson, one of the formulas that was derived for the cosine of a double angle is: $\cos 2\theta = 1 - 2\sin^2 \theta$. Set $\theta = \frac{\alpha}{2}$, so the equation above becomes $\cos 2\frac{\alpha}{2} = 1 - 2\sin^2 \frac{\alpha}{2}$.

Solving this for $\sin \frac{\alpha}{2}$, we get:

$$\begin{aligned}
\cos 2\frac{\alpha}{2} &= 1 - 2\sin^2 \frac{\alpha}{2} \\
\cos \alpha &= 1 - 2\sin^2 \frac{\alpha}{2} \\
2\sin^2 \frac{\alpha}{2} &= 1 - \cos \alpha \\
\sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\
\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}}
\end{aligned}$$

$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin \frac{\alpha}{2} = -\sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

Example 1: Determine the exact value of $\sin 15^\circ$.

Solution: Using the half angle identity, $\alpha = 30^\circ$, and 15° is located in the first quadrant. Therefore, $\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$.

$$\begin{aligned}\sin 15^\circ &= \sqrt{\frac{1-\cos 30^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}\end{aligned}$$

Plugging this into a calculator, $\sqrt{\frac{2-\sqrt{3}}{4}} \approx 0.2588$. Using the sine function on your calculator will validate that this answer is correct.

Example 2: Use the half angle identity to find exact value of $\sin 112.5^\circ$

Solution: since $\sin\frac{225^\circ}{2} = \sin 112.5^\circ$, use the half angle formula for sine, where $\alpha = 225^\circ$. In this example, the angle 112.5° is a second quadrant angle, and the sin of a second quadrant angle is positive.

$$\begin{aligned}\sin 112.5^\circ &= \sin\frac{225^\circ}{2}\\ &= \pm\sqrt{\frac{1-\cos 225^\circ}{2}}\\ &= +\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}}\\ &= \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}}\\ &= \sqrt{\frac{2+\sqrt{2}}{4}}\end{aligned}$$

One of the other formulas that was derived for the cosine of a double angle is:

$\cos 2\theta = 2\cos^2\theta - 1$. Set $\theta = \frac{\alpha}{2}$, so the equation becomes $\cos 2\frac{\alpha}{2} = -1 + 2\cos^2\frac{\alpha}{2}$. Solving this for $\cos\frac{\alpha}{2}$, we get:

$$\begin{aligned}\cos 2\frac{\alpha}{2} &= 2\cos^2\frac{\alpha}{2} - 1\\ \cos\alpha &= 2\cos^2\frac{\alpha}{2} - 1\\ 2\cos^2\frac{\alpha}{2} &= 1+\cos\alpha\\ \cos^2\frac{\alpha}{2} &= \frac{1+\cos\alpha}{2}\\ \cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\end{aligned}$$

$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

Example 3: Given that the $\cos\theta = \frac{3}{4}$, and that θ is a fourth quadrant angle, find $\cos\frac{1}{2}\theta$

Solution: Because θ is in the fourth quadrant, the half angle would be in the second quadrant, making the cosine of the half angle negative.

$$\begin{aligned}\cos\frac{\theta}{2} &= -\sqrt{\frac{1+\cos\theta}{2}}\\ &= -\sqrt{\frac{1+\frac{3}{4}}{2}}\\ &= -\sqrt{\frac{\frac{7}{4}}{2}}\\ &= -\sqrt{\frac{7}{8}} = -\frac{\sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{14}}{4}\end{aligned}$$

Example 4: Use the half angle formula for the cosine function to prove that the following expression is an identity: $2\cos^2\frac{x}{2}-\cos x=1$

Solution: Use the formula $\cos\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{2}}$ and substitute it on the left-hand side of the expression.

$$\begin{aligned}2\left(\sqrt{\frac{1+\cos\theta}{2}}\right)^2-\cos\theta &= 1\\ 2\left(\frac{1+\cos\theta}{2}\right)-\cos\theta &= 1\\ 1+\cos\theta-\cos\theta &= 1\\ 1 &= 1\end{aligned}$$

The half angle identity for the tangent function begins with the reciprocal identity for tangent.

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}\Rightarrow\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

The half angle formulas for sine and cosine are then substituted into the identity.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}\\ &= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\end{aligned}$$

At this point, you can multiply by either $\frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}}$ or $\frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}}$. We will show both, because they produce different answers.

$$\begin{aligned} &= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}} \\ &= \frac{1-\cos\alpha}{\sqrt{1-\cos^2\alpha}} \\ &= \frac{1-\cos\alpha}{\sqrt{\sin^2\alpha}} \\ &= \frac{1-\cos\alpha}{\sin\alpha} \end{aligned} \qquad \text{or} \qquad \begin{aligned} &= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}} \\ &= \frac{\sqrt{1-\cos^2\alpha}}{1+\cos\alpha} \\ &= \frac{\sqrt{\sin^2\alpha}}{1+\cos\alpha} \\ &= \frac{\sin\alpha}{1+\cos\alpha} \end{aligned}$$

So, the two half angle identities for tangent are $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$ and $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$.

Example 5: Use the half-angle identity for tangent to determine an exact value for $\tan\frac{7\pi}{12}$.

Solution:

$$\begin{aligned} \tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} \\ \tan\frac{7\pi}{12} &= \frac{1-\cos\frac{7\pi}{6}}{\sin\frac{7\pi}{6}} \\ \tan\frac{7\pi}{12} &= \frac{1+\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \\ \tan\frac{7\pi}{12} &= -2-\sqrt{3} \end{aligned}$$

Example 6: Prove the following identity: $\tan x = \frac{1-\cos 2x}{\sin 2x}$

Solution: Substitute the double angle formulas for $\cos 2x$ and $\sin 2x$.

$$\begin{aligned} \tan x &= \frac{1-\cos 2x}{\sin 2x} \\ &= \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} \\ &= \frac{1-1+2\sin^2 x}{2\sin x\cos x} \\ &= \frac{2\sin^2 x}{2\sin x\cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

Solving Trigonometric Equations Using Half Angle Formulas

Example 7: Solve the trigonometric equation $\sin^2\theta = 2\sin^2\frac{\theta}{2}$ over the interval $[0, 2\pi)$.

Solution:

$$\begin{aligned}
\sin^2\theta &= 2\sin^2\frac{\theta}{2} \\
\sin^2\theta &= 2\left(\frac{1-\cos\theta}{2}\right) && \text{Half angle identity} \\
1-\cos^2\theta &= 1-\cos\theta && \text{Pythagorean identity} \\
\cos\theta-\cos^2\theta &= 0 \\
\cos\theta(1-\cos\theta) &= 0
\end{aligned}$$

Then $\cos\theta = 0$ or $1-\cos\theta = 0$, which is $\cos\theta = 1$.

$\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$, or 2π.

Points to Consider

- Can you derive a third or fourth angle formula?
- How do $\frac{1}{2}\sin x$ and $\sin\frac{1}{2}x$ differ? Is there a formula for $\frac{1}{2}\sin x$?

Review Questions

1. Find the exact value of:
   a. $\cos 112.5^\circ$
   b. $\sin 105^\circ$
   c. $\tan\frac{7\pi}{8}$
   d. $\tan\frac{\pi}{8}$
   e. $\sin 67.5^\circ$
   f. $\tan 165^\circ$
2. If $\sin\theta = \frac{7}{25}$ and θ is in Quad II, find $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, \tan\frac{\theta}{2}$
3. Prove the identity: $\tan\frac{b}{2} = \frac{\sec b}{\sec b\csc b+\csc b}$
4. Verify the identity: $\cot\frac{c}{2} = \frac{\sin c}{1-\cos c}$
5. Prove that $\sin x\tan\frac{x}{2}+2\cos x = 2\cos^2\frac{x}{2}$
6. If $\sin u = -\frac{8}{13}$, find $\cos\frac{u}{2}$
7. Solve $2\cos^2\frac{x}{2} = 1$ for $0 \le x < 2\pi$
8. Solve $\tan\frac{a}{2} = 4$ for $0^\circ \le a < 360^\circ$
9. Solve the trigonometric equation $\cos\frac{x}{2} = 1+\cos x$ such that $0 \le x < 2\pi$.
10. Prove $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$.

3.7 Products, Sums, Linear Combinations, and Applications

Learning Objectives

- Use the transformation formulas to go from product to sum and sum to product.
- Derive multiple angle formulas.
- Use linear combinations to solve trigonometric equations.
- Apply trigonometric equations to real-life situations.

Sum to Product Formulas for Sine and Cosine

In some problems, the product of two trigonometric functions is more conveniently found by the sum of two trigonometric functions by use of identities such as this one:

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$$

This can be verified by using the sum and difference formulas:

$$\begin{aligned}
2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} &= 2\left[\sin\left(\tfrac{\alpha}{2}+\tfrac{\beta}{2}\right)\cos\left(\tfrac{\alpha}{2}-\tfrac{\beta}{2}\right)\right]\\
&= 2\left[\left(\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\left(\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\right]\\
&= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\cos^2\tfrac{\beta}{2}+\sin^2\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\beta}{2}\cos^2\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\alpha}{2}\sin^2\tfrac{\beta}{2}\cos\tfrac{\alpha}{2}\right]\\
&= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\left(\sin^2\tfrac{\beta}{2}+\cos^2\tfrac{\beta}{2}\right)+\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\left(\sin^2\tfrac{\alpha}{2}+\cos^2\tfrac{\alpha}{2}\right)\right]\\
&= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}+\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\right]\\
&= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\
&= \sin\left(2\cdot\frac{\alpha}{2}\right)+\sin\left(2\cdot\frac{\beta}{2}\right)\\
&= \sin\alpha+\sin\beta
\end{aligned}$$

The following variations can be derived similarly:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\times\cos\frac{\alpha+\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\times\sin\frac{\alpha-\beta}{2}$$

Example 1: Change $\sin 5x - \sin 9x$ into a product.

Solution: Use the formula $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\times\cos\frac{\alpha+\beta}{2}$.

$$\begin{aligned}\sin 5x - \sin 9x &= 2\sin\frac{5x-9x}{2}\cos\frac{5x+9x}{2}\\ &= 2\sin(-2x)\cos 7x\\ &= -2\sin 2x\cos 7x\end{aligned}$$

Example 2: Change $\cos(-3x)+\cos 8x$ into a product.

Solution: Use the formula $\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$.

$$\begin{aligned}\cos(-3x)+\cos(8x) &= 2\cos\frac{-3x+8x}{2}\cos\frac{-3x-8x}{2}\\ &= 2\cos(2.5x)\cos(-5.5x)\\ &= 2\cos(2.5)\cos(5.5x)\end{aligned}$$

Example 3: Change $2\sin 7x\cos 4x$ to a sum.

Solution: This is the reverse of what was done in the previous two examples. Looking at the four formulas above, take the one that has sine and cosine as a product, $\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$. Therefore, $7x = \frac{\alpha+\beta}{2}$ and $4x = \frac{\alpha-\beta}{2}$.

$$7x = \frac{\alpha+\beta}{2} \qquad \text{and} \qquad 4x = \frac{\alpha-\beta}{2}$$

$$14x = \alpha+\beta \qquad\qquad 8x = \alpha-\beta$$
$$\alpha = 14x-\beta \qquad \text{so} \qquad 8x = [14x-\beta]-\beta$$

$$-6x = -2\beta$$
$$3x = \beta$$

$$\alpha = 14x-3x$$
$$\alpha = 11x$$

So, this translates to $\sin(11x)+\sin(3x)$. A shortcut for this problem, would be to notice that the sum of $7x$ and $4x$ is $11x$ and the difference is $3x$.

Product to Sum Formulas for Sine and Cosine

There are two formulas for transforming a product of sine or cosine into a sum or difference. First, let's look at the product of the sine of two angles. To do this, start with cosine.

$$\begin{aligned}
\cos(a-b) = \cos a\cos b + \sin a\sin b \text{ and } \cos(a+b) &= \cos a\cos b - \sin a\sin b\\
\cos(a-b) - \cos(a+b) &= \cos a\cos b + \sin a\sin b - (\cos a\cos b - \sin a\sin b)\\
\cos(a-b) - \cos(a+b) &= \cos a\cos b + \sin a\sin b - \cos a\cos b + \sin a\sin b\\
\cos(a-b) - \cos(a+b) &= 2\sin a\sin b\\
\frac{1}{2}[\cos(a-b) - \cos(a+b)] &= \sin a\sin b
\end{aligned}$$

The following product to sum formulas can be derived using the same method:

$$\begin{aligned}
\cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]\\
\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\
\cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]
\end{aligned}$$

Example 4: Change $\cos 2x\cos 5y$ to a sum.

Solution: Use the formula $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$. Set $\alpha = 2x$ and $\beta = 5y$.

$$\cos 2x\cos 5y = \frac{1}{2}[\cos(2x-5y)+\cos(2x+5y)]$$

Example 5: Change $\frac{\sin 11z+\sin z}{2}$ to a product.

Solution: Use the formula $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$. Therefore, $\alpha+\beta = 11z$ and $\alpha-\beta = z$. Solve the second equation for α and plug that into the first.

$$\begin{aligned}
\alpha = z+\beta \rightarrow (z+\beta)+\beta &= 11z \qquad\qquad \text{and} \quad \alpha = z+5z = 6z\\
z+2\beta &= 11z\\
2\beta &= 10z\\
\beta &= 5z
\end{aligned}$$

$\frac{\sin 11z+\sin z}{2} = \sin 6z\cos 5z$. Again, the sum of $6z$ and $5z$ is $11z$ and the difference is z.

Solving Equations with Product and Sum Formulas

Example 6: Solve $\sin 4x + \sin 2x = 0$.

Solution: Use the formula $\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$.

$$\begin{aligned} \sin 4x + \sin 2x &= 0 \qquad & \text{So, } \sin 3x = 0 \text{ and } \cos x = 0 \rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \\ 2\sin 3x \cos x &= 0 & 3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi \\ \sin 3x \cos x &= 0 & x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Example 7: Solve $\cos 5x + \cos x = \cos 2x$.

Solution: Use the formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$.

$$\begin{aligned} \cos 5x + \cos x &= \cos 2x \\ 2\cos 3x \cos 2x &= \cos 2x \\ 2\cos 3x \cos 2x - \cos 2x &= 0 \\ \cos 2x(2\cos 3x - 1) &= 0 \end{aligned}$$

$$\swarrow \qquad \searrow$$

$$\begin{aligned} \cos 2x = 0 \qquad & \qquad 2\cos 3x - 1 = 0 \\ & \qquad 2\cos 3x = 1 \\ 2x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} & \quad \cos 3x = \frac{1}{2} \\ x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad & \qquad 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \\ & \qquad x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9} \end{aligned}$$

Triple-Angle Formulas and Beyond

By combining the sum formula and the double angle formula, formulas for triple angles and more can be found.

Example 8: Find the formula for $\sin 3x$

Solution: Use both the double angle formula and the sum formula.

$$\begin{aligned} \sin 3x &= \sin(2x + x) \\ &= \sin(2x)\cos x + \cos(2x)\sin x \\ &= (2\sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x \\ &= 2\sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x \\ &= 3\sin x \cos^2 x - \sin^3 x \\ &= 3\sin x(1 - \sin^2 x) - \sin^3 x \\ &= 3\sin x - 4\sin^3 x \end{aligned}$$

Example 9: Find the formula for $\cos 4x$

Solution: Using the same method from the previous example, you can obtain this formula.

$$\begin{aligned}\cos 4x &= \cos(2x+2x)\\ &= \cos^2 2x - \sin^2 2x\\ &= (\cos^2 x - \sin^2 x)^2 - (2\sin x\cos x)^2\\ &= \cos^4 x - 2\sin^2 x\cos^2 x + \sin^4 x - 4\sin^2 x\cos^2 x\\ &= \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x\\ &= \cos^4 x - 6(1-\cos^2 x)\cos^2 x + (1-\cos^2 x)^2\\ &= 1 - 8\cos^2 x + 8\cos^4 x\end{aligned}$$

Linear Combinations

Here, we take an equation which takes a linear combination of sine and cosine and converts it into a simpler cosine function.

$A\cos x + B\sin x = C\cos(x-D)$, where $C = \sqrt{A^2+B^2}$, $\cos D = \frac{A}{C}$ and $\sin D = \frac{B}{C}$.

Example 10: Transform $3\cos 2x - 4\sin 2x$ into the form $C\cos(2x-D)$

Solution: $A = 3$ and $B = -4$, so $C = \sqrt{3^2+(-4)^2} = 5$. Therefore $\cos D = \frac{3}{5}$ and $\sin D = -\frac{4}{5}$ which makes the reference angle is -53.1° or -0.927 radians. since cosine is positive and sine is negative, the angle must be a fourth quadrant angle. D must therefore be 306.9° or 5.36 radians. The final answer is $3\cos 2x - 4\sin 2x = 5\cos(2x-5.36)$.

Example 11: Solve $5\cos x + 12\sin x = 6$.

Solution: First, transform the left-hand side into the form $C\cos(x-D)$. $A = 5$ and $B = 12$, so $C = \sqrt{5^2+12^2} = 13$. From this $\cos D = \frac{5}{13}$ and $\sin D = \frac{12}{13}$, which makes the angle in the first quadrant and 1.176 radians. Now, our equation looks like this: $13\cos(x-1.176) = 6$ and we can solve for x.

$$\begin{aligned}\cos(x-1.176) &= \frac{6}{13}\\ x-1.176 &= \cos^{-1}\left(\frac{6}{13}\right)\\ x-1.176 &= 1.09\\ x &= 2.267 \text{ radians}\end{aligned}$$

Applications & Technology

Example 12: The range of a small rocket that is launched with an initial velocity v at an angle with θ the horizontal is given by $R(range) = \frac{v^2(velocity)}{g(9.8m/s^2)}\sin 2\theta$. If the rocket is launched with an initial velocity of 15 m/s, what angle is needed to reach a range of 20 m?

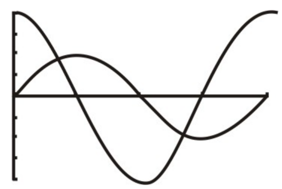
Solution: Plug in 15 m/s for v and 20 m for the range to solve for the angle.

$$\begin{aligned}
20 &= \frac{15^2}{9.8}\sin 2\theta \\
20 &= 22.96\sin 2\theta \\
0.87\bar{1} &= \sin 2\theta \\
\sin^{-1}(0.87\bar{1}) &= 2\theta \\
60.59^\circ, 119.41^\circ &= 2\theta \\
30.3^\circ, 59.7^\circ &= \theta
\end{aligned}$$

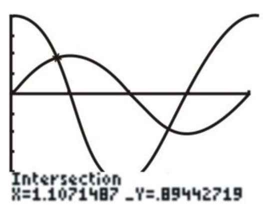
You can also use the TI-83 to solve trigonometric equations. It is sometimes easier than solving the equation algebraically. Just be careful with the directions and make sure your final answer is in the form that is called for. You calculator cannot put radians in terms of π.

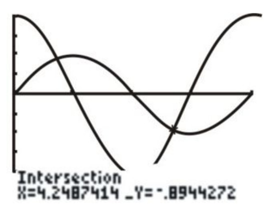
Example 13: Solve $\sin x = 2\cos x$ such that $0 \le x \le 2\pi$ using a graphing calculator.

Solution: In $y =$, graph $y1 = \sin x$ and $y2 = 2\cos x$.



Next, use **CALC** to find the intersection points of the graphs.





Review Questions

1. Express the sum as a product: $\sin 9x + \sin 5x$
2. Express the difference as a product: $\cos 4y - \cos 3y$
3. Verify the identity (using sum-to-product formula): $\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$
4. Express the product as a sum: $\sin(6\theta)\sin(4\theta)$
5. Transform to the form $C\cos(x-D)$
    a. $5\cos x - 5\sin x$
    b. $-15\cos 3x - 8\sin 3x$
6. Solve $\sin 11x - \sin 5x = 0$ for all solutions $0 \le x < 2\pi$.
7. Solve $\cos 4x + \cos 2x = 0$ for all solutions $0 \le x < 2\pi$.
8. Solve $\sin 5x + \sin x = \sin 3x$ for all solutions $0 \le x < 2\pi$.
9. In the study of electronics, the function $f(t) = \sin(200t + \pi) + \sin(200t - \pi)$ is used to analyze frequency. Simplify this function using the sum-to-product formula.
10. Derive a formula for $\tan 4x$.

11. A spring is being moved up and down. Attached to the end of the spring is an object that undergoes a vertical displacement. The displacement is given by the equation $y = 3.50\sin t + 1.20\sin 2t$. Find the first two values of t (in seconds) for which $y = 0$.

Chapter Summary

Here are the identities studied in this chapter:

Quotient & Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

Even & Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x \qquad \sec(-x) = \sec x \qquad \cot(-x) = -\cot x$$

Co-Function Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$
$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Sum and Difference Identities

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double Angle Identities

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$
$$\sin(2\alpha) = 2\sin\alpha\cos\beta$$
$$\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Half Angle Identities

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}=\frac{\sin\alpha}{1+\cos\alpha}$$

Product to Sum & Sum to Product Identities

$$\sin a+\sin b=2\sin\frac{a+b}{2}\cos\frac{a-b}{2} \qquad \sin a\sin b=\frac{1}{2}[\cos(a-b)-\cos(a+b)]$$
$$\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2} \qquad \cos a\cos b=\frac{1}{2}[\cos(a-b)+\cos(a+b)]$$
$$\cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a-b}{2} \qquad \sin a\cos b=\frac{1}{2}[\sin(a+b)+\sin(a-b)]$$
$$\cos a-\cos b=2-2\sin\frac{a+b}{2}\sin\frac{a-b}{2} \qquad \cos a\sin b=\frac{1}{2}[\sin(a+b)-\sin(a-b)]$$

Linear Combination Formula

$A\cos x+B\sin x=C\cos(x-D)$, where $C=\sqrt{A^2+B^2}, \cos D=\frac{A}{C}$ and $\sin D=\frac{B}{C}$

Review Questions

1. Find the sine, cosine, and tangent of an angle with terminal side on $(-8,15)$.
2. If $\sin a=\frac{\sqrt{5}}{3}$ and $\tan a<0$, find $\sec a$.
3. Simplify: $\frac{\cos^4 x-\sin^4 x}{\cos^2 x-\sin^2 x}$.
4. Verify the identity: $\frac{1+\sin x}{\cos x\sin x}=\sec x(\csc x+1)$

For problems 5-8, find all the solutions in the interval $[0,2\pi)$.

5. $\sec\left(x+\frac{\pi}{2}\right)+2=0$
6. $8\sin\left(\frac{x}{2}\right)-8=0$
7. $2\sin^2 x+\sin 2x=0$
8. $3\tan^2 2x=1$
9. Solve the trigonometric equation $1-\sin x=\sqrt{3}\sin x$ over the interval $[0,\pi]$.
10. Solve the trigonometric equation $2\cos 3x-1=0$ over the interval $[0,2\pi]$.
11. Solve the trigonometric equation $2\sec^2 x-\tan^4 x=3$ for all real values of x.

Find the exact value of:

12. $\cos 157.5^\circ$
13. $\sin\frac{13\pi}{12}$
14. Write as a product: $4(\cos 5x+\cos 9x)$
15. Simplify: $\cos(x-y)\cos y-\sin(x-y)\sin y$
16. Simplify: $\sin\left(\frac{4\pi}{3}-x\right)+\cos\left(x+\frac{5\pi}{6}\right)$
17. Derive a formula for $\sin 6x$.
18. If you solve $\cos 2x=2\cos^2 x-1$ for $\cos^2 x$, you would get $\cos^2 x=\frac{1}{2}(\cos 2x+1)$. This new formula is used to reduce powers of cosine by substituting in the right part of the equation for $\cos^2 x$. Try writing $\cos^4 x$ in terms of the first power of cosine.

19. If you solve $\cos 2x = 1 - 2\sin^2 x$ for $\sin^2 x$, you would get $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Similar to the new formula above, this one is used to reduce powers of sine. Try writing $\sin^4 x$ in terms of the first power of cosine.
20. Rewrite in terms of the first power of cosine:
    a. $\sin^2 x \cos^2 2x$
    b. $\tan^4 2x$

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9701 .

CHAPTER 4

Inverse Trigonometric Functions

Chapter Outline

4.1 BASIC INVERSE TRIGONOMETRIC FUNCTIONS

4.2 GRAPHING INVERSE TRIGONOMETRIC FUNCTIONS

4.3 INVERSE TRIGONOMETRIC PROPERTIES

4.4 APPLICATIONS & MODELS

4.1 Basic Inverse Trigonometric Functions

Introduction

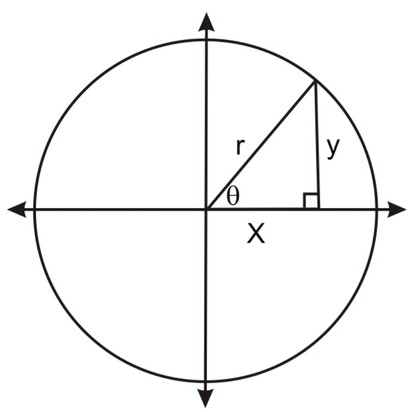
Recall that an inverse function is a reflection of the function over the line $y = x$. In order to find the inverse of a function, you must switch the x and y values and then solve for y. A function has an inverse if and only if it has exactly one output for every input and exactly one input for every output. All of the trig functions fit these criteria over a specific range. In this chapter, we will explore inverse trig functions and equations.

Learning Objectives

- Understand and evaluate inverse trigonometric functions.
- Extend the inverse trigonometric functions to include the $\csc^{-1}$, $\sec^{-1}$ and $\cot^{-1}$ functions.
- Apply inverse trigonometric functions to the critical values on the unit circle.

Defining the Inverse of the Trigonometric Ratios

Recall from Chapter 1, the ratios of the six trig functions and their inverses, with regard to the unit circle.

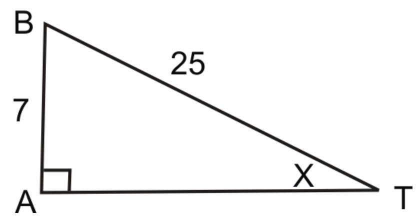


$$\sin\theta = \frac{y}{r} \rightarrow \sin^{-1}\frac{y}{r} = \theta \qquad \cos\theta = \frac{x}{r} \rightarrow \cos^{-1}\frac{x}{r} = \theta$$
$$\tan\theta = \frac{y}{x} \rightarrow \tan^{-1}\frac{y}{x} = \theta \qquad \cot\theta = \frac{x}{y} \rightarrow \cot^{-1}\frac{x}{y} = \theta$$
$$\csc\theta = \frac{r}{y} \rightarrow \csc^{-1}\frac{r}{y} = \theta \qquad \sec\theta = \frac{r}{x} \rightarrow \sec^{-1}\frac{r}{x} = \theta$$

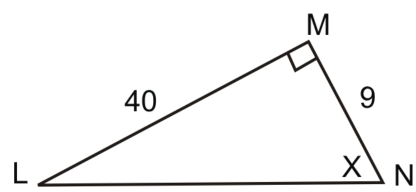
These ratios can be used to find any θ in standard position or in a triangle.

Example 1: Find the measure of the angles below.

a.



b.



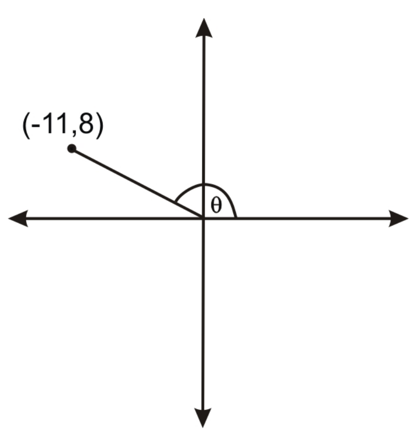
Solution: For part a, you need to use the sine function and part b utilizes the tangent function. Because both problems require you to solve for an angle, the inverse of each must be used.

a. $\sin x = \frac{7}{25} \rightarrow \sin^{-1} \frac{7}{25} = x \rightarrow x = 16.26^\circ$

b. $\tan x = \frac{40}{9} \rightarrow \tan^{-1} \frac{40}{9} = x \rightarrow x = 77.32^\circ$

The trigonometric value $\tan\theta = \frac{40}{9}$ of the angle is known, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. (Directions for how to find inverse function values in the graphing calculator are in Chapter 1). The inverse of the tangent function is read "tangent inverse" and is also called the arctangent relation. The inverse of the cosine function is read "cosine inverse" and is also called the arccosine relation. The inverse of the sine function is read "sine inverse" and is also called the arcsine relation.

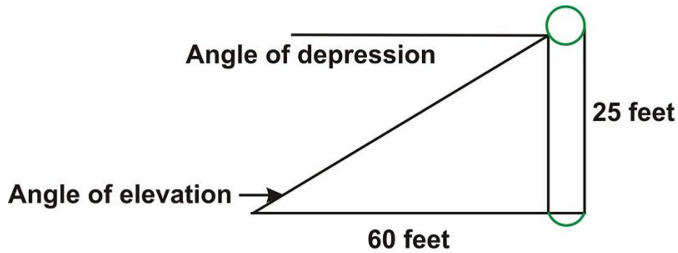
Example 2: Find the angle, θ, in standard position.



Solution: The $\tan\theta = \frac{y}{x}$ or, in this case, $\tan\theta = \frac{8}{-11}$. Using the inverse tangent, you get $\tan^{-1} -\frac{8}{11} = -36.03^\circ$. This means that the reference angle is 36.03°. This value of 36.03° is the angle you also see if you move counterclockwise from the -x axis. To find the corresponding angle in the second quadrant (which is the same as though you started at the +x axis and moved counterclockwise), subtract 36.03° from 180°, yielding 143.97°.

Recall that inverse trigonometric functions are also used to find the angle of depression or elevation.

Example 3: A new outdoor skating rink has just been installed outside a local community center. A light is mounted on a pole 25 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the skating rink. If the end of the rink is 60 feet from the pole, at what angle of depression should the light be installed?



θ

Solution: In this diagram, the angle of depression, which is located outside of the triangle, is not known. Recall, the angle of depression equals the angle of elevation. For the angle of elevation, the pole where the light is located is the opposite and is 25 feet high. The length of the rink is the adjacent side and is 60 feet in length. To calculate the measure of the angle of elevation the trigonometric ratio for tangent can be applied.

$$\tan\theta = \frac{25}{60}$$
$$\tan\theta = 0.4166$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(0.4166)$$
$$\theta = 22.6^\circ$$

The angle of depression at which the light must be placed to light the rink is 22.6°

Exact Values for Inverse Sine, Cosine, and Tangent

Recall the unit circle and the critical values. With the inverse trigonometric functions, you can find the angle value (in either radians or degrees) when given the ratio and function. Make sure that you find all solutions within the given interval.

Example 4: Find the exact value of each expression without a calculator, in $[0, 2\pi)$.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\tan^{-1}\sqrt{3}$

Solution: These are all values from the special right triangles and the unit circle.

a. Recall that $-\frac{\sqrt{3}}{2}$ is from the $30-60-90$ triangle. The reference angle for sin and $\frac{\sqrt{3}}{2}$ would be 60°. Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

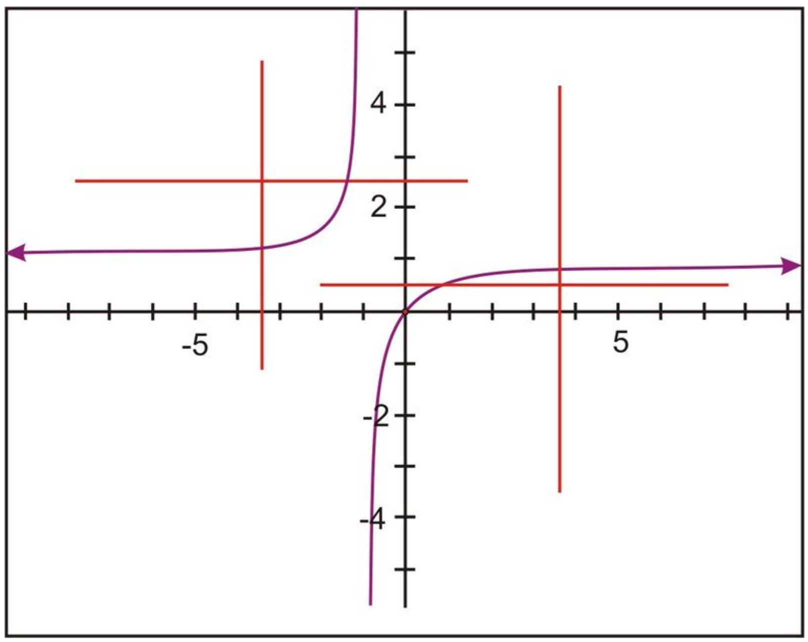
b. $-\frac{\sqrt{2}}{2}$ is from an isosceles right triangle. The reference angle is then 45°. Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

c. $\sqrt{3}$ is also from a $30-60-90$ triangle. Tangent is $\sqrt{3}$ for the reference angle 60°. Tangent is positive in the first and third quadrants, so the answer would be $\frac{\pi}{3}$ or $\frac{4\pi}{3}$.

Notice how each one of these examples yield two answers. This poses a problem when finding a singular inverse for each of the trig functions. Therefore, we need to restrict the domain in which the inverses can be found, which will be addressed in the next section. Unless otherwise stated, all angles are in radians.

Finding Inverses Algebraically

In the Prerequisite Chapter, you learned that each function has an inverse relation and that this inverse relation is a function only if the original function is one-to-one. A function is one-to-one when its graph passes both the vertical and the horizontal line test. This means that every vertical and horizontal line will intersect the graph in exactly one place.



This is the graph of $f(x) = \frac{x}{x+1}$. The graph suggests that f is one-to-one because it passes both the vertical and the horizontal line tests. To find the inverse of f, switch **the** x **and** y **and solve for** y.

First, switch x and y.

$$x = \frac{y}{y+1}$$

Next, multiply both sides by $(y+1)$.

$$\begin{aligned}(y+1)x &= \frac{y}{y+1}(y+1)\\ x(y+1) &= y\end{aligned}$$

Then, apply the distributive property and put all the y terms on one side so you can pull out the y.

$$\begin{aligned} xy + x &= y \\ xy - y &= -x \\ y(x-1) &= -x \end{aligned}$$

Divide by $(x-1)$ to get y by itself.

$$y = \frac{-x}{x-1}$$

Finally, multiply the right side by $\frac{-1}{-1}$.

$$y = \frac{x}{1-x}$$

Therefore the inverse of f is $f^{-1}(x) = \frac{x}{1-x}$.

The symbol f^{-1} is read "f inverse" and is not the reciprocal of f.

Example 5: Find the inverse of $f(x) = \frac{1}{x-5}$ algebraically.

Solution: To find the inverse algebraically, switch $f(x)$ to y and then switch x and y.

$$\begin{aligned} y &= \frac{1}{x-5} \\ x &= \frac{1}{y-5} \\ x(y-5) &= 1 \\ xy - 5x &= 1 \\ xy &= 5x + 1 \\ y &= \frac{5x+1}{x} \end{aligned}$$

Example 6: Find the inverse of $f(x) = 5\sin^{-1}\left(\frac{2}{x-3}\right)$

Solution:

a.

$$\begin{aligned}
f(x) &= 5\sin^{-1}\left(\frac{2}{x-3}\right) \\
x &= 5\sin^{-1}\left(\frac{2}{y-3}\right) \\
\frac{x}{5} &= \sin^{-1}\left(\frac{2}{y-3}\right) \\
\sin\frac{x}{5} &= \left(\frac{2}{y-3}\right) \\
(y-3)\sin\frac{x}{5} &= 2 \\
(y-3) &= \frac{2}{\sin\frac{x}{5}} \\
y &= \frac{2}{\sin\frac{x}{5}} + 3
\end{aligned}$$

Example 7: Find the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x+4)$

Solution:

$$\begin{aligned}
x &= 4\tan^{-1}(3y+4) \\
\frac{x}{4} &= \tan^{-1}(3y+4) \\
\tan\frac{x}{4} &= 3y+4 \\
\tan\frac{x}{4} - 4 &= 3y \\
\frac{\tan\frac{x}{4} - 4}{3} &= y \\
f^{-1}(x) &= \frac{\tan\frac{x}{4} - 4}{3}
\end{aligned}$$

Points to Consider

- What is the difference between an inverse and a reciprocal?
- Considering that most graphing calculators do not have $\csc, \sec$ or $\cot$ buttons, how would you find the inverse of each of these?
- Besides algebraically, is there another way to find the inverse?

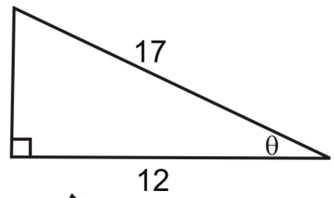
Review Questions

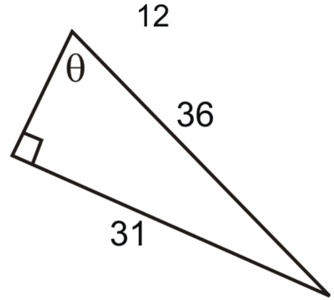
1. Use the special triangles or the unit circle to evaluate each of the following:
   a. $\cos 120^\circ$
   b. $\csc \frac{3\pi}{4}$
   c. $\tan \frac{5\pi}{3}$

2. Find the exact value of each inverse function, without a calculator in $[0, 2\pi)$:

   a. $\cos^{-1}(0)$
   b. $\tan^{-1}\left(-\sqrt{3}\right)$
   c. $\sin^{-1}\left(-\frac{1}{2}\right)$

Find the value of the missing angle.

3. 

4. 

5. What is the value of the angle with its terminal side passing through (-14, -23)?
6. A 9-foot ladder is leaning against a wall. If the foot of the ladder is 4 feet from the base of the wall, what angle does the ladder make with the floor?

Find the inverse of the following functions.

7. $f(x) = 2x^3 - 5$
8. $y = \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x - 5\right)$
9. $g(x) = 2\sin(x-1) + 4$
10. $h(x) = 5 - \cos^{-1}(2x+3)$

4.2 Graphing Inverse Trigonometric Functions

Learning Objectives

- Understand the meaning of restricted domain as it applies to the inverses of the six trigonometric functions.
- Apply the domain, range and quadrants of the six inverse trigonometric functions to evaluate expressions.

Finding the Inverse by Mapping

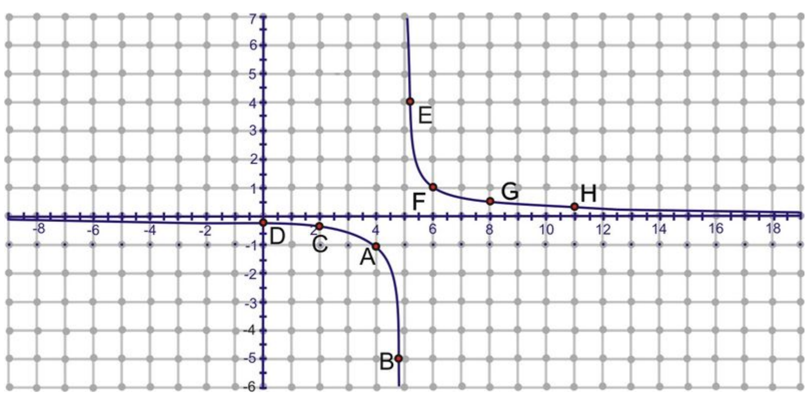
Determining an inverse function algebraically can be both involved and difficult, so it is useful to know how to map f to f^{-1}. The graph of f can be used to produce the graph of f^{-1} by applying the inverse reflection principle:

The points (a,b) and (b,a) in the coordinate plane are symmetric with respect to the line $y = x$.

The points (a,b) and (b,a) are reflections of each other across the line $y = x$.

Example 1: Find the inverse of $f(x) = \frac{1}{x-5}$ mapping.

Solution: From the last section, we know that the inverse of this function is $y = \frac{5x+1}{x}$. To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot. Note: The coordinates of some of the points are rounded.



A: (4, -1)

B: (4.8, -5)

C: (2, -0.3)

D: (0, -0.2)

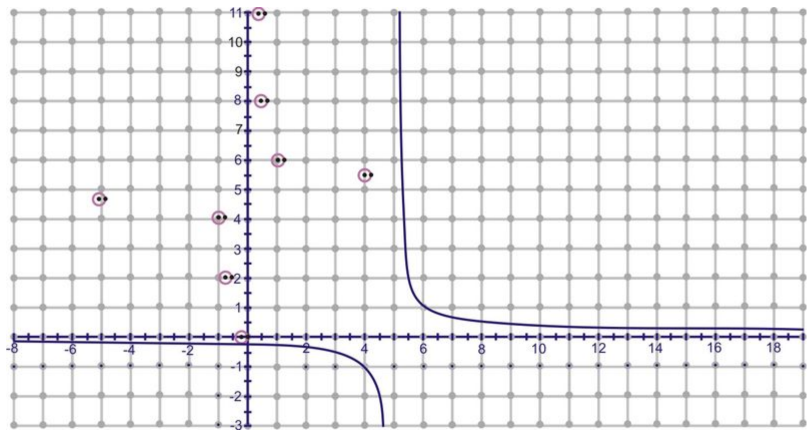
E: (5.3, 3.3)

F: (6, 1)

G: (8, 0.3)

H: (11, 0.2)

Now, take these eight points, switch the x and y and plot (y,x). Connect them to make the inverse function.



$A^{-1}:\ (-1,4)$

$B^{-1}:\ (-5,4.8)$

$C^{-1}:\ (-0.3,2)$

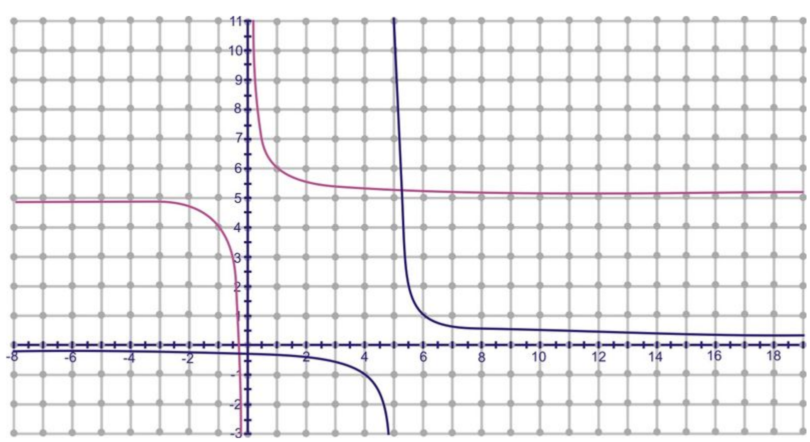
$D^{-1}:\ (-0.2,0)$

$E^{-1}:\ (3.3,5.3)$

$F^{-1}:\ (1,6)$

$G^{-1}:\ (0.3,8)$

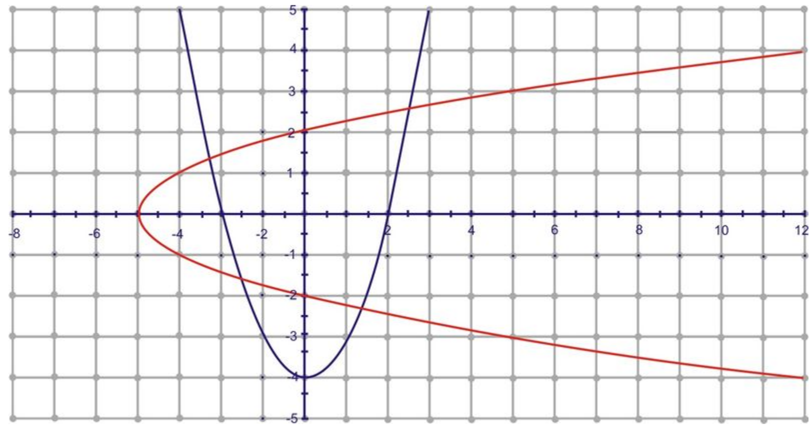
$H^{-1}:\ (0.2,11)$



Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

Example 2: Find the inverse of $f(x) = x^2 - 4$.

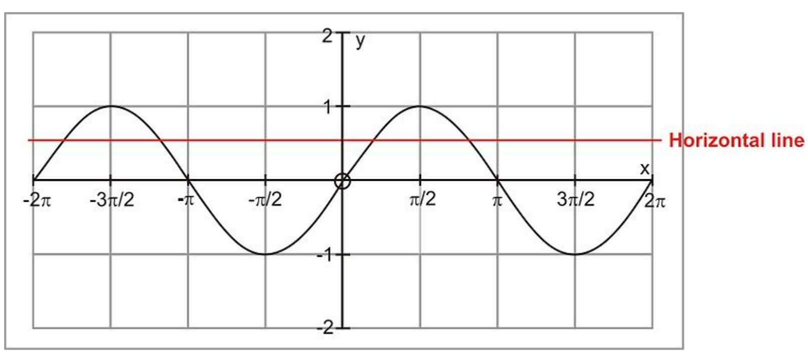
Solution: Let's use the graphic approach for this one. The function is graphed in blue and its inverse is red.



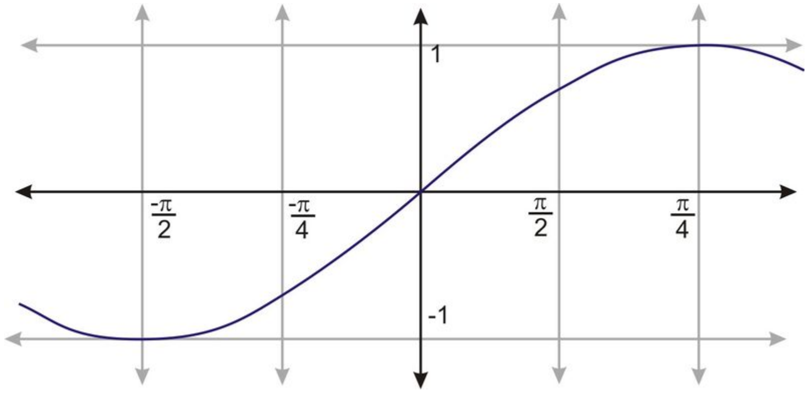
Clearly, the inverse relation is not a function because it does not pass the vertical line test. This is because all parabolas fail the horizontal line test. To "make" the inverse a function, we restrict the domain of the original function. For parabolas, this is fairly simple. To find the inverse of this function algebraically, we get $f^{-1}(x) = \sqrt{x+4}$. Technically, however, the inverse is $\pm\sqrt{x+4}$ because the square root of any number could be positive or negative. So, the inverse of $f(x) = x^2 - 4$ is both parts of the square root equation, $\sqrt{x+4}$ and $-\sqrt{x+4}$. $\sqrt{x+4}$ will yield the top portion of the horizontal parabola and $-\sqrt{x+4}$ will yield the bottom half. Be careful, because if you just graph $f^{-1}(x) = \sqrt{x+4}$ in your graphing calculator, it will only graph the top portion of the inverse.

This technique of sectioning the inverse is applied to finding the inverse of trigonometric functions because it is periodic.

Finding the Inverse of the Trigonometric Functions



In order to consider the inverse of this function, we need to restrict the domain so that we have a section of the graph that is one-to-one. If the domain of f is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ a new function $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. is defined. This new function is one-to-one and takes on all the values that the function $f(x) = \sin x$ takes on. Since the restricted domain is smaller, $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ takes on all values once and only once.

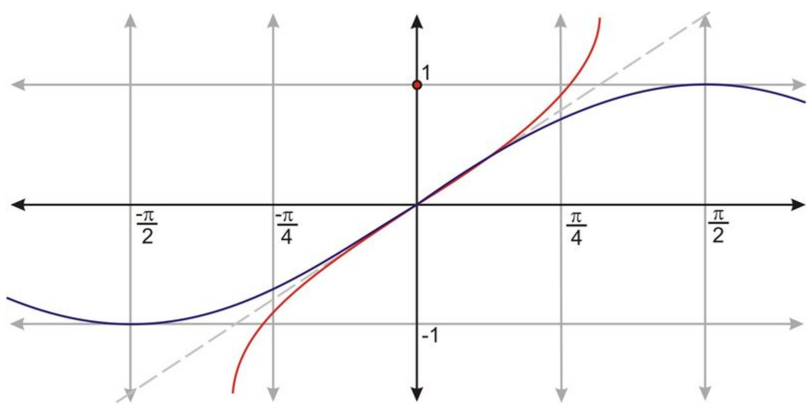


In the previous lesson the inverse of $f(x)$ was represented by the symbol $f^{-1}(x)$, and $y = f^{-1}(x) \Leftrightarrow f(y) = x$. The inverse of $\sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ will be written as $\sin^{-1} x$. or $\arcsin x$.

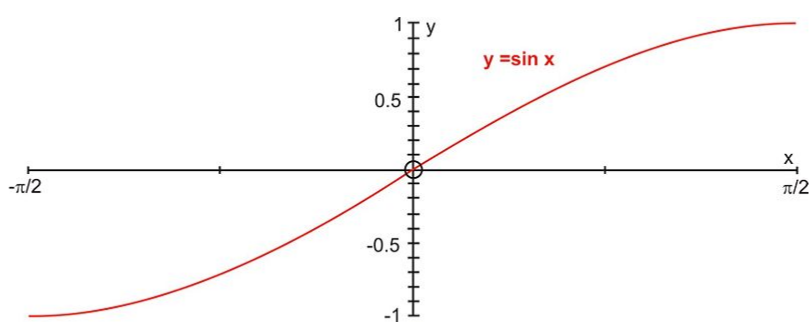
$$\left\{ \begin{matrix} y = \sin^{-1} x \\ or \\ y = \arcsin x \end{matrix} \right\} \Leftrightarrow \sin y = x$$

In this lesson we will use both $\sin^{-1} x$ and $\arcsin x$ and both are read as "the inverse sine of x" or "the number between $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ whose sine is x."

The graph of $y = \sin^{-1} x$ is obtained by applying the inverse reflection principle and reflecting the graph of $y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ in the line $y = x$. The domain of $y = \sin x$ becomes the range of $y = \sin^{-1} x$, and hence the range of $y = \sin x$ becomes the domain of $y = \sin^{-1} x$.

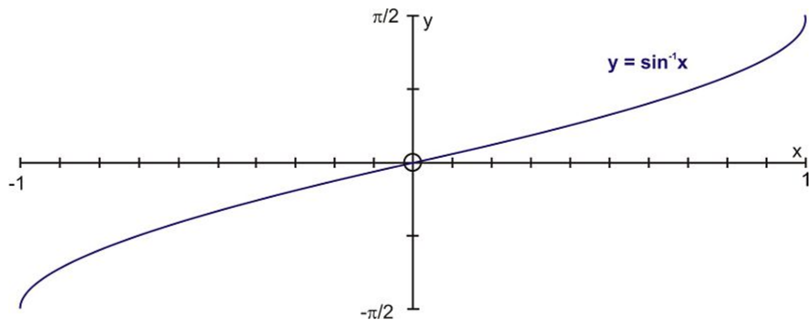


Another way to view these graphs is to construct them on separate grids. If the domain of $y = \sin x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse sine function $y = \sin^{-1} x$ is the inverse of the restricted section of the sine function.



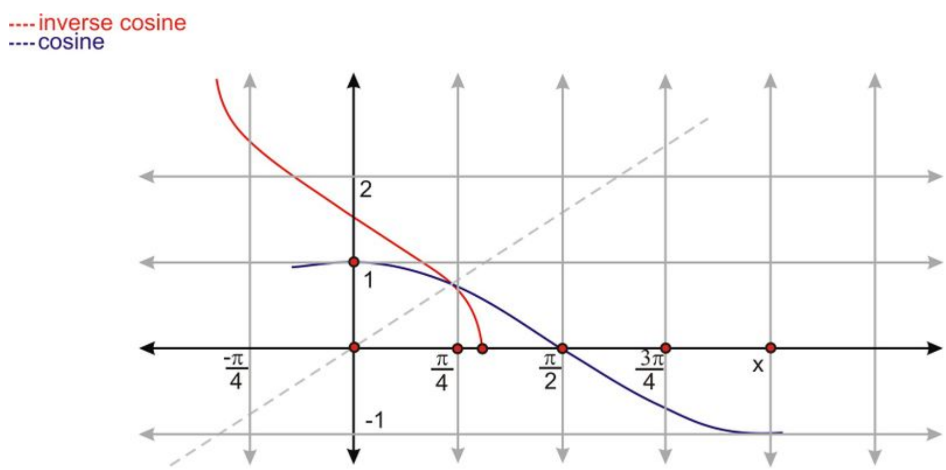
The domain of $y = \sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and the range is** [-1, 1].

The restriction of $y = \sin x$ is a one-to-one function and it has an inverse that is shown below.

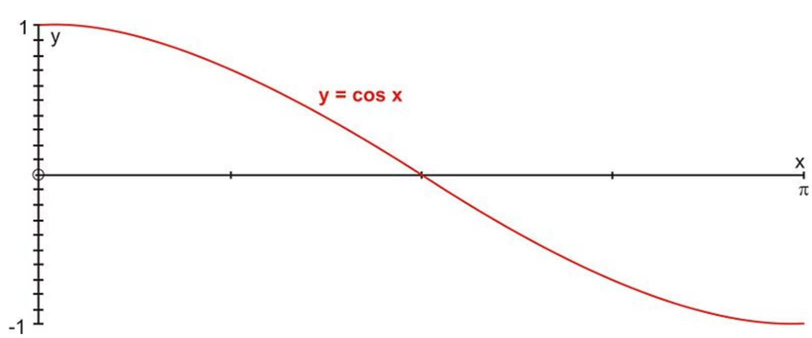


The domain of $y = \sin^{-1}$ **is** [-1, 1] **and the range is** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse functions for cosine and tangent are defined by following the same process as was applied for the inverse sine function. However, in order to create one-to-one functions, different intervals are used. The cosine function is restricted to the interval $0 \le x \le \pi$ and the new function becomes $y = \cos x, 0 \le x \le \pi$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$ The result is the graph of $y = \cos^{-1} x$ (also expressed as $y = \arccos x$).

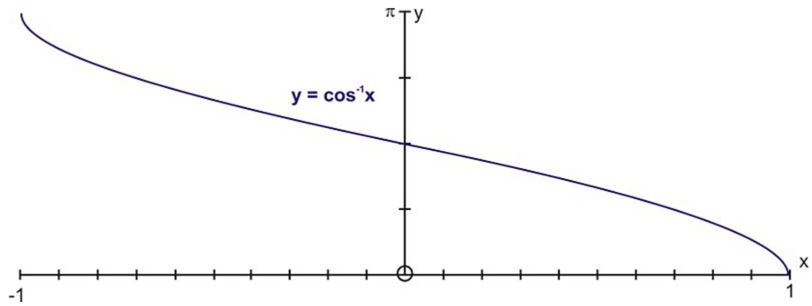


Again, construct these graphs on separate grids to determine the domain and range. If the domain of $y = \cos x$ is restricted to the interval $[0, \pi]$, the result is a restricted one-to one function. The inverse cosine function $y = \cos^{-1} x$ is the inverse of the restricted section of the cosine function.



The domain of $y = \cos x$ is $[0, \pi]$ **and the range is** [-1, 1].

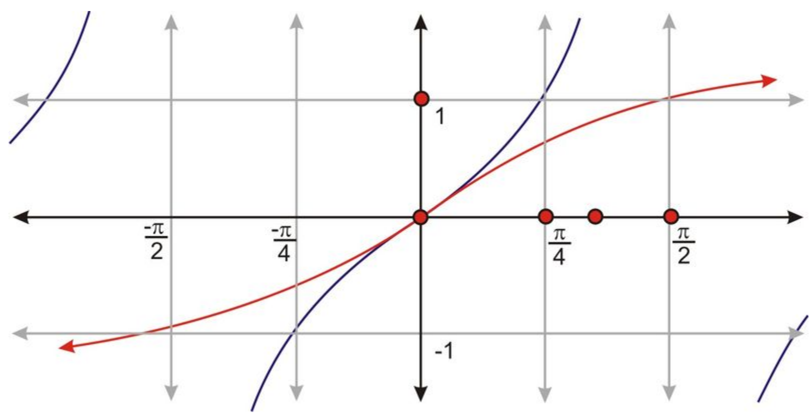
The restriction of $y = \cos x$ is a one-to-one function and it has an inverse that is shown below.



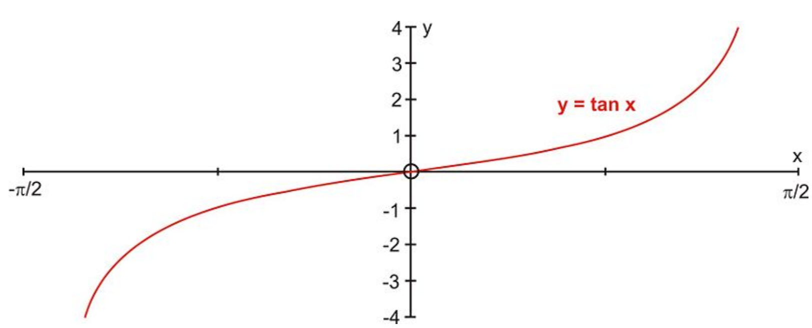
The statements $y = \cos x$ **and** $x = \cos y$ **are equivalent for** $y-$ **values in the restricted domain** $[0, \pi]$ **and** $x-$ **values between** -1 **and** 1.

The domain of $y = \cos^{-1} x$ is [-1, 1] **and the range is** $[0, \pi]$.

The tangent function is restricted to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the new function becomes $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$. The result is the graph of $y = \tan^{-1} x$ (also expressed as $y = \arctan x$).

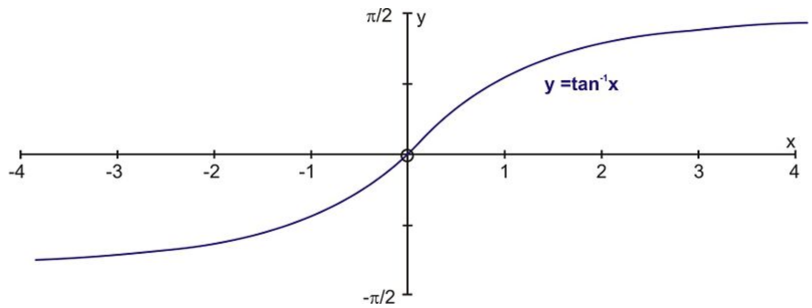


Graphing the two functions separately will help us to determine the domain and range. If the domain of $y = \tan x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse tangent function $y = \tan^{-1} x$ is the inverse of the restricted section of the tangent function.



The domain of $y = \tan x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and the range is** $[-\infty, \infty]$.

The restriction of $y = \tan x$ is a one-to-one function and it has an inverse that is shown below.



The statements $y = \tan x$ **and** $x = \tan y$ **are equivalent for** $y-$ **values in the restricted domain** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and** $x-$ **values between** -4 **and** +4.

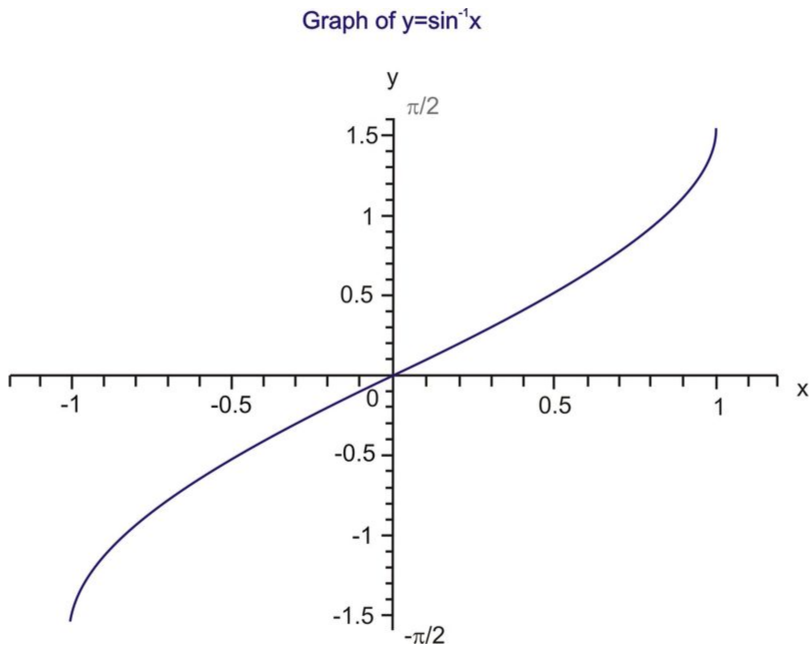
The domain of $y = \tan^{-1} x$ is $[-\infty, \infty]$ **and the range is** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

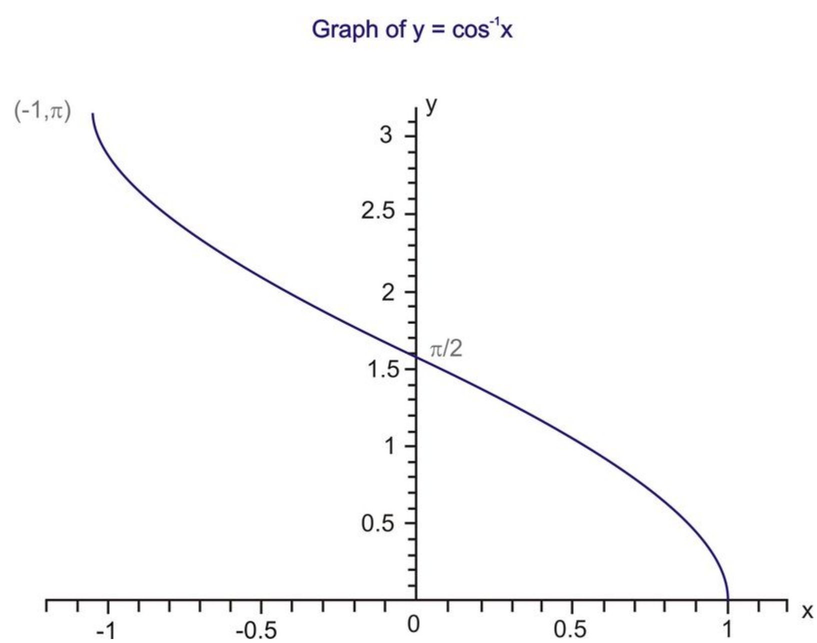
The above information can be readily used to evaluate inverse trigonometric functions without the use of a calculator. These calculations are done by applying the restricted domain functions to the unit circle. To summarize:

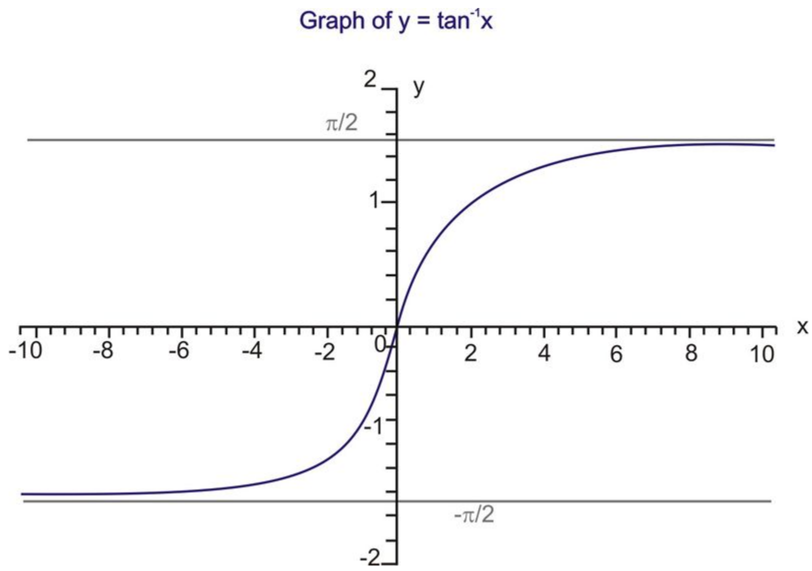
TABLE 4.1:

Restricted Domain Function	Inverse Trigonometric Function	Domain	Range	Quadrants
$y = \sin x$		$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	[-1, 1]	1 AND 4
	$y = \arcsin x$	[-1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
	$y = \sin^{-1} x$			
$y = \cos x$		$[0, \pi]$	[-1, 1]	1 AND 2
	$y = \arccos x$	[-1, 1]	$[0, \pi]$	
	$y = \cos^{-1} x$			
$y = \tan x$		$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$	1 AND 4
	$y = \arctan x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
	$y = \tan^{-1} x$			

Now that the three trigonometric functions and their inverses have been summarized, let's take a look at the graphs of these inverse trigonometric functions.


Graph of y=sin-1x
y
π/2
1.5
1
0.5
-1
-0.5
0
0.5
1
x
-0.5
-1
-1.5
-π/2


Graph of y = cos-1x
(-1,π)
y
3
2.5
2
π/2
1.5
1
0.5
-1
-0.5
0
0.5
1
x



Points to Consider

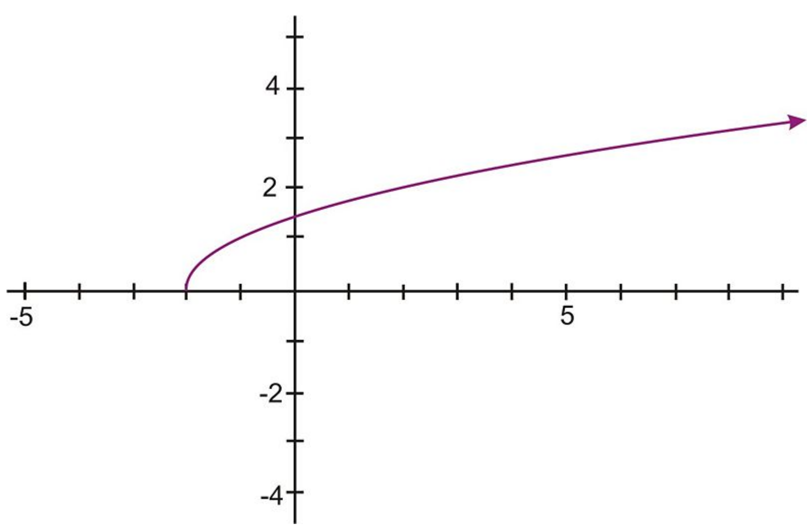
- What are the restricted domains for the inverse relations of the trigonometric functions?
- Can the values of the special angles of the unit circle be applied to the inverse trigonometric functions?

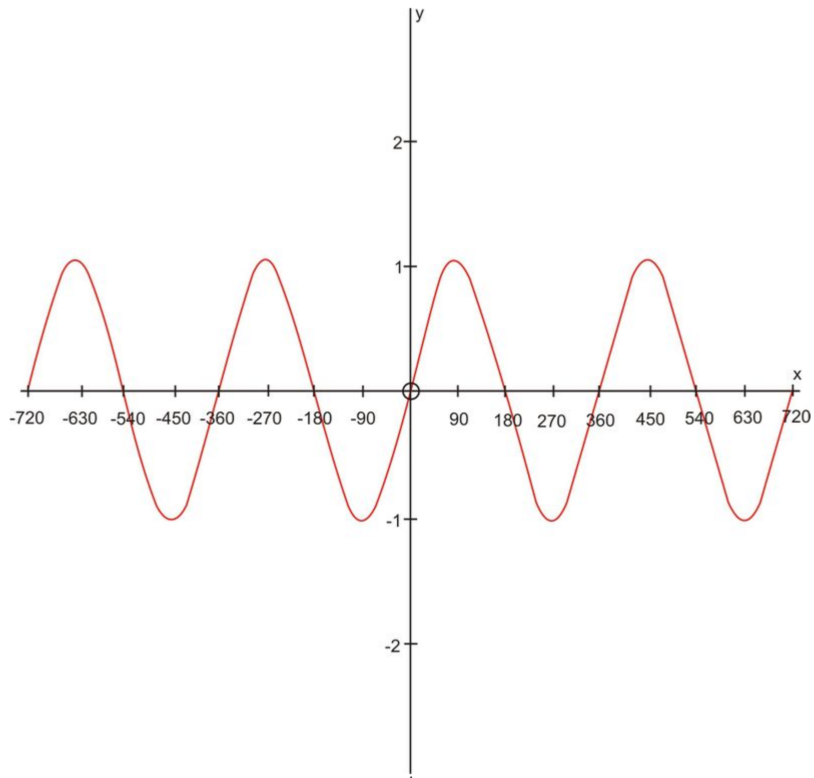
Review Questions

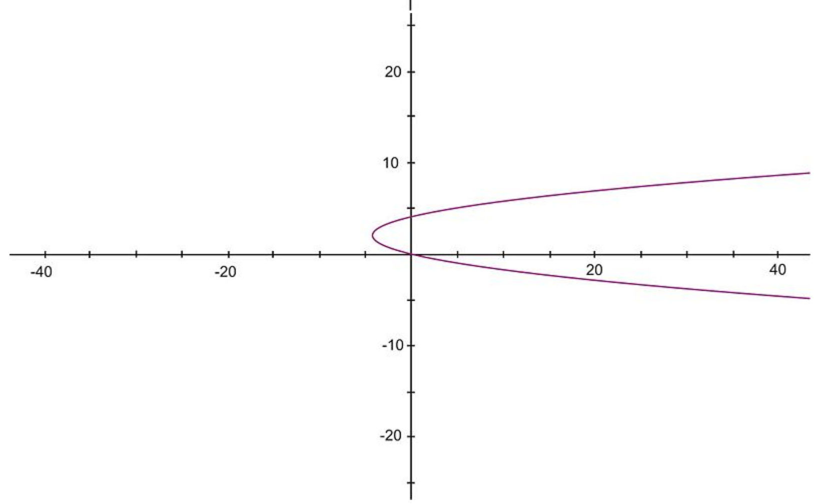
Study each of the following graphs and answer these questions:

(a) Is the graphed relation a function?

(b) Does the relation have an inverse that is a function?

1. 

2. 

3. 

Find the inverse of the following functions using the mapping principle.

4. $f(x) = x^2 + 2x - 15$
5. $y = 1 + 2\sin x$
6. Sketch a graph of $y = \frac{1}{2}\cos^{-1}(3x+1)$. Sketch $y = \cos^{-1} x$ on the same set of axes and compare how the two differ.
7. Sketch a graph of $y = 3 - \tan^{-1}(x-2)$. Sketch $y = \tan^{-1} x$ on the same set of axes and compare how the two differ.
8. Graph $y = 2\sin^{-1}(2x)$
9. Graph $y = 4 + \cos^{-1}\frac{1}{3}x$
10. Remember that sine and cosine are out of phase with each other, $\sin x = \cos\left(x - \frac{\pi}{2}\right)$. Find the inverse of $y = \cos\left(x - \frac{\pi}{2}\right)$. Is the inverse of $y = \cos\left(x - \frac{\pi}{2}\right)$ the same as $y = \sin^{-1} x$? Why or why not?

4.3 Inverse Trigonometric Properties

Learning Objectives

- Relate the concept of inverse functions to trigonometric functions.
- Reduce the composite function to an algebraic expression involving no trigonometric functions.
- Use the inverse reciprocal properties.
- Compose each of the six basic trigonometric functions with $\tan^{-1} x$.

Composing Trig Functions and their Inverses

In the Prerequisite Chapter, you learned that for a function $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined. If this property is applied to the trigonometric functions, the following equations will be true whenever they are defined:

$$\sin(\sin^{-1}(x)) = x \qquad \cos(\cos^{-1}(x)) = x \qquad \tan(\tan^{-1}(x)) = x$$

As well, you learned that $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined. If this property is applied to the trigonometric functions, the following equations that deal with finding an inverse trig. function of a trig. function, will only be true for values of x within the restricted domains.

$$\sin^{-1}(\sin(x)) = x \qquad \cos^{-1}(\cos(x)) = x \qquad \tan^{-1}(\tan(x)) = x$$

These equations are better known as composite functions and are composed of one trigonometric function in conjunction with another different trigonometric function. The composite functions will become algebraic functions and will not display any trigonometry. Let's investigate this phenomenon.

Example 1: Find $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$.

Solution: We know that $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, within the defined restricted domain. Then, we need to find $\sin\frac{\pi}{4}$, which is $\frac{\sqrt{2}}{2}$. So, the above properties allow for a short cut. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$, think of it like the sine and sine inverse cancel each other out and all that is left is the $\frac{\sqrt{2}}{2}$.

Composing Trigonometric Functions

Besides composing trig functions with their own inverses, you can also compose any trig functions with any inverse. When solving these types of problems, start with the function that is composed inside of the other and work your way out. Use the following examples as a guideline.

Example 2: Without using technology, find the exact value of each of the following:

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

c. $\cos(\tan^{-1}(-1))$

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

Solution: For all of these types of problems, the answer is restricted to the inverse functions' ranges.

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$: First find $\tan^{-1}\sqrt{3}$, which is $\frac{\pi}{3}$. Then find $\cos\frac{\pi}{3}$. Your final answer is $\frac{1}{2}$. Therefore, $\cos\left(\tan^{-1}\sqrt{3}\right) = \frac{1}{2}$.

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

c. $\cos(\tan^{-1}(-1)) = \cos^{-1}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Inverse Reciprocal Functions

We already know that the cosecant function is the reciprocal of the sine function. This will be used to derive the reciprocal of the inverse sine function.

$$\begin{aligned} y &= \sin^{-1} x \\ x &= \sin y \\ \frac{1}{x} &= \csc y \\ \csc^{-1}\frac{1}{x} &= y \\ \csc^{-1}\frac{1}{x} &= \sin^{-1} x \end{aligned}$$

Because cosecant and secant are inverses, $\sin^{-1}\frac{1}{x} = \csc^{-1} x$ is also true.

The inverse reciprocal identity for cosine and secant can be proven by using the same process as above. However, remember that these inverse functions are defined by using restricted domains and the reciprocals of these inverses must be defined with the intervals of domain and range on which the definitions are valid.

$$\sec^{-1}\frac{1}{x} = \cos^{-1} x \leftrightarrow \cos^{-1}\frac{1}{x} = \sec^{-1} x$$

Tangent and cotangent have a slightly different relationship. Recall that the graph of cotangent differs from tangent by a reflection over the $y-$axis and a shift of $\frac{\pi}{2}$. As an equation, this can be written as $\cot x = -\tan\left(x - \frac{\pi}{2}\right)$. Taking the inverse of this function will show the inverse reciprocal relationship between arccotangent and arctangent.

$$\begin{aligned} y &= \cot^{-1} x \\ y &= -\tan^{-1}\left(x - \frac{\pi}{2}\right) \\ x &= -\tan\left(y - \frac{\pi}{2}\right) \\ -x &= \tan\left(y - \frac{\pi}{2}\right) \\ \tan^{-1}(-x) &= y - \frac{\pi}{2} \\ \frac{\pi}{2} + \tan^{-1}(-x) &= y \\ \frac{\pi}{2} - \tan^{-1} x &= y \end{aligned}$$

Remember that tangent is an odd function, so that $\tan(-x) = -\tan(x)$. Because tangent is odd, its inverse is also odd. So, this tells us that $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ and $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$. You will determine the domain and range of all of these functions when you graph them in the exercises for this section. To graph arcsecant, arccosecant, and arccotangent in your calculator you will use these conversion identities: $\sec^{-1} x = \cos^{-1} \frac{1}{x}, \csc^{-1} x = \sin^{-1} \frac{1}{x}, \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$. Note: It is also true that $\cot^{-1} x = \tan^{-1} \frac{1}{x}$.

Now, let's apply these identities to some problems that will give us an insight into how they work.

Example 3: Evaluate $\sec^{-1} \sqrt{2}$

Solution: Use the inverse reciprocal property. $\sec^{-1} x = \cos^{-1} \frac{1}{x} \rightarrow \sec^{-1} \sqrt{2} = \cos^{-1} \frac{1}{\sqrt{2}}$. Recall that $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, $\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$, and we know that $\cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Therefore, $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$.

Example 4: Find the exact value of each expression within the restricted domain, without a calculator.

a. $\sec^{-1} \sqrt{2}$

b. $\cot^{-1}\left(-\sqrt{3}\right)$

c. $\csc^{-1} \frac{2\sqrt{3}}{3}$

Solution: For each of these problems, first find the reciprocal and then determine the angle from that.

a. $\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$ From the unit circle, we know that the answer is $\frac{\pi}{4}$.

b. $\cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\sqrt{3}\right)$ From the unit circle, the answer is $\frac{5\pi}{6}$.

c. $\csc^{-1} \frac{2\sqrt{3}}{3} = \sin^{-1} \frac{\sqrt{3}}{2}$ Within our interval, there are is one answer, $\frac{\pi}{3}$.

Example 5: Using technology, find the value in radian measure, of each of the following:

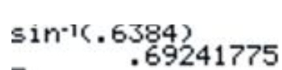
a. $\arcsin 0.6384$

b. $\arccos(-0.8126)$

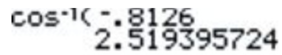
c. $\arctan(-1.9249)$

Solution:

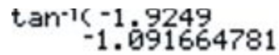
a.



b.



c.



Make sure that your calculator's MODE is RAD (radians).

Composing Inverse Reciprocal Trig Functions

In this subsection, we will combine what was learned in the previous two sections. Here are a few examples:

Example 6: Without a calculator, find $\cos\left(\cot^{-1}\sqrt{3}\right)$.

Solution: First, find $\cot^{-1}\sqrt{3}$, which is also $\tan^{-1}\frac{\sqrt{3}}{3}$. This is $\frac{\pi}{6}$. Now, find $\cos\frac{\pi}{6}$, which is $\frac{\sqrt{3}}{2}$. So, our answer is $\frac{\sqrt{3}}{2}$.

Example 7: Without a calculator, find $\sec^{-1}\left(\csc\frac{\pi}{3}\right)$.

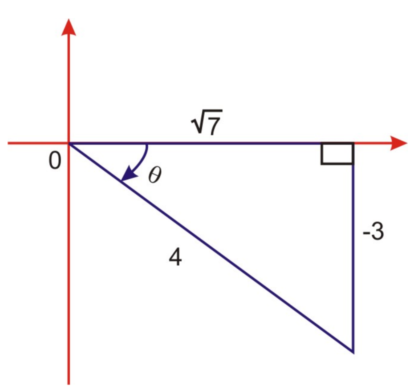
Solution: First, $\csc\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Then $\sec^{-1}\frac{2\sqrt{3}}{3} = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

Example 8: Evaluate $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

Solution: Even though this problem is not a critical value, it can still be done without a calculator. Recall that sine is the opposite side over the hypotenuse of a triangle. So, 3 is the opposite side and 5 is the hypotenuse. This is a Pythagorean Triple, and thus, the adjacent side is 4. To continue, let $\theta = \sin^{-1}\frac{3}{5}$ or $\sin\theta = \frac{3}{5}$, which means θ is in the Quadrant 1 (from our restricted domain, it cannot also be in Quadrant II). Substituting in θ we get $\cos\left(\sin^{-1}\frac{3}{5}\right) = \cos\theta$ and $\cos\theta = \frac{4}{5}$.

Example 9: Evaluate $\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$

Solution: Even though $\frac{3}{4}$ does not represent two lengths from a Pythagorean Triple, you can still use the Pythagorean Theorem to find the missing side. $(-3)^2 + b^2 = 4^2$, so $b = \sqrt{16-9} = \sqrt{7}$. From the restricted domain, sine inverse is negative in the 4^{th} Quadrant. To illustrate:

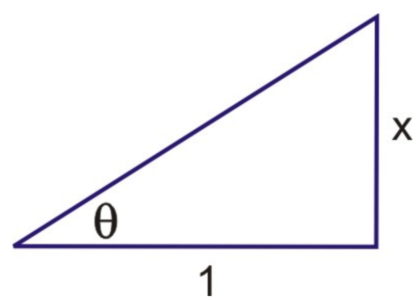


Let

$$\theta = \sin^{-1}\left(-\frac{3}{4}\right)$$
$$\sin\theta = -\frac{3}{4}$$
$$\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right) = \tan\theta$$
$$\tan\theta = \frac{-3}{\sqrt{7}} \text{ or } \frac{-3\sqrt{7}}{7}$$

Trigonometry in Terms of Algebra

All of the trigonometric functions can be rewritten in terms of only x, when using one of the inverse trigonometric functions. Starting with tangent, we draw a triangle where the opposite side (from θ) is defined as x and the adjacent side is 1. The hypotenuse, from the Pythagorean Theorem would be $\sqrt{x^2+1}$. Substituting $\tan^{-1} x$ for θ, we get:



$$\tan\theta = \frac{x}{1}$$
$$\tan\theta = x \qquad hypotenuse = \sqrt{x^2+1}$$
$$\theta = \tan^{-1} x$$

$$\sin(\tan^{-1} x) = \sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad \csc(\tan^{-1} x) = \csc\theta = \frac{\sqrt{x^2+1}}{x}$$
$$\cos(\tan^{-1} x) = \cos\theta = \frac{1}{\sqrt{x^2+1}} \qquad \sec(\tan^{-1} x) = \sec\theta = \sqrt{x^2+1}$$
$$\tan(\tan^{-1} x) = \tan\theta = x \qquad \cot(\tan^{-1} x) = \cot\theta = \frac{1}{x}$$

Example 10: Find $\sin(\tan^{-1} 3x)$.

Solution: Instead of using x in the ratios above, use $3x$.

$$\sin(\tan^{-1} 3x) = \sin\theta = \frac{3x}{\sqrt{(3x)^2+1}} = \frac{3x}{\sqrt{9x^2+1}}$$

Example 11: Find $\sec^2(\tan^{-1}x)$.

Solution: This problem might be better written as $[\sec(\tan^{-1}x)]^2$. Therefore, all you need to do is square the ratio above.

$$[\sec(\tan^{-1}x)]^2 = \left(\sqrt{x^2+1}\right)^2 = x^2+1$$

You can also write the all of the trig functions in terms of arcsine and arccosine. However, for each inverse function, there is a different triangle. You will derive these formulas in the exercise for this section.

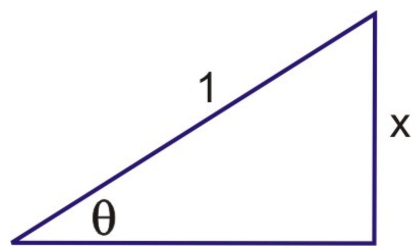
Points to Consider

- Is it possible to graph these composite functions? What happens when you graph $y = \sin(\cos^{-1}x)$ in your calculator?
- Do exact values of functions of inverse functions exist if any value is used?

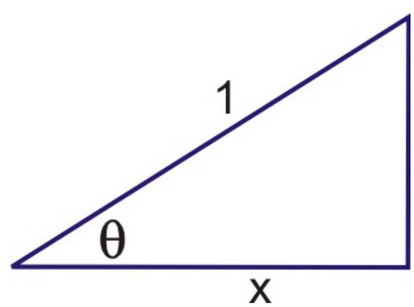
Review Questions

1. Find the exact value of the functions, without a calculator, over their restricted domains.
   a. $\cos^{-1}\frac{\sqrt{3}}{2}$
   b. $\sec^{-1}\sqrt{2}$
   c. $\sec^{-1}\left(-\sqrt{2}\right)$
   d. $\sec^{-1}(-2)$
   e. $\cot^{-1}(-1)$
   f. $\csc^{-1}\left(\sqrt{2}\right)$
2. Use your calculator to find:
   a. $\arccos(-0.923)$
   b. $\arcsin 0.368$
   c. $\arctan 5.698$
3. Find the exact value of the functions, without a calculator, over their restricted domains.
   a. $\csc\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$
   b. $\sec^{-1}(\tan(\cot^{-1}1))$
   c. $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$
   d. $\cot\left(\sec^{-1}\frac{2\sqrt{3}}{3}\right)$
4. Using your graphing calculator, graph $y = \sec^{-1}x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)
5. Using your graphing calculator, graph $y = \csc^{-1}x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)

6. Using your graphing calculator, graph $y = \cot^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)
7. Evaluate:
   a. $\sin\left(\cos^{-1} \frac{5}{13}\right)$
   b. $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right)$
   c. $\cos\left(\csc^{-1} \frac{25}{7}\right)$
8. Express each of the following functions as an algebraic expression involving no trigonometric functions.
   a. $\cos^2(\tan^{-1} x)$
   b. $\cot(\tan^{-1} x^2)$
9. To find trigonometric functions in terms of sine inverse, use the following triangle.



   a. Determine the sine, cosine and tangent in terms of arcsine.
   b. Find $\tan(\sin^{-1} 2x^3)$.
10. To find the trigonometric functions in terms of cosine inverse, use the following triangle.



   a. Determine the sine, cosine and tangent in terms of arccosine.
   b. Find $\sin^2\left(\cos^{-1} \frac{1}{2}x\right)$.

4.4 Applications & Models

Learning Objectives

- Apply inverse trigonometric functions to real life situations.

The following problems are real-world problems that can be solved using the trigonometric functions. In everyday life, indirect measurement is used to obtain answers to problems that are impossible to solve using measurement tools. However, mathematics will come to the rescue in the form of trigonometry to calculate these unknown measurements.

Example 1: On a cold winter day the sun streams through your living room window and causes a warm, toasty atmosphere. This is due to the angle of inclination of the sun which directly affects the heating and the cooling of buildings. Noon is when the sun is at its maximum height in the sky and at this time, the angle is greater in the summer than in the winter. Because of this, buildings are constructed such that the overhang of the roof can act as an awning to shade the windows for cooling in the summer and yet allow the sun's rays to provide heat in the winter. In addition to the construction of the building, the angle of inclination of the sun varies according to the latitude of the building's location.

If the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun $= 90^\circ - \text{latitude} + -23.5^\circ \cdot \cos\left[(N+10)\frac{360}{365}\right]$ where N represents the number of the day of the year that corresponds to the date of the year. Note: This formula is accurate to $\pm\frac{1^\circ}{2}$

a. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 42°, March 10^{th}, the 69^{th} day of the year.

Solution:

$$\begin{aligned}
\text{Angle of sun} &= 90^\circ - 42^\circ + -23.5^\circ \cdot \cos\left[(69+10)\frac{360}{365}\right]\\
\text{Angle of sun} &= 48^\circ + -23.5^\circ(0.2093)\\
\text{Angle of sun} &= 48^\circ - 4.92^\circ\\
\text{Angle of sun} &= 43.08^\circ
\end{aligned}$$

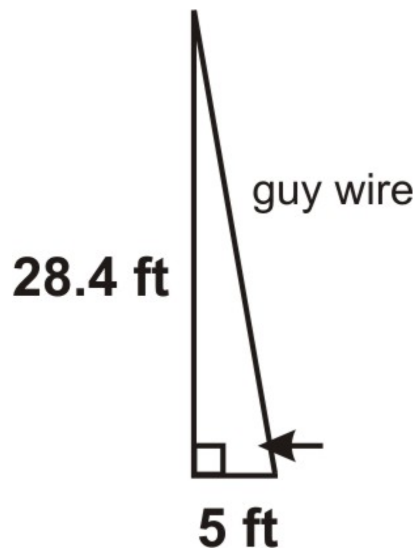
b. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 20°, September 21^{st}.

Solution:

$$\begin{aligned}
\text{Angle of sun} &= 90^\circ - 20^\circ + -23.5^\circ \cdot \cos\left[(264+10)\frac{360}{365}\right]\\
\text{Angle of sun} &= 70^\circ + -23.5^\circ(0.0043)\\
\text{Angle of sun} &= 70.10^\circ
\end{aligned}$$

Example 2: A tower, 28.4 feet high, must be secured with a guy wire anchored 5 feet from the base of the tower. What angle will the guy wire make with the ground?

Solution: Draw a picture.



$$\tan\theta = \frac{opp.}{adj.}$$
$$\tan\theta = \frac{28.4}{5}$$
$$\tan\theta = 5.68$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(5.68)$$
$$\theta = 80.02^\circ$$

The following problem that involves functions and their inverses will be solved using the property $f(f^{-1}(x)) = f^{-1}(f(x))$. In addition, technology will also be used to complete the solution.

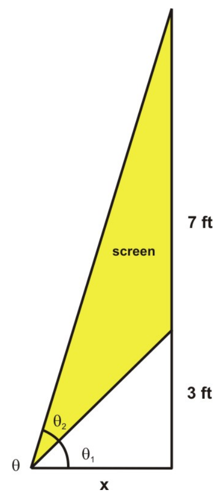
Example 3: In the main concourse of the local arena, there are several viewing screens that are available to watch so that you do not miss any of the action on the ice. The bottom of one screen is 3 feet above eye level and the screen itself is 7 feet high. The angle of vision (inclination) is formed by looking at both the bottom and top of the screen.

a. Sketch a picture to represent this problem.

b. Calculate the measure of the angle of vision that results from looking at the bottom and then the top of the screen. At what distance from the screen does the maximum value for the angle of vision occur?

Solution:

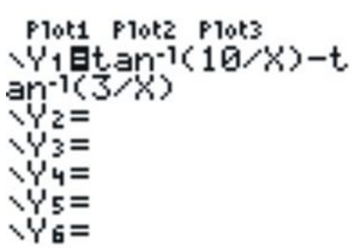
a.

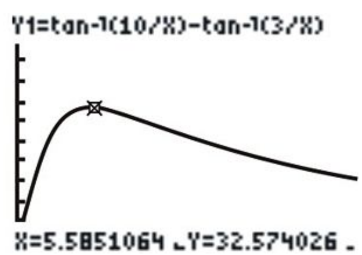


b.

$$\theta_2 = \tan\theta - \tan\theta_1$$
$$\tan\theta = \frac{10}{x} \text{ and } \tan\theta_1 = \frac{3}{x}$$
$$\theta_2 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$$

To determine these values, use a graphing calculator and the trace function to determine when the actual maximum occurs.

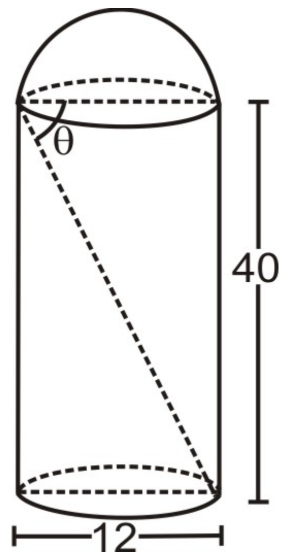




From the graph, it can be seen that the maximum occurs when $x \approx 5.59\ ft$. and $\theta \approx 32.57°$.

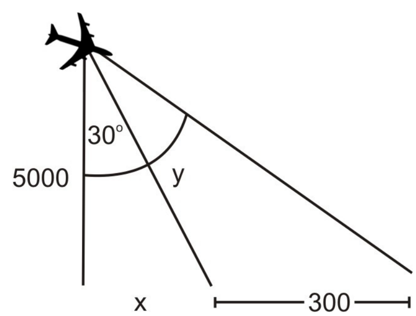
Example 4: A silo is 40 feet high and 12 feet across. Find the angle of depression from the top edge of the silo to the floor of the opposite edge.

Solution: $\tan\theta = \frac{40}{12} \rightarrow \theta = \tan^{-1}\frac{40}{12} = 73.3°$



Example 5: The pilot of an airplane flying at an elevation of 5000 feet sights two towers that are 300 feet apart. If the angle between the point directly below him and the base of the tower closer to him is 30°, determine the angle *y* between the bases of the two towers.

Solution: Draw a picture. First we need to find *x* in order to find *y*.



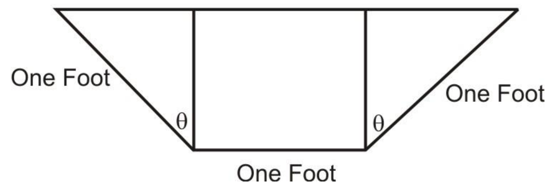
$$\begin{aligned}
\tan 30^\circ &= \frac{x}{5000} \rightarrow x = 5000 \tan 30^\circ \rightarrow x = 2886.75 \\
\tan(30^\circ + y) &= \frac{2886.75 + 300}{5000} \\
\tan(30^\circ + y) &= \frac{3186.75}{5000} \\
30^\circ + y &= \tan^{-1} \frac{3186.75}{5000} \\
30^\circ + y &= 32.51^\circ \\
y &= 2.51^\circ
\end{aligned}$$

Which means that the bases of the two towers are 2.51° apart.

Review Questions

1. The intensity of a certain type of polarized light is given by the equation $I = I_0 \sin 2\theta \cos 2\theta$. Solve for θ.

2. The following diagram represents the ends of a water trough. The ends are actually isosceles trapezoids, and the length of the trough from end-to-end is ten feet. Determine the maximum volume of the trough and the value of θ that maximizes that volume.



3. A boat is docked at the end of a 10 foot pier. The boat leaves the pier and drops anchor 230 feet away 3 feet straight out from shore (which is perpendicular to the pier). What was the bearing of the boat from a line drawn from the end of the pier through the foot of the pier?
4. The electric current in a certain circuit is given by $i = I_m[\sin(wt+\alpha)\cos\varphi + \cos(wt+\alpha)\sin\varphi]$ Solve for t.
5. Using the formula from Example 1, determine the measurement of the sun's angle of inclination for a building located at a latitude of:
   a. 64° on the 16^{th} of November
   b. 15° on the 8^{th} of August
6. A ship leaves port and travels due east 15 nautical miles, then changes course to $N\ 20^\circ\ W$ and travels 40 more nautical miles. Find the bearing to the port of departure.
7. Find the maximum displacement for the simple harmonic motion described by $d = 4\cos \pi t$.
8. The pilot of an airplane flying at an elevation of 10,000 feet sights two towers that are 500 feet apart. If the angle between the point directly below him and the base of the tower closer to him is 18°, determine the angle y between the bases of the two towers.

Chapter Summary

In this chapter, we studied all aspects of inverse trigonometric functions. First, we defined the function by finding inverses algebraically. Second, we analyzed the graphs of inverse functions. We needed to restrict the domain of the trigonometric functions in order to take the inverse of each of them. This is because they are periodic and did not pass the horizontal line test. Then, we learned about the properties of the inverse functions, mostly composing a trig function and an inverse. Finally, we applied the principles of inverse trig functions to real-life situations.

Chapter Vocabulary

Arccosecant
: Read "cosecant inverse" and also written $\csc^{-1}$. The domain of this function is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$.

Arccosine
: Read "cosine inverse" and also written $\cos^{-1}$. The domain of this function is [-1, 1]. The range is $[0, \pi]$.

Arccotangent
: Read "cotangent inverse" and also written $\cot^{-1}$. The domain of this function is all reals. The range is $(0, \pi)$.

Arcsecant

Read "secant inverse" and also written $\sec^{-1}$. The domain of this function is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[0,\pi], y \neq \frac{\pi}{2}$.

Arcsine

Read "sine inverse" and also written $\sin^{-1}$. The domain of this function is [-1, 1]. The range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Arctangent

Read "tangent inverse" and also written $\tan^{-1}$. The domain of this function is all reals. The range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Composite Function

The final result from when one function is plugged into another, $f(g(x))$.

Harmonic Motion

A motion that is consistent and periodic, in a sinusoidal pattern. The general equation is $x(t) = A\cos(2\pi ft + \varphi)$ where A is the amplitude, f is the frequency, and φ is the phase shift.

Horizontal Line Test

The test applied to a function to see if it has an inverse. Continually draw horizontal lines across the function and if a horizontal line touches the function more than once, it does not have an inverse.

Inverse Function

Two functions that are symmetric over the line $y = x$.

Inverse Reflection Principle

The points (a,b) and (b,a) in the coordinate plane are symmetric with respect to the line $y = x$. The points (a,b) and (b,a) are reflections of each other across the line $y = x$.

Invertible

If a function has an inverse, it is invertible.

One-to-One Function

A function, where, for every x value, there is EXACTLY one $y-$value. These are the only invertible functions.

Review Questions

1. Find the exact value of the following expressions:
   a. $\csc^{-1}(-2)$
   b. $\cos^{-1}\frac{\sqrt{3}}{2}$
   c. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
   d. $\sec^{-1}\left(-\sqrt{2}\right)$
   e. $\arcsin 0$
   f. $\arctan 1$
2. Use your calculator to find the value of each of the following expressions:

a. $\arccos \frac{3}{5}$
b. $\csc^{-1} 2.25$
c. $\tan^{-1} 8$
d. $\arcsin(-0.98)$
e. $\cot^{-1}\left(-\frac{9}{40}\right)$
f. $\sec^{-1} \frac{6}{5}$

3. Find the exact value of the following expressions:

a. $\cos\left(\sin^{-1} \frac{\sqrt{2}}{2}\right)$
b. $\tan(\cot^{-1} 1)$
c. $\csc\left(\sec^{-1} \frac{2\sqrt{3}}{3}\right)$
d. $\sin\left(\arccos \frac{12}{13}\right)$
e. $\tan\left(\arcsin \frac{5}{7}\right)$
f. $\sec^{-1}\left(\csc \frac{\pi}{6}\right)$

4. Find the inverse of each of the following:

a. $f(x) = 5 + \cos(2x - 1)$
b. $g(x) = -4\sin^{-1}(x+3)$

5. Sketch a graph of each of the following:

a. $y = 3 - \arcsin\left(\frac{1}{2}x + 1\right)$
b. $f(x) = 2\tan^{-1}(3x - 4)$
c. $h(x) = \sec^{-1}(x - 1) + 2$
d. $y = 1 + 2\arccos 2x$

6. Using the triangles from Section 4.3, find the following:

a. $\sin(\cos^{-1} x^3)$
b. $\tan^2\left(\sin^{-1} \frac{x^2}{3}\right)$
c. $\cos^4(\arctan(2x)^2)$

7. A ship leaves port and travels due west 20 nautical miles, then changes course to $E40^\circ S$ and travels 65 more nautical miles. Find the bearing to the port of departure.
8. Using the formula from Example 1 in Section 4.4, determine the measurement of the sun's angle of inclination for a building located at a latitude of 36° on the 12^{th} of May.
9. Find the inverse of $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$. HINT: Set $a = \sin x$ and $b = \sin y$ and rewrite $\cos x$ and $\cos y$ in terms of sine.
10. Find the inverse of $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$. HINT: Set $a = \cos x$ and $b = \cos y$ and rewrite $\sin x$ and $\sin y$ in terms of sine.

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9702 .

CHAPTER 5 Triangles and Vectors

Chapter Outline

5.1 THE LAW OF COSINES
5.2 AREA OF A TRIANGLE
5.3 THE LAW OF SINES
5.4 THE AMBIGUOUS CASE
5.5 GENERAL SOLUTIONS OF TRIANGLES
5.6 VECTORS
5.7 COMPONENT VECTORS

5.1 The Law of Cosines

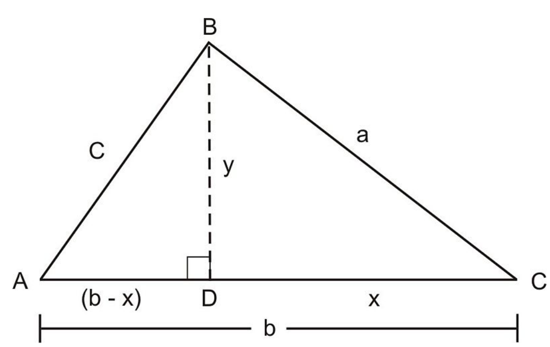
Introduction

This chapter takes concepts that had only been applied to right triangles and interprets them so that they can be used for any type of triangle. First, the laws of sines and cosines take the Pythagorean Theorem and ratios and apply them to any triangle. The second half of the chapter introduces and manipulates vectors. Vectors can be added, subtracted, multiplied and divided.

Learning Objectives

- Understand how the Law of Cosines is derived.
- Apply the Law of Cosines when you know two sides and the included angle of an oblique (non-right) triangle (SAS).
- Apply the Law of Cosines when you know all three sides of an oblique triangle.
- Identify accurate drawings of oblique triangles.
- Use the Law of Cosines in real-world and applied problems.

Derive the Law of Cosines



$\triangle ABC$ contains an altitude BD that extends from B and intersects AC. We will refer to the length of BD as y. The sides of $\triangle ABC$ measure a units, b units, and c units. If DC is x units long, then AD measures $(b-x)$ units.

Using the Pythagorean Theorem we know that:

$c^2 = y^2 + (b-x)^2$	Pythagorean Theorem
$c^2 = y^2 + b^2 - 2bx + x^2$	Expand $(b-x)^2$
$c^2 = a^2 + b^2 - 2bx$	$a^2 = y^2 + x^2$ by Pythagorean Theorem
$c^2 = a^2 + b^2 - 2b(a\cos C)$	$\cos C = \frac{x}{a}$, so $a\cos C = x$ (cross multiply)
$c^2 = a^2 + b^2 - 2ab\cos C$	Simplify

We can use a similar process to derive all three forms of the **Law of Cosines:**

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note that if either $\angle A, \angle B$ or $\angle C$ is 90° then $\cos 90^\circ = 0$ and the Law of Cosines is identical to the Pythagorean Theorem.

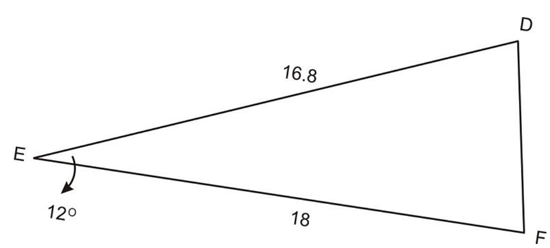
The Law of Cosines is one tool we use in certain situations involving all triangles: right, obtuse, and acute. It is a general statement relating the lengths of the sides of any general triangle to the cosine of one of its angles. There are two situations in which we can and want to use the Law of Cosines:

1. When we know two sides and the included angle in an oblique triangle and want to find the third side (SAS).
2. When we know all three sides in an oblique triangle and want to find one of the angles (SSS).

Case #1: Finding the Side of an Oblique Triangle

One case where we can use the Law of Cosines is when we know two sides and the included angle in a triangle (SAS) and want to find the third side.

Example 1: Using $\triangle DEF, \angle E = 12^\circ, d = 18$, and $f = 16.8$. Find e.

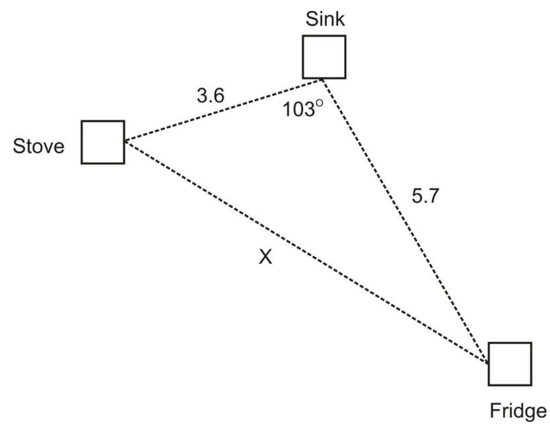


Solution: Since $\triangle DEF$ isn't a right triangle, we cannot use the Pythagorean Theorem or trigonometry functions to find the third side. However, we can use our newly derived Law of Cosines.

$e^2 = 18^2 + 16.8^2 - 2(18)(16.8)\cos 12$	Law of Cosines
$e^2 = 324 + 282.24 - 2(18)(16.8)\cos 12$	Simplify squares
$e^2 = 324 + 282.24 - 591.5836689$	Multiply
$e^2 = 14.6563311$	Add and subtract from left to right
$e \approx 3.8$	Square root

*Note that the negative answer is thrown out as having no geometric meaning in this case.

Example 2: An architect is designing a kitchen for a client. When designing a kitchen, the architect must pay special attention to the placement of the stove, sink, and refrigerator. In order for a kitchen to be utilized effectively, these three amenities must form a triangle with each other. This is known as the "work triangle." By design, the three parts of the work triangle must be *no less than 3 feet apart and no more than 7 feet apart*. Based on the dimensions of the current kitchen, the architect has determined that the sink will be 3.6 feet away from the stove and 5.7 feet away from the refrigerator. If the sink forms a 103° angle with the stove and the refrigerator, will the distance between the stove and the refrigerator remain within the confines of the work triangle?



Solution: In order to find the distance from the sink to the refrigerator, we need to find side x. To find side x, we will use the Law of Cosines because we are dealing with an obtuse triangle (and thus have no right angles to work with). We know the length two sides: the sink to the stove and the sink to the refrigerator. We also know the included angle (the angle between the two known lengths) is 103°. This means we have the SAS case and can apply the Law of Cosines.

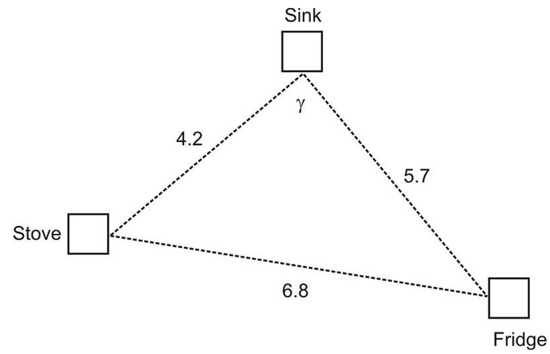
$$
\begin{aligned}
x^2 &= 3.6^2 + 5.7^2 - 2(3.6)(5.7)\cos 103 && \text{Law of Cosines} \\
x^2 &= 12.96 + 32.49 - 2(3.6)(5.7)\cos 103 && \text{Simplify squares} \\
x^2 &= 12.96 + 32.49 + 9.23199127 && \text{Multiply} \\
x^2 &= 54.68199127 && \text{Evaluate} \\
x &\approx 7.4 && \text{Square root}
\end{aligned}
$$

No, this triangle does not conform to the definition of a work triangle. The sink and the refrigerator are too far apart by 0.4 feet.

Case #2: Finding any Angle of a Triangle

Another situation where we can apply the Law of Cosines is when we know all three sides in a triangle (SSS) and we need to find one of the angles. The Law of Cosines allows us to find any of the three angles in the triangle. First, we will look at how to apply the Law of Cosines in this case, and then we will look at a real-world application.

Example 3: Continuing on from Example 2, if the architect moves the stove so that it is 4.2 feet from the sink and makes the fridge 6.8 feet from the stove, how does this affect the angle the sink forms with the stove and the refrigerator?

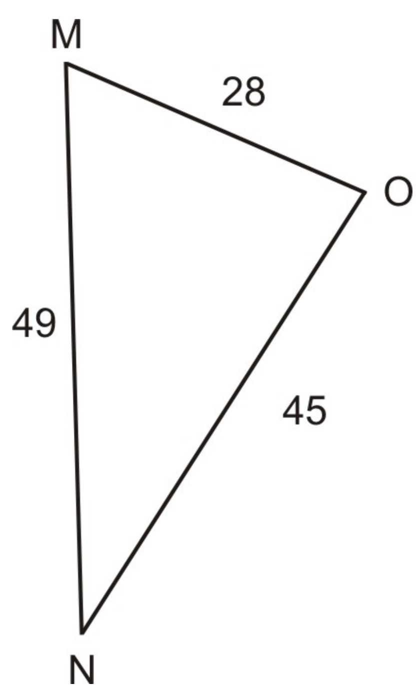


Solution: In order to find how the angle is affected, we will again need to utilize the Law of Cosines, but because we do not know the measures of any of the angles, we solve for Y.

$6.8^2 = 4.2^2 + 5.7^2 - 2(4.2)(5.7)\cos Y$	Law of Cosines
$46.24 = 17.64 + 32.49 - 2(4.2)(5.7)\cos Y$	Simplify squares
$46.24 = 17.64 + 32.49 - 47.88\cos Y$	Multiply
$46.24 = 50.13 - 47.88\cos Y$	Add
$-3.89 = -47.88\cos Y$	Subtract
$0.0812447786 = \cos Y$	Divide
$85.3^\circ \approx Y$	$\cos^{-1}(0.081244786)$

The new angle would be 85.3°, which means it would be 17.7° less than the original angle.

Example 4: In oblique $\triangle MNO, m = 45, n = 28$, and $o = 49$. Find $\angle M$.

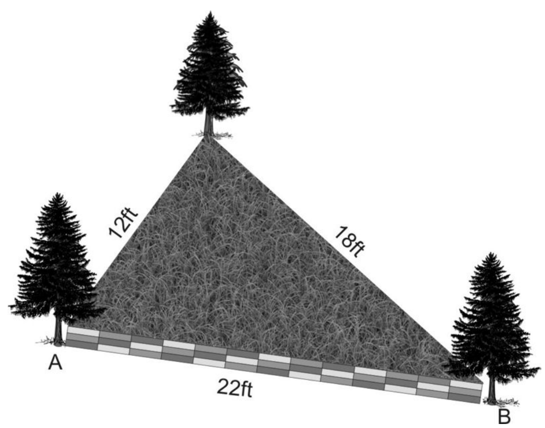


Solution: Since we know all three sides of the triangle, we can use the Law of Cosines to find $\angle M$.

$45^2 = 28^2 + 49^2 - 2(28)(49)\cos M$	Law of Cosines
$2025 = 784 + 2401 - 2(28)(49)\cos M$	Simplify squares
$2025 = 784 + 2401 - 2744\cos M$	Multiply
$2025 = 3185 - 2744\cos M$	Add
$-1160 = -2744\cos M$	Subtract 3185
$0.422740525 = \cos M$	Divide by -2744
$65^\circ \approx M$	$\cos^{-1}(0.422740525)$

It is important to note that we could use the Law of Cosines to find $\angle N$ or $\angle O$ also.

Example 5: Sam is building a retaining wall for a garden that he plans on putting in the back corner of his yard. Due to the placement of some trees, the dimensions of his wall need to be as follows: side $1 = 12ft$, side $2 = 18ft$, and side $3 = 22ft$. At what angle do side 1 and side 2 need to be? Side 2 and side 3? Side 1 and side 3?



Solution: Since we know the measures of all three sides of the retaining wall, we can use the Law of Cosines to find the measures of the angles formed by adjacent walls. We will refer to the angle formed by side 1 and side 2 as $\angle A$, the angle formed by side 2 and side 3 as $\angle B$, and the angle formed by side 1 and side 3 as $\angle C$. First, we will find $\angle A$.

$22^2 = 12^2 + 18^2 - 2(12)(18)\cos A$	Law of Cosines
$484 = 144 + 324 - 2(12)(18)\cos A$	Simplify squares
$484 = 144 + 324 - 432\cos A$	Multiply
$484 = 468 - 432\cos A$	Add
$16 = -432\cos A$	Subtract 468
$-0.037037037 \approx \cos A$	Divide by -432
$92.1^\circ \approx A$	$\cos^{-1}(-0.037037037)$

Next we will find the measure of $\angle B$ also by using the Law of Cosines.

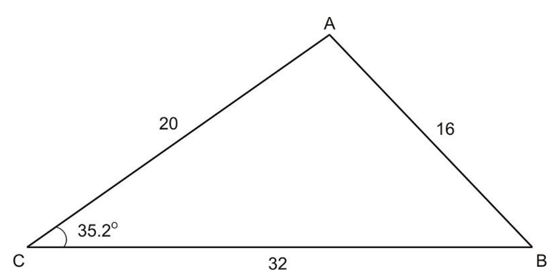
$$
\begin{aligned}
18^2 &= 12^2+22^2-2(12)(22)\cos B && \text{Law of Cosines}\\
324 &= 144+484-2(12)(22)\cos B && \text{Simplify squares}\\
324 &= 144+484-528\cos B && \text{Multiply}\\
324 &= 628-528\cos B && \text{Add}\\
-304 &= -528\cos B && \text{Subtract 628}\\
0.575757576 &= \cos B && \text{Divide by } -528\\
54.8^\circ &\approx B && \cos^{-1}(0.575757576)
\end{aligned}
$$

Now that we know two of the angles, we can find the third angle using the Triangle Sum Theorem, $\angle C = 180 - (92.1+54.8) = 33.1^\circ$.

Identify Accurate Drawings of General Triangles

The Law of Cosines can also be used to verify that drawings of oblique triangles are accurate. In a right triangle, we might use the Pythagorean Theorem to verify that all three sides are the correct length, or we might use trigonometric ratios to verify an angle measurement. However, when dealing with an obtuse or acute triangle, we must rely on the Law of Cosines.

Example 6: In $\triangle ABC$ at the right, $a = 32, b = 20$, And $c = 16$. Is the drawing accurate if it labels $\angle C$ as 35.2°? If not, what should $\angle C$ measure?



Solution: We will use the Law of Cosines to check whether or not $\angle C$ is 35.2°.

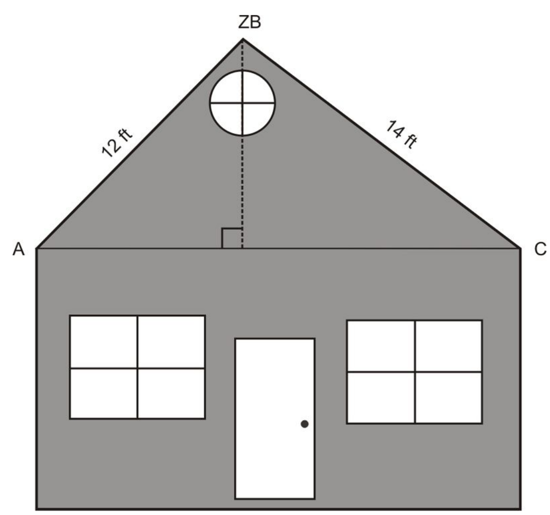
$$
\begin{aligned}
16^2 &= 20^2+32^2-2(20)(32)\cos 35.2 && \text{Law of Cosines}\\
256 &= 400+1024-2(20)(32)\cos 35.2 && \text{Simply squares}\\
256 &= 400+1024-1045.94547 && \text{Multiply}\\
256 &\neq 378.05453 && \text{Add and subtract}
\end{aligned}
$$

Since $256 \neq 378.05453$, we know that $\angle C$ is not 35.2°. Using the Law of Cosines, we can figure out the correct measurement of $\angle C$.

$16^2 = 20^2 + 32^2 - 2(20)(32)\cos C$	Law of Cosines
$256 = 400 + 1024 - 2(20)(32)\cos C$	Simplify Squares
$256 = 400 + 1024 - 1280\cos C$	Multiply
$256 = 1424 - 1280\cos C$	Add
$-1168 = -1280\cos C$	Subtract 1424
$0.9125 = \cos C$	Divide
$24.1^\circ \approx \angle C$	$\cos^{-1}(0.9125)$

For some situations, it will be necessary to utilize not only the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios to verify that a triangle or quadrilateral has been drawn accurately.

Example 7: A builder received plans for the construction of a second-story addition on a house. The diagram shows how the architect wants the roof framed, while the length of the house is 20 ft. The builder decides to add a perpendicular support beam from the peak of the roof to the base. He estimates that new beam should be 8.3 feet high, but he wants to double-check before he begins construction. Is the builder's estimate of 8.3 feet for the new beam correct? If not, how far off is he?



Solution: If we knew either $\angle A$ or $\angle C$, we could use trigonometric ratios to find the height of the support beam. However, neither of these angle measures are given to us. Since we know all three sides of $\triangle ABC$, we can use the Law of Cosines to find one of these angles. We will find $\angle A$.

$14^2 = 12^2 + 20^2 - 2(12)(20)\cos A$	Law of Cosines
$196 = 144 + 400 - 480\cos A$	Simplify
$196 = 544 - 480\cos A$	Add
$-348 = -480\cos A$	Subtract
$0.725 = \cos A$	Divide
$43.5^\circ \approx \angle A$	$\cos^{-1}(0.725)$

Now that we know $\angle A$, we can use it to find the length of BD.

$$\sin 43.5 = \frac{x}{12}$$
$$12 \sin 43.5 = x$$
$$8.3 \approx x$$

Yes, the builder's estimate of 8.3 feet for the support beam is accurate.

Points to Consider

- How is the Pythagorean Theorem a special case of the Law of Cosines?
- In the SAS case, is it possible to use the Law of Cosines to find all missing sides and angles?
- In which cases can we **not** use the Law of Cosines? Explain.
- Give an example of three side lengths that do not form a triangle.

Review Questions

1. Using each figure and the given information below, decide which side(s) or angle(s) you could find using the Law of Cosines.

TABLE 5.1:

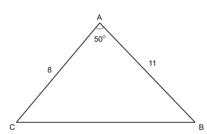
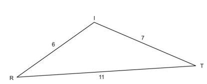
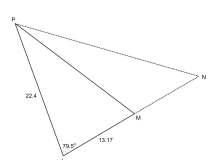
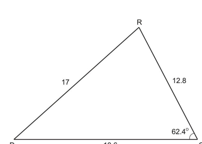
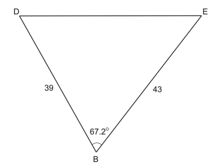
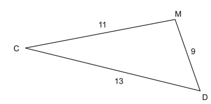
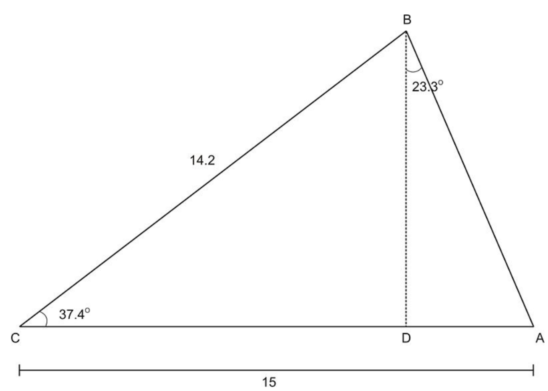
Given Information	**Figure**	**What can you find?**
a. $\angle A = 50^\circ, b = 8, c = 11$		
b. $t = 6, r = 7, i = 11$		
c. $\angle L = 79.5^\circ, m = 22.4, p = 13.17$		
d. $q = 17, d = 12.8, r = 18.6, \angle Q = 62.4^\circ$		

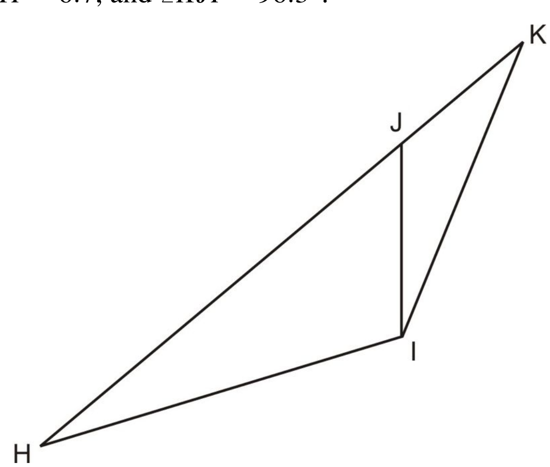
TABLE 5.1: (continued)

Given Information	Figure	What can you find?
e. $\angle B = 67.2^\circ, d = 43, e = 39$		
f. $c = 9, d = 11, m = 13$		

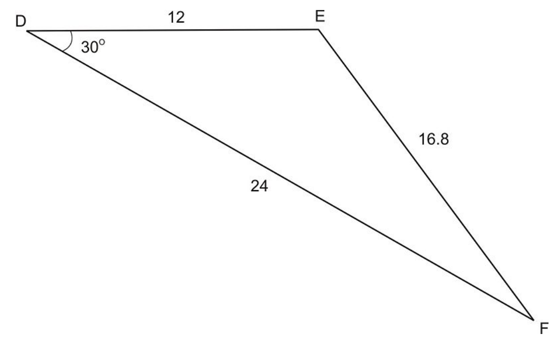
2. Using the figures and information from the chart above, use the Law of Cosines to find the following:
   a. side a
   b. the largest angle
   c. side l
   d. the smallest angle
   e. side b
   f. the second largest angle
3. In $\triangle CIR, c = 63, i = 52$, and $r = 41.9$. Find the measure of all three angles.
4. Find AD using the Pythagorean Theorem, Law of Cosines, trig functions, or any combination of the three.



5. Find HK using the Pythagorean Theorem, Law of Cosines, trig functions, or any combination of the three if $JK = 3.6, KI = 5.2, JI = 1.9, HI = 6.7$, and $\angle KJI = 96.3^\circ$.

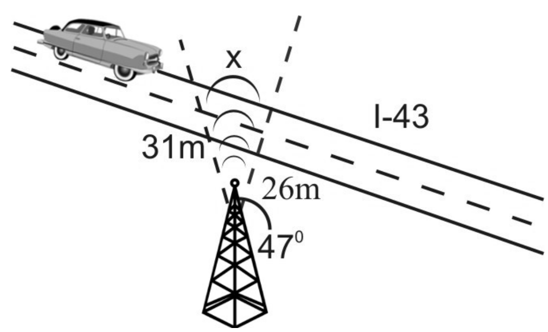


6. Use the Law of Cosines to determine whether or not the following triangle is drawn accurately. If not, determine how much side d is off by.



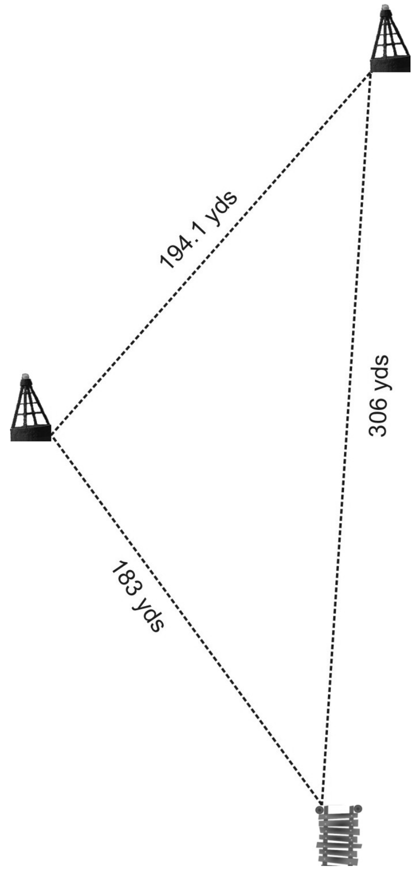
7. A businessman is traveling down Interstate 43 and has intermittent cell phone service. There is a transmission tower near Interstate 43. The range of service from the tower forms a 47° angle and the range of service is 26 miles to one section of I-43 and 31 miles to another point on I-43.

a. If the businessman is traveling at a speed of 45 miles per hour, how long will he have service for?
b. If he slows down to 35mph, how much longer will he be able to have service?

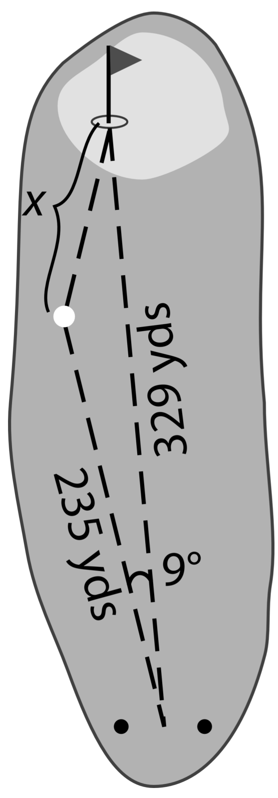


8. A dock is being built so that it is 183 yards away from one buoy and 306 yards away from a second buoy. The two buoys are 194.1 yards apart.

a. What angle does the dock form with the two buoys?
b. If the second buoy is moved so that it is 329 yards away from the dock and 207 yards away from the first buoy, how does this affect the angle formed by the dock and the two buoys?

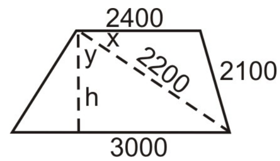


9. A golfer hits the ball from the 18^{th} tee. His shot is a 235 yard hook (curves to the left) 9° from the path straight to the flag on the green. If the tee is 329 yards from the flag, how far is the ball away from the flag?



10. Given the numbers 127, 210 and 17 degrees write a problem that uses the Law of Cosines.

11. The sides of a triangle are 15, 27 and 39. What is its area?
12. A person inherits a piece of land in the shape of a trapezoid as shown, with the side lengths being in feet. What is the area of the piece of land? How many acres is it?



5.2 Area of a Triangle

Learning Objectives

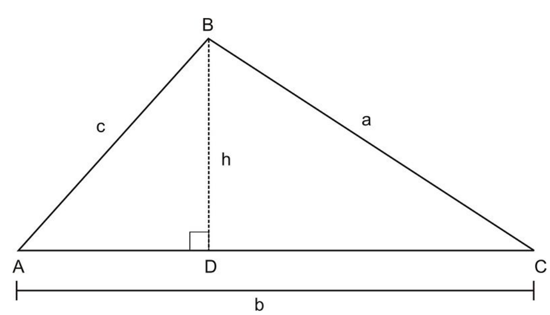
- Apply the area formula to triangles where you know two sides and the included angle.
- Apply the area formula to triangles where you know all three sides, Heron's Formula.
- Use the area formulas in real-world and applied problems.

In this section, we will look at how we can derive a new formula using the area formula that we already know and the sine function. This new formula will allow us to find the area of a triangle when we don't know the height. We will also look at when we can use this formula and how to apply it to real-world situations.

Deriving an Alternate Formula to the Triangle Area Equation

We can use the area formula from Geometry, $A = \frac{1}{2}bh$, as well as the sine function, to derive a new formula that can be used when the height, or altitude, is unknown.

In $\triangle ABC$ below, BD is altitude from B to AC. We will refer to the length of BD as h since it also represents the height of the triangle. Also, we will refer to the area of the triangle as K to avoid confusing the area with $\angle A$.



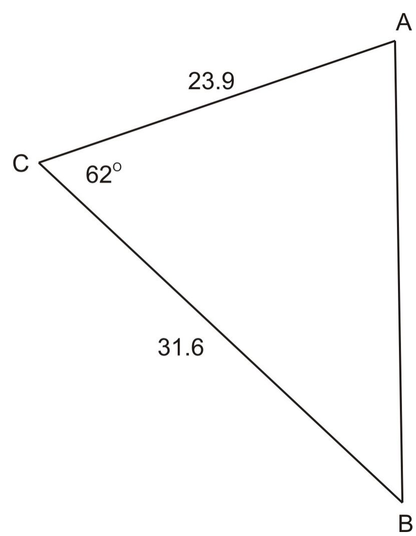
$k = \frac{1}{2}bh$	Area of a triangle
$k = \frac{1}{2}b(c\sin A)$	$\sin A = \frac{h}{c}$ therefore $c\sin A = h$
$k = \frac{1}{2}bc\sin A$	Simplify

We can use a similar method to derive all three forms of the area formula, regardless of the angle:

$$K = \frac{1}{2}bc\sin A$$
$$K = \frac{1}{2}ac\sin B$$
$$K = \frac{1}{2}ab\sin C$$

The formula $K = \frac{1}{2}\,bc\sin A$ requires us to know two sides and the included angle (SAS) in a triangle. Once we know these three things, we can easily calculate the area of an oblique triangle.

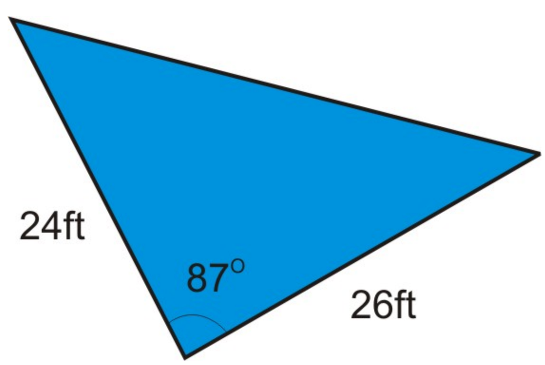
Example 1: In $\triangle ABC, \angle C = 62^\circ, b = 23.9$, and $a = 31.6$. Find the area of the triangle.



Solution: Using our new formula, $K = \frac{1}{2}\,ab\sin C$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(31.6)(23.9)\sin 62$$
$$K \approx 333.4$$

Example 2: The Pyramid Hotel recently installed a triangular pool. One side of the pool is 24 feet, another side is 26 feet, and the angle in between the two sides is 87°. If the hotel manager needs to order a cover for the pool, and the cost is \$35 per square foot, how much can he expect to spend?



Solution: In order to find the cost of the cover, we first need to know the area of the cover. Once we know how many square feet the cover is, we can calculate the cost. In the illustration above, you can see that we know two of the sides and the included angle. This means we can use the formula $K = \frac{1}{2}\, bc \sin A$.

$$K = \frac{1}{2}\,(24)(26)\sin 87$$
$$K \approx 311.6$$
$$311.6\; sq.ft. \times \$35/sq.ft. = \$10,905.03$$

The cost of the cover will be $\$10,905.03$.

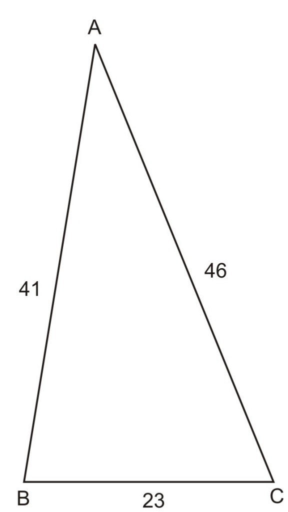
Find the Area Using Three Sides: Heron's Formula

In the last section, we learned how to find the area of an oblique triangle when we know two sides and the included angle using the formula $K = \frac{1}{2}\, bc \sin A$. We could also find the area of a triangle in which we know all three sides by first using the Law of Cosines to find one of the angles and then using the formula $K = \frac{1}{2}\, bc \sin A$. While this process works, it is time-consuming and requires a lot of calculation. Fortunately, we have another formula, called Heron's Formula, which allows us to calculate the area of a triangle when we know all three sides. It is derived from $K = \frac{1}{2}\, bc \sin A$, the Law of Cosines and the Pythagorean Identity.

Heron's Formula

$K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$ or half of the perimeter of the triangle.

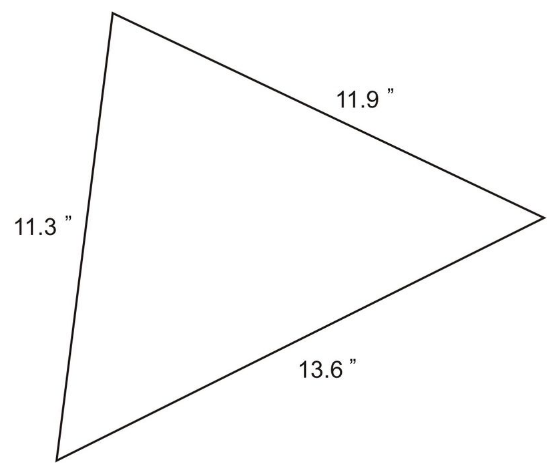
Example 3: In $\triangle ABC, a = 23, b = 46$, and $C = 41$. Find the area of the triangle.



Solution: First, you need to find s: $s = \frac{1}{2}(23+41+46) = 55$. Now, plug s and the three sides into Heron's Formula and simplify.

$$K = \sqrt{55(55-23)(55-46)(55-41)}$$
$$K = \sqrt{55(32)(9)(14)}$$
$$k = \sqrt{221760}$$
$$K \approx 470.9$$

Example 4: A handyman is installing a tile floor in a kitchen. Since the corners of the kitchen are not exactly square, he needs to have special triangular shaped tile made for the corners. One side of the tile needs to be 11.3", the second side needs to be 11.9",and the third side is 13.6". If the tile costs $4.89 per square foot, and he needs four of them, how much will it cost to have the tiles made?



Solution: In order to find the cost of the tiles, we first need to find the area of one tile. Since we know the measurements of all three sides, we can use Heron's Formula to calculate the area.

$$s = \frac{1}{2}(11.3+11.9+13.6) = 18.4$$
$$K = \sqrt{18.4(18.4-11.3)(18.4-11.9)(18.4-13.6)}$$
$$K = \sqrt{18.4(7.1)(6.5)(4.8)}$$
$$K = \sqrt{4075.968}$$
$$K \approx 63.843\ in^2$$

The area of one tile is 63.843 square inches. The cost of the tile is given to us in square feet, while the area of the tile is in square inches. In order to find the cost of one tile, we must first convert the area of the tile into square feet.

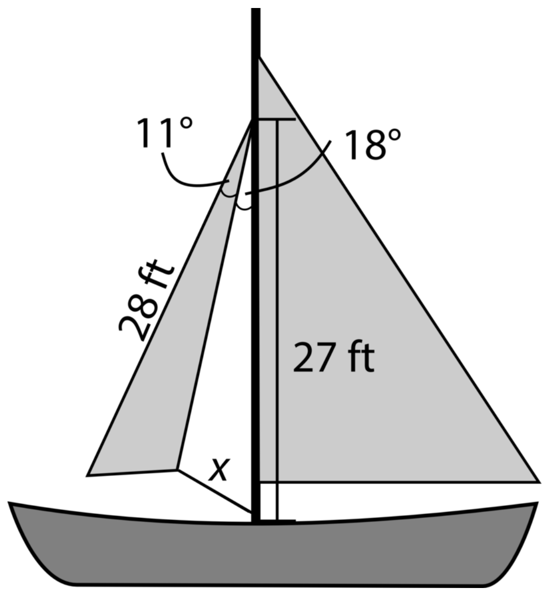
$$1 \text{ square foot} = 12in \times 12in = 144in^2$$
$$\frac{63.843}{144} = 0.443\ ft^2 \quad \text{Covert square inches into square feet}$$
$$0.443\ ft^2 \times 4.89 = 2.17 \quad \text{Multiply by the cost of the tile.}$$
$$2.17 \times 4 = 8.68$$

The cost for four tiles would be $8.68.

Finding a Part of the Triangle, Given the Area

We have already looked at two examples of situations where we can apply the two new area formulas we learned in this section. In this section, we will look at another real-world application where we know the area but need to find another part of the triangle, as well as an application involving a quadrilateral.

Example 5: The jib sail on a sailboat came untied and the rope securing it was lost. If the area of the jib sail is 56.1 square feet, use the figure and information below to find the length of the rope.



Solution: Since we know the area, one of the sides, and one angle of the jib sail, we can use the formula $K = \frac{1}{2}\,bc\sin A$ to find the side of the jib sail that is attached to the mast. We will call this side y.

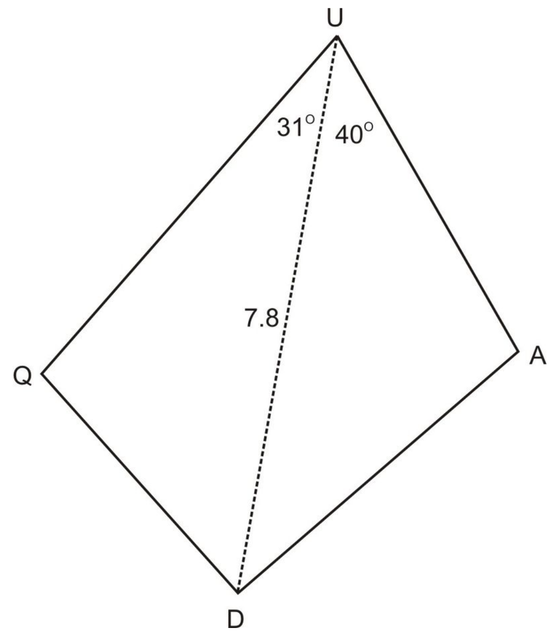
$$\begin{aligned} 56.1 &= \frac{1}{2}\,28(y)\sin 11 \\ 56.1 &= 2.671325935\,y \\ 21.0 &= y \end{aligned}$$

Now that we know side y, we know two sides and the included angle in the triangle formed by the mast, the rope, and the jib sail. We can now use the Law of Cosines to calculate the length of the rope.

$$\begin{aligned} x^2 &= 21^2 + 27^2 - 2(21)(27)\cos 18 \\ x^2 &= 91.50191052 \\ x &\approx 9.6\ ft \end{aligned}$$

The length of the rope is approximately 9.6 feet.

Example 6: In quadrilateral $QUAD$ below, the area of $\triangle QUD = 5.64$, the area of $\triangle UAD = 6.39, \angle QUD = 31^\circ, \angle DUA = 40^\circ$, and $UD = 7.8$. Find the perimeter of $QUAD$.



Solution: In order to find the perimeter of $QUAD$, we need to know sides QU, QD, UA, and AD. Since we know the area, one side, and one angle in each of the triangles, we can use $K = \frac{1}{2}bc\sin A$ to figure out QU and UA.

$$5.64 = \frac{1}{2}(7.8)(QU)\sin 31 \qquad 6.39 = \frac{1}{2}(7.8)UA\sin 40$$
$$2.8 \approx QU \qquad 2.5 \approx UA$$

Now that we know QU and UA, we know two sides and the included angle in each triangle (SAS). This means that we can use the Law of Cosines to find the other two sides, QD and DA. First we will find QD and DA.

$$QD^2 = 2.8^2 + 7.8^2 - 2(2.8)(7.8)\cos 31 \qquad DA^2 = 2.5^2 + 7.8^2 - 2(2.5)(7.8)\cos 40$$
$$QD^2 = 31.23893231 \qquad DA^2 = 37.21426672$$
$$QD \approx 5.6 \qquad DA \approx 6.1$$

Finally, we can calculate the perimeter since we have found all four sides of the quadrilateral.

$$pQUAD = 2.8 + 5.6 + 6.1 + 2.5 = 17$$

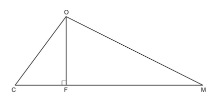
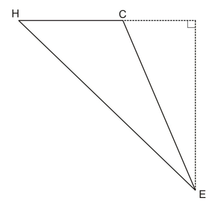
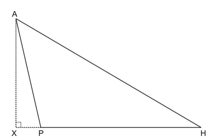
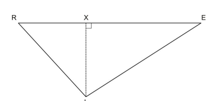
Points to Consider

- Why can't s (half of the perimeter) in Heron's Formula be smaller than any of the three sides in the triangle?
- How could we find the area of a triangle is AAS, SSA, and ASA cases?
- Is it possible to figure out the length of the third side of a triangle if we know the other two sides and the area?

Review Questions

1. Using the figures and given information below, determine which formula you would need to use in order to find the area of each triangle ($A = \frac{1}{2}bh, K = \frac{1}{2}bc\sin A$, or Heron's Formula).

TABLE 5.2:

Given	Figure	Formula
a. $CF = 3, FM = 8$, and $CO = 5$		
b. $HC = 4.1, CE = 7.4$, and $HE = 9.6$		
c. $AP = 59.8, PH = 86.3, \angle APH = 103^\circ$		
d. $RX = 11.1, XE = 18.9, \angle R = 41^\circ$		

2. Find the area of all of the triangles in the chart above to the nearest tenth.
3. Using the given information and the figures below, decide which area formula you would need to use to find each side, angle, or area.

TABLE 5.3:

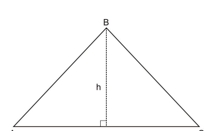
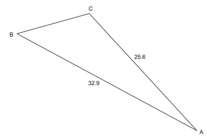
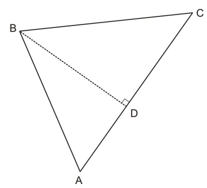
Given	Figure	Find	Formula
a. Area $= 1618.98, b = 36.3$		h	

TABLE 5.3: (continued)

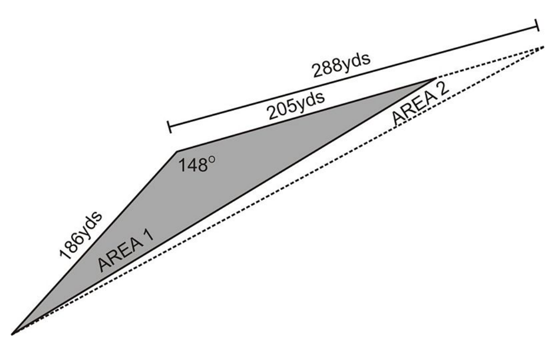
Given	Figure	Find	Formula
b. Area $= 387.6, b = 25.6, c = 32.9$		$\angle A$	
c. Area $\triangle ABD = 16.96, AD = 3.2, \angle DBC = 49.6^\circ$		Area of $\triangle ABC$	

4. Using the figures and information from the table above, find the angle, side, or area requested.
5. The Pyramid Hotel is planning on repainting the exterior of the building. The building has four sides that are isosceles triangles with bases measuring 590 ft and legs measuring 375 ft.

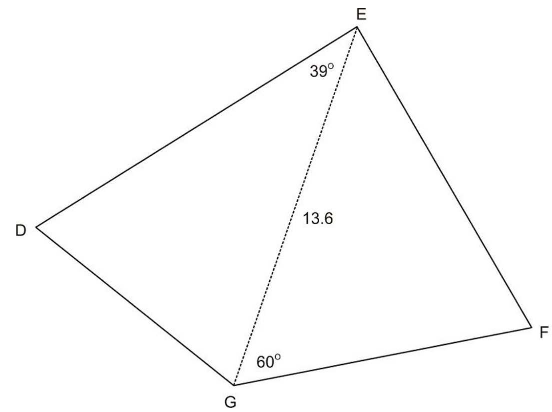
   a. What is the total area that needs to be painted?
   b. If one gallon of paint covers 25 square feet, how many gallons of paint are needed?

6. A contractor needs to replace a triangular section of roof on the front of a house. The sides of the triangle are 8.2 feet, 14.6 feet, and 16.3 feet. If one bundle of shingles covers $33\frac{1}{3}$ square feet and costs \$15.45, how many bundles does he need to purchase? How much will the shingles cost him? How much of the bundle will go to waste?
7. A farmer needs to replant a triangular section of crops that died unexpectedly. One side of the triangle measures 186 yards, another measures 205 yards, and the angle formed by these two sides is 148°.

   a. What is the area of the section of crops that needs to be replanted?
   b. The farmer goes out a few days later to discover that more crops have died. The side that used to measure 205 yards now measures 288 yards. How much has the area that needs to be replanted increased by?



8. Find the perimeter of the quadrilateral at the left If the area of $\triangle DEG = 56.5$ and the area of $\triangle EGF = 84.7$.



9. In $\triangle ABC, BD$ is an altitude from B to AC. The area of $\triangle ABC = 232.96, AB = 16.2$, and $AD = 14.4$. Find DC.
10. Show that in any triangle $DEF, d^2+e^2+f^2 = 2(ef\cos D + df\cos E + de\cos F)$.

5.3 The Law of Sines

Learning Objectives

- Understand how both forms of the Law of Sines are obtained.
- Apply the Law of Sines when you know two angles and a non-included side and if you know two angles and the included side.
- Use the Law of Sines in real-world and applied problems.

We have learned about the Law of Cosines, which is a generalization of the Pythagorean Theorem for non-right triangles. We know that we can use the Law of Cosines when:

1. We know two sides of a triangle and the included angle (SAS) or
2. We know all three sides of the triangle (SSS)

But, what happens if the triangle we are working with doesn't fit either of those scenarios? Here we introduce the Law of Sines.

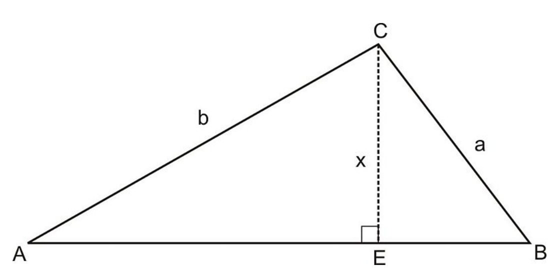
The Law of Sines is a statement about the relationship between the sides and the angles in any triangle. While the Law of Sines will yield one correct answer in many situations, there are times when it is ambiguous, meaning that it can produce more than one answer. We will explore the ambiguity of the Law of Sines in the next section.

We can use the Law of Sines when:

1. We know two angles and a non-included side (AAS) or
2. We know two angles and the included side (ASA)

Deriving the Law of Sines

$\triangle ABC$ contains altitude CE, which extends from C and intersects AB. We will refer to the length of altitude CE as x.



We know that $\sin A = \frac{x}{b}$ and $\sin B = \frac{x}{a}$, by the definition of sine. If we cross-multiply both equations and substitute, we will have the Law of Sines.

$$b(\sin A) = x \qquad \text{and} \qquad a(\sin B) = x$$
$$\searrow \qquad\qquad \swarrow$$
$$b(\sin A) = a(\sin B)$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B}$$

Extending these ratios to angle C and side c, we arrive at both forms of the Law of Sines:

Form 1 : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

(sines over sides)

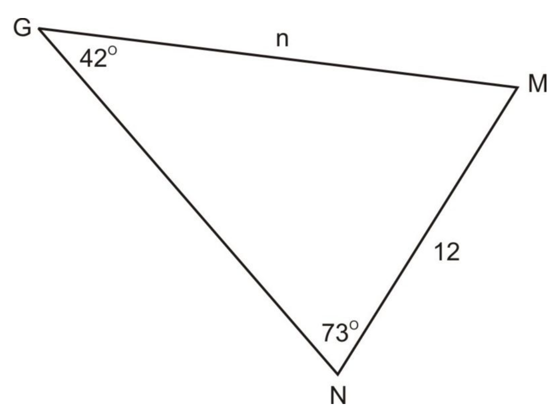
Form 2 : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

(sides over sines)

AAS (Angle-Angle-Side)

One case where we can to use the Law of Sines is when we know two of the angles in a triangle and a non-included side (AAS).

Example 1: Using $\triangle GMN, \angle G = 42^\circ, \angle N = 73^\circ$ and $g = 12$. Find n.



Since we know two angles and one non-included side (g), we can find the other non-included side (n).

$$\frac{\sin 73^\circ}{n} = \frac{\sin 42^\circ}{12}$$
$$n \sin 42^\circ = 12 \sin 73^\circ$$
$$n = \frac{12 \sin 73^\circ}{\sin 42^\circ}$$
$$n \approx 17.15$$

Example 2: Continuing on from Example 1, find $\angle M$ and m.

Solution: $\angle M$ is simply $180^\circ - 42^\circ - 73^\circ = 65^\circ$. To find side m, you can now use either the Law of Sines or Law of Cosines. Considering that the Law of Sines is a bit simpler and new, let's use it. It does not matter which side and opposite angle you use in the ratio with $\angle M$ and m.

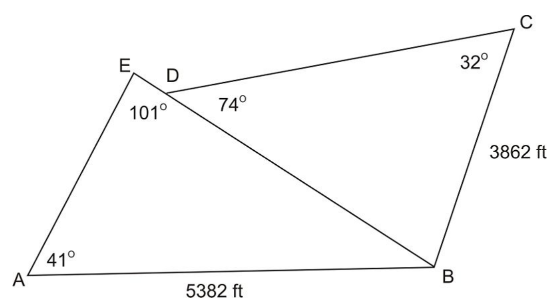
Option 1: $\angle G$ and g

$$\begin{aligned}\frac{\sin 65^\circ}{m} &= \frac{\sin 42^\circ}{12}\\ m\sin 42^\circ &= 12\sin 65^\circ\\ m &= \frac{12\sin 65^\circ}{\sin 42^\circ}\\ m &\approx 16.25\end{aligned}$$

Option 2: $\angle N$ and n

$$\begin{aligned}\frac{\sin 65^\circ}{m} &= \frac{\sin 73^\circ}{17.15}\\ m\sin 73^\circ &= 17.15\sin 65^\circ\\ m &= \frac{17.15\sin 65^\circ}{\sin 73^\circ}\\ m &\approx 16.25\end{aligned}$$

Example 3: A business group wants to build a golf course on a plot of land that was once a farm. The deed to the land is old and information about the land is incomplete. If AB is 5382 feet, BC is 3862 feet, $\angle AEB$ is 101°, $\angle BDC$ is 74°, $\angle EAB$ is 41° and $\angle DCB$ is 32°, what are the lengths of the sides of each triangular piece of land? What is the total area of the land?



Solution: Before we can figure out the area of the land, we need to figure out the length of each side. In triangle ABE, we know two angles and a non-included side. This is the AAS case. First, we will find the third angle in triangle ABE by using the Triangle Sum Theorem. Then, we can use the Law of Sines to find both AE and EB.

$$\begin{aligned}\angle ABE &= 180 - (41 + 101) = 38^\circ\\ \frac{\sin 101}{5382} &= \frac{\sin 38}{AE}\\ AE(\sin 101) &= 5382(\sin 38)\\ AE &= \frac{5382(\sin 38)}{\sin 101}\\ AE &= 3375.5\ feet\end{aligned}$$

$$\begin{aligned}\frac{\sin 101}{5382} &= \frac{\sin 41}{EB}\\ EB(\sin 101) &= 5382(\sin 41)\\ EB &= \frac{5382(\sin 41)}{\sin 101}\\ EB &\approx 3597.0\ feet\end{aligned}$$

Next, we need to find the missing side lengths in triangle DCB. In this triangle, we again know two angles and a non-included side (AAS), which means we can use the Law of Sines. First, let's find $\angle DBC = 180 - (74 + 32) = 74^\circ$. Since both $\angle BDC$ and $\angle DBC$ measure 74°, triangle DCB is an isosceles triangle. This means that since BC is 3862 feet, DC is also 3862 feet. All we have left to find now is DB.

$$\begin{aligned}
\frac{\sin 74}{3862} &= \frac{\sin 32}{DB} \\
DB(\sin 74) &= 3862(\sin 32) \\
DB &= \frac{3862(\sin 32)}{\sin 74} \\
DB &\approx 2129.0\ feet
\end{aligned}$$

Finally, we need to calculate the area of each triangle and then add the two areas together to get the total area. From the last section, we learned two area formulas, $K = \frac{1}{2}\, bc \sin A$ and Heron's Formula. In this case, since we have enough information to use either formula, we will use $K = \frac{1}{2}\, bc \sin A$ since it is less computationally intense.

First, we will find the area of triangle ABE.

Triangle ABE:

$$\begin{aligned}
K &= \frac{1}{2}(3375.5)(5382)\sin 41 \\
K &= 5,959,292.8\ ft^2
\end{aligned}$$

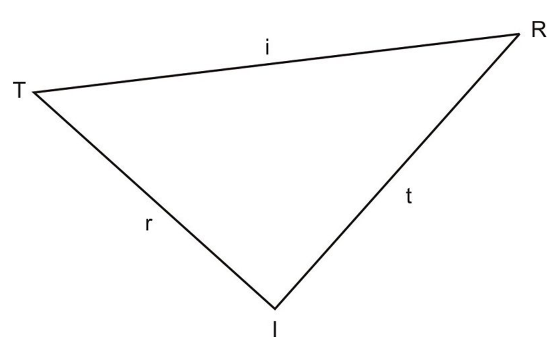
Triangle DBC:

$$\begin{aligned}
K &= \frac{1}{2}(3862)(3862)\sin 32 \\
K &= 3,951,884.6\ ft^2
\end{aligned}$$

The total area is $5,959,292.8 + 3,951,884.6 = 9,911,177.4\ ft^2$.

ASA (Angle-Side-Angle)

The second case where we use the Law of Sines is when we know two angles in a triangle and the **included** side (ASA). For instance, in $\triangle TRI$:



$\angle T$, $\angle R$, and i are known

$\angle T$, $\angle I$,and r are known

$\angle R$, $\angle I$, and t are known

In this case, the Law of Sines allows us to find either of the non-included sides.

Example 4: (Use the picture above) In $\triangle TRI, \angle T = 83^\circ, \angle R = 24^\circ$, and $i = 18.5$. Find the measure of t.

Solution: Since we know two angles and the included side, we can find either of the non-included sides using the Law of Sines. Since we already know two of the angles in the triangle, we can find the third angle using the fact that the sum of all of the angles in a triangle must equal 180°.

$$\begin{aligned}\angle I &= 180 - (83 + 24)\\ \angle I &= 180 - 107\\ \angle I &= 73^\circ\end{aligned}$$

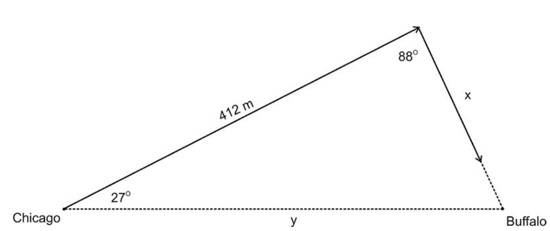
Now that we know $\angle I = 73^\circ$, we can use the Law of Sines to find t.

$$\begin{aligned}\frac{\sin 73}{18.5} &= \frac{\sin 83}{t}\\ t(\sin 73) &= 18.5(\sin 83)\\ t &= \frac{18.5(\sin 83)}{\sin 73}\\ t &\approx 19.2\end{aligned}$$

Notice how we wait until the last step to input the values into the calculator. This is so our answer is as accurate as possible.

Example 5: In order to avoid a large and dangerous snowstorm on a flight from Chicago to Buffalo, pilot John starts out 27° off of the normal flight path. After flying 412 miles in this direction, he turns the plane toward Buffalo. The angle formed by the first flight course and the second flight course is 88°. For the pilot, two issues are pressing:

1. What is the total distance of the modified flight path?
2. How much further did he travel than if he had stayed on course?



Solution, Part 1: In order to find the total distance of the modified flight path, we need to know side x. To find side x, we will need to use the Law of Sines. Since we know two angles and the included side, this is an ASA case. Remember that in the ASA case, we need to first find the third angle in the triangle.

$$\begin{aligned} MissingAngle &= 180-(27+88) = 65^\circ && \text{The sum of angles in a triangle is 180} \\ \frac{\sin 65}{412} &= \frac{\sin 27}{x} && \text{Law of Sines} \\ x(\sin 65) &= 412(\sin 27) && \text{Cross multiply} \\ x &= \frac{412(\sin 27)}{\sin 65} && \text{Divide by sin 65} \\ x &\approx 206.4 \ miles \end{aligned}$$

The total distance of the modified flight path is $412+206.4 = 618.4$ *miles*.

Solution, Part 2: To find how much farther John had to travel, we need to know the distance of the original flight path, y. We can use the Law of Sines again to find y.

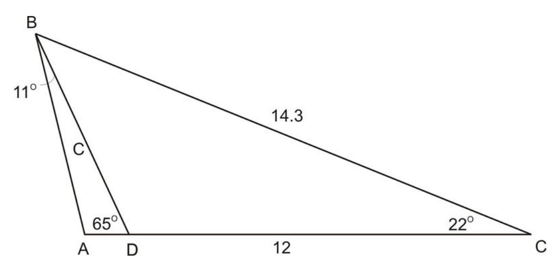
$$\begin{aligned} \frac{\sin 65}{412} &= \frac{\sin 88}{y} && \text{Law of Sines} \\ y(\sin 65) &= 412(\sin 88) && \text{Cross multiply} \\ y &= \frac{412(\sin 88)}{\sin 65} && \text{Divide by } \sin 65 \\ y &\approx 454.3 \ miles \end{aligned}$$

John had to travel $618.4-454.3 = 164.1$ *miles* farther.

Solving Triangles

The Law of Sines can be applied in many ways. Below are some examples of the different ways and situations to which we may apply the Law of Sines. In many ways, the Law of Sines is much easier to use than the Law of Cosines since there is much less computation involved.

Example 6: In the figure below, $\angle C = 22^\circ, BC = 12, DC = 14.3, \angle BDA = 65^\circ$, and $\angle ABD = 11^\circ$. Find AB.



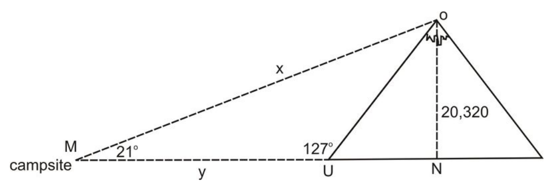
Solution: In order to find AB, we need to know one side in $\triangle ABD$. In $\triangle BCD$, we know two sides and an angle, which means we can use the Law of Cosines to find BD. In this case, we will refer to side BD as c.

$$\begin{aligned} c^2 &= 12^2+14.3^2-2(12)(14.3)\cos 22 && \text{Law of Cosines} \\ c^2 &\approx 30.28 \\ c &\approx 5.5 \end{aligned}$$

Now that we know $BD \approx 5.5$, we can use the Law of Sines to find AB. In this case, we will refer to AB as x.

$$\begin{aligned}
\angle A &= 180-(11+65) = 104^\circ && \text{Triangle Sum Theorem}\\
\frac{\sin 104}{5.5} &= \frac{\sin 65}{x} && \text{Law of Sines}\\
x &= \frac{5.5 \sin 65}{\sin 104} && \text{Cross multiply and divide by } \sin 104\\
x &\approx 5.14
\end{aligned}$$

Example 7: A group of forest rangers are hiking through Denali National Park towards Mt. McKinley, the tallest mountain in North America. From their campsite, they can see Mt. McKinley, and the angle of elevation from their campsite to the summit is 21°. They know that the slope of mountain forms a 127° angle with ground and that the vertical height of Mt. McKinley is 20,320 feet. How far away is their campsite from the base of the mountain? If they can hike 2.9 miles in an hour, how long will it take them to get the base?



Solution: As you can see from the figure above, we have two triangles to deal with here: a right triangle ($\triangle MON$) and non-right triangle ($\triangle MOU$). In order to find the distance from the campsite to the base of the mountain, y, we first need to find one side of our non-right triangle, $\triangle MOU$. If we look at $\angle M$ in $\triangle MNO$, we can see that side ON is our opposite side and side x is our hypotenuse. Remember that the sine function is opposite/hypotenuse. Therefore we can find side x using the sine function.

$$\begin{aligned}
\sin 21^\circ &= \frac{20320}{x}\\
x \sin 21^\circ &= 20320\\
x &= \frac{20320}{\sin 21^\circ}\\
x &\approx 56701.5
\end{aligned}$$

Now that we know side x, we know two angles and the non-included side in $\triangle MOU$. We can use the Law of Sines to solve for side y. First, $\angle MOU = 180^\circ - 127^\circ - 21^\circ = 32^\circ$ by the Triangle Sum Theorem.

$$\begin{aligned}
\frac{\sin 127^\circ}{56701.5} &= \frac{\sin 32^\circ}{y}\\
y \sin 127^\circ &= 56701.5 \sin 32^\circ\\
y &= \frac{56701.5 \sin 32^\circ}{\sin 127^\circ}\\
x &\approx 37623.2 \text{ or } 7.1 \; miles
\end{aligned}$$

If they can hike 2.9 miles per hour, then they will hike the 7.1 miles in 2.45 hours, or 2 hours and 27 minutes.

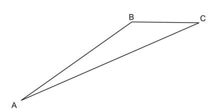
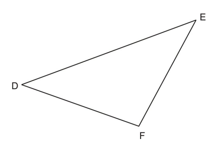
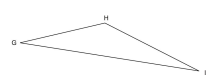
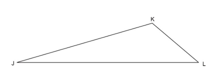
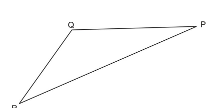
Points to Consider

- Are there any situations where we might not be able to use the Law of Sines or the Law of Cosines?
- Considering what you already know about the sine function, is it possible for two angles to have the same sine? How might this affect using the Law of Sines to solve for an angle?
- By using both the Law of Sines and the Law of Cosines, it is possible to solve any triangle we are given?

Review Questions

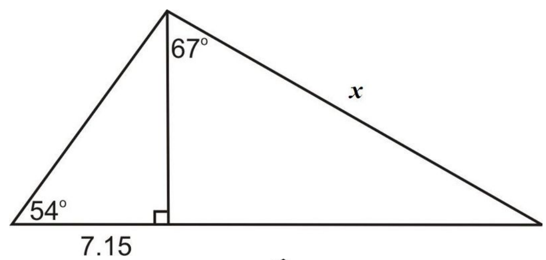
1. In the table below, you are given a figure and information known about that figure. Decide if each situation represents the AAS case or the ASA case.

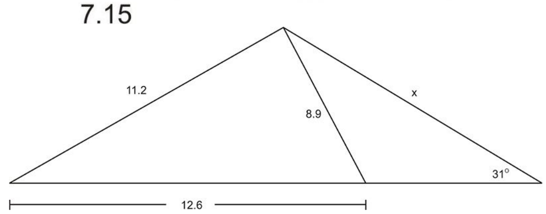
TABLE 5.4:

Given	Figure	Case
a. $b = 16, A = 11.7^\circ, C = 23.8^\circ$		
b. $e = 214.9, D = 39.7^\circ, E = 41.3^\circ$		
c. $G = 22^\circ, I = 18^\circ, H = 140^\circ$		
d. $k = 6.3, J = 16.2^\circ, L = 40.3^\circ$		
e. $M = 31^\circ, O = 9^\circ, m = 15$		
f. $Q = 127^\circ, R = 21.8^\circ, r = 3.62$		

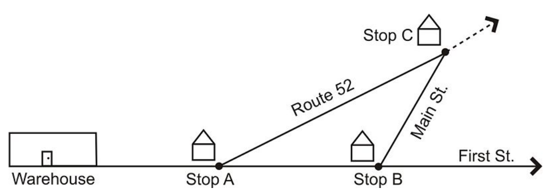
2. Even though ASA and AAS triangles represent two different cases of the Law of Sines, what do they both have in common?

3. Using the figures and the given information from the table above, find the following if possible:
   a. side a
   b. side d
   c. side i
   d. side l
   e. side o
   f. side q
4. In $\triangle GHI, \angle I = 21.3^\circ, \angle H = 62.1^\circ$, and $i = 108$. Find g and h.
5. Use the Law of Sines to show that $\frac{a}{b} = \frac{\sin A}{\sin B}$ is true.
6. Use the Law of Sines, the Law of Cosines, and trigonometry functions to solve for x.

a. 

b. 

7. In order to avoid a storm, a pilot starts out 11° off path. After he has flown 218 miles, he turns the plane toward his destination. The angle formed between his first path and his second path is 105°. If the plane traveled at an average speed of 495 miles per hour, how much longer did the modified flight take?
8. A delivery truck driver has three stops to make before she must return to the warehouse to pick up more packages. The warehouse, Stop A, and Stop B are all on First Street. Stop A is on the corner of First Street and Route 52, which intersect at a 41° angle. Stop B is on the corner of First Street and Main Street, which intersect at a 103° angle. Stop C is at the intersection of Main Street and Route 52. The driver knows that Stop A and Stop B are 12.3 miles apart and that the warehouse is 1.1 miles from Stop A. If she must be back to the warehouse by 10:00 a.m., travels at a speed of 45 MPH, and takes 2 minutes to deliver each package, at what time must she leave?



5.4 The Ambiguous Case

Learning Objectives

- Find possible triangles given two sides and an angle (SSA).
- Use the Law of Cosines and Sines in various ambiguous cases.

In previous sections, we learned about the Law of Cosines and the Law of Sines. We learned that we can use the Law of Cosines when:

1. we know all three sides of a triangle (SSS) and
2. we know two sides and the included angle (SAS).

We learned that we can use the Law of Sines when:

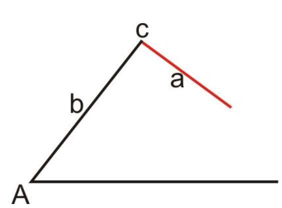
1. we know two angles and a non-included side (AAS) and
2. we know two angles and the included side (ASA).

However, we have not explored how to approach a triangle when we know two sides and a *non*-included angle (SSA). In this section, we will look at why the SSA case is called the ambiguous case, the possible triangles formed by the SSA case, and how to apply the Law of Sines and the Law of Cosines when we encounter the SSA case.

Possible Triangles with SSA

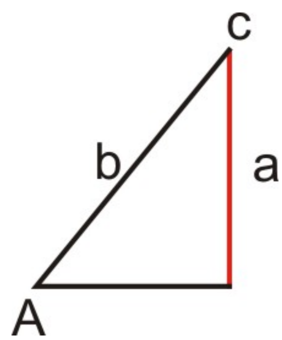
In Geometry, you learned that two sides and a non-included angle do not necessarily define a unique triangle. Consider the following cases given a, b, and $\angle A$:

Case 1: No triangle exists $(a < b)$



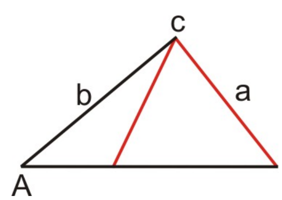
In this case $a < b$ and side a is too short to reach the base of the triangle. Since no triangle exists, there is no solution.

Case 2: One triangle exists $(a < b)$



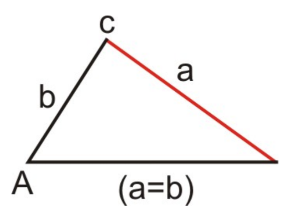
In this case, $a < b$ and side a is perpendicular to the base of the triangle. Since this situation yields exactly one triangle, there is exactly one solution.

Case 3: Two triangles exist $(a < b)$



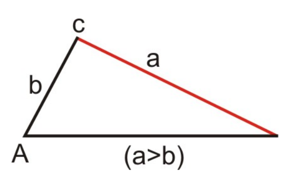
In this case, $a < b$ and side a meets the base at exactly two points. Since two triangles exist, there are two solutions.

Case 4: One triangle exists $(a = b)$



In this case $a = b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 5: One triangle exists $(a > b)$



In this case, $a > b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 3 is referred to as the Ambiguous Case because there are two possible triangles and two possible solutions. One way to check to see how many possible solutions (if any) a triangle will have is to compare sides a and b. If you are faced with the first situation, where $a < b$, we can still tell how many solutions there will be by using a and $b \sin A$.

TABLE 5.5:

	If:	Then:
a.	$a < b$	No solution, one solution, two solutions
i.	$a < b\sin A$	No solution
ii.	$a = b\sin A$	One solution
iii.	$a > b\sin A$	Two solutions
b.	$a = b$	One solution
c.	$a > b$	One solution

Example 1: Determine if the sides and angle given determine no, one or two triangles. All sets contain an angle, its opposite side and the side between them.

a. $a = 5, b = 8, A = 62.19^\circ$

b. $c = 14, b = 10, B = 15.45^\circ$

c. $d = 16, g = 11, D = 44.94^\circ$

d. $a = 9, b = 7, B = 51.06^\circ$

Solution: Even though a, b and $\angle A$ are not used in every example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the angle.

a. $5 < 8, 8\sin 62.19^\circ = 7.076$. So $5 < 7.076$, which means there is no solution.

b. $10 < 14, 14\sin 15.45^\circ = 3.73$. So $10 > 3.73$, which means there are two solutions.

c. $16 > 11$, there is one solution.

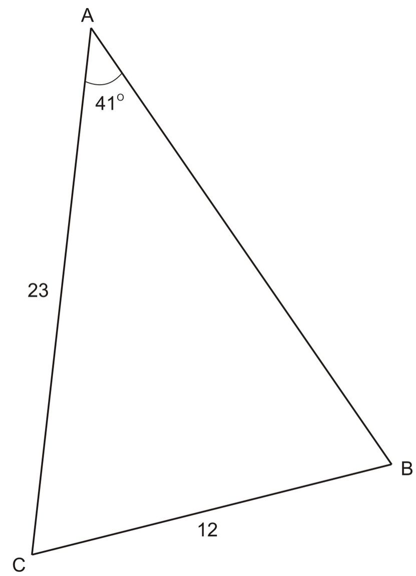
d. $7 < 9, 9\sin 51.06^\circ = 7.00$. So $7 = 7$, which means there is one solution.

In the next two sections we will look at how to use the Law of Cosines and the Law of Sines when faced with the various cases above.

Using the Law of Sines

In triangle ABC below, we know two sides and a non-included angle. Remember that the Law of Sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. Since we know a, b, and $\angle A$, we can use the Law of Sines to find $\angle B$. However, since this is the SSA case, we have to watch out for the Ambiguous case. Since $a < b$, we could be faced with either Case 1, Case 2, or Case 3 above.

Example 2: Find $\angle B$.



Solution: Use the Law of Sines to determine the angle.

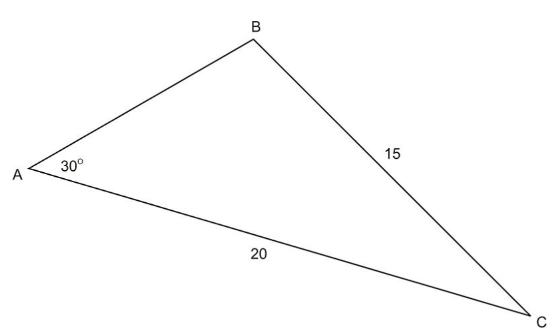
$$\frac{\sin 41}{12} = \frac{\sin B}{23}$$
$$23 \sin 41 = 12 \sin B$$
$$\frac{23 \sin 41}{12} = \sin B$$
$$1.257446472 = \sin B$$

Since no angle exists with a sine greater than 1, there is no solution to this problem.

We also could have compared a and $b \sin A$ beforehand to see how many solutions there were to this triangle.

$a = 12, b \sin A = 15.1$: since $12 < 15.1, a < b \sin A$ which tells us there are no solutions.

Example 3: In triangle $ABC, a = 15, b = 20$, and $\angle A = 30^\circ$. Find $\angle B$.



Solution: Again in this case, $a < b$ and we know two sides and a non-included angle. By comparing a and $b \sin A$, we find that $a = 15, b \sin A = 10$. Since $15 > 10$ we know that there will be two solutions to this problem.

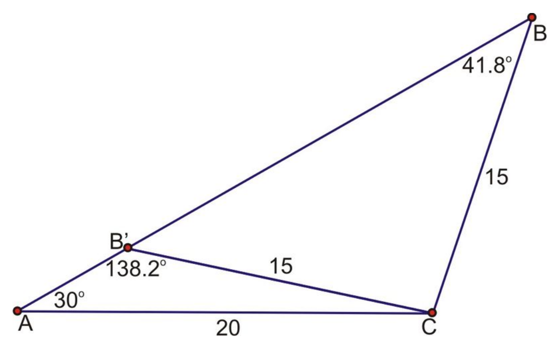
$$\frac{\sin 30}{15} = \frac{\sin B}{20}$$
$$20 \sin 30 = 15 \sin B$$
$$\frac{20 \sin 30}{15} = \sin B$$
$$0.6666667 = \sin B$$
$$\angle B = 41.8^\circ$$

There are two angles less than 180° with a sine of 0.6666667, however. We found the first one, 41.8°, by using the inverse sine function. To find the second one, we will subtract 41.8° from 180°, $\angle B = 180^\circ - 41.8^\circ = 138.2^\circ$.

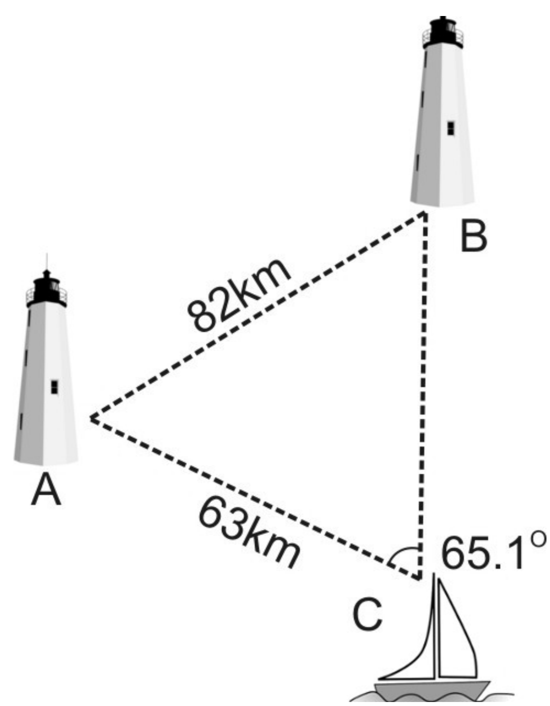
To check to make sure 138.2° is a solution, we will use the Triangle Sum Theorem to find the third angle. Remember that all three angles must add up to 180°.

$$180^\circ - (30^\circ + 41.8^\circ) = 108.2^\circ \quad \text{or} \quad 180^\circ - (30^\circ + 138.2^\circ) = 11.8^\circ$$

This problem yields two solutions. Either $\angle B = 41.8^\circ$ or 138.2°.



Example 4: A boat leaves lighthouse A and travels 63km. It is spotted from lighthouse B, which is 82km away from lighthouse A. The boat forms an angle of 65.1° with both lighthouses. How far is the boat from lighthouse B?



Solution: In this problem, we again have the SSA angle case. In order to find the distance from the boat to the lighthouse (a) we will first need to find the measure of angle A. In order to find angle A, we must first use the Law of Sines to find angle B. Since $c > b$, this situation will yield exactly one answer for the measure of angle B.

$$\begin{aligned}\frac{\sin 65.1^\circ}{82} &= \frac{\sin B}{63}\\ \frac{63\sin 65.1^\circ}{82} &= \sin B\\ 0.6969 &\approx \sin B\\ \angle B &= 44.2^\circ\end{aligned}$$

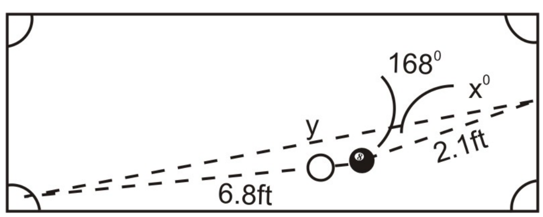
Now that we know the measure of angle B, we can find the measure of angle A, $\angle A = 180^\circ - 65.1^\circ - 44.2^\circ = 70.7^\circ$. Finally, we can use $\angle A$ to find side a.

$$\begin{aligned}\frac{\sin 65.1^\circ}{82} &= \frac{\sin 70.7^\circ}{a}\\ \frac{82\sin 70.7^\circ}{\sin 65.1^\circ} &= a\\ a &= 85.3\end{aligned}$$

The boat is approximately 85.3 km away from lighthouse B.

Using the Law of Cosines

Example 5: In a game of pool, a player must put the eight ball into the bottom left pocket of the table. Currently, the eight ball is 6.8 feet away from the bottom left pocket. However, due to the position of the cue ball, she must bank the shot off of the right side bumper. If the eight ball is 2.1 feet away from the spot on the bumper she needs to hit and forms a 168° angle with the pocket and the spot on the bumper, at what angle does the ball need to leave the bumper?



Note: This is actually a trick shot performed by spinning the eight ball, and the eight ball will not actually travel in straight-line trajectories. However, to simplify the problem, assume that it travels in straight lines.

Solution: In the scenario above, we have the SAS case, which means that we need to use the Law of Cosines to begin solving this problem. The Law of Cosines will allow us to find the distance from the spot on the bumper to the pocket (y). Once we know y, we can use the Law of Sines to find the angle (X).

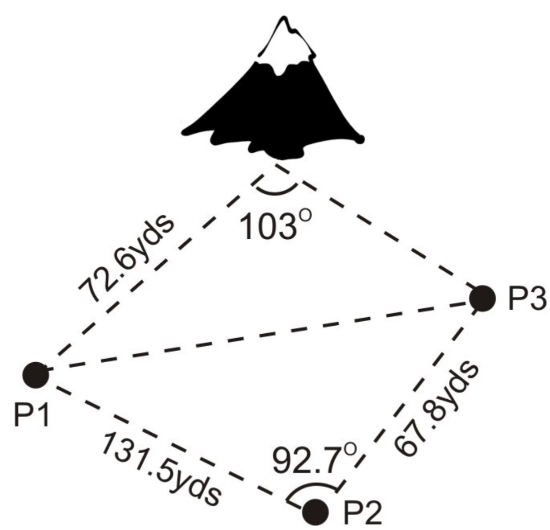
$$\begin{aligned}y^2 &= 6.8^2 + 2.1^2 - 2(6.8)(2.1)\cos 168^\circ\\ y^2 &= 78.59\\ y &= 8.86\ feet\end{aligned}$$

The distance from the spot on the bumper to the pocket is 8.86 feet. We can now use this distance and the Law of Sines to find angle X. Since we are finding an angle, we are faced with the SSA case, which means we could have no solution, one solution, or two solutions. However, since we know all three sides this problem will yield only one solution.

$$\begin{aligned}\frac{\sin 168^\circ}{8.86} &= \frac{\sin X}{6.8}\\ \frac{6.8\sin 168^\circ}{8.86} &= \sin X\\ 0.1596 &\approx \sin B\\ \angle B &= 8.77^\circ\end{aligned}$$

In the previous example, we looked at how we can use the Law of Sines and the Law of Cosines together to solve a problem involving the SSA case. In this section, we will look at situations where we can use not only the Law of Sines and the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios. We will also look at another real-world application involving the SSA case.

Example 6: Three scientists are out setting up equipment to gather data on a local mountain. Person 1 is 131.5 yards away from Person 2, who is 67.8 yards away from Person 3. Person 1 is 72.6 yards away from the mountain. The mountains forms a 103° angle with Person 1 and Person 3, while Person 2 forms a 92.7° angle with Person 1 and Person 3. Find the angle formed by Person 3 with Person 1 and the mountain.



Solution: In the triangle formed by the three people, we know two sides and the included angle (SAS). We can use the Law of Cosines to find the remaining side of this triangle, which we will call x. Once we know x, we will two sides and the non-included angle (SSA) in the triangle formed by Person 1, Person 2, and the mountain. We will then be able to use the Law of Sines to calculate the angle formed by Person 3 with Person 1 and the mountain, which we will refer to as Y.

To find x:

$$\begin{aligned}x^2 &= 131.5^2+67.8^2-2(131.5)(67.8)\cos 92.7\\ x^2 &= 22729.06397\\ x &= 150.8\ yds\end{aligned}$$

Now that we know $x = 150.8$, we can use the Law of Sines to find Y. Since this is the SSA case, we need to check to see if we will have no solution, one solution, or two solutions. Since $150.8 > 72.6$, we know that we will have only one solution to this problem.

$$\frac{\sin 103}{150.8} = \frac{\sin Y}{72.6}$$
$$\frac{72.6 \sin 103}{150.8} = \sin Y$$
$$0.4690932805 = \sin Y$$
$$28.0 \approx \angle Y$$

Points to Consider

- Why is there only one possible solution to the SSA case if $a > b$?
- Explain why $b > a > b \sin A$ yields two possible solutions to a triangle.
- If we have a SSA angle case with two possible solutions, how can we check both solutions to make sure they are correct?

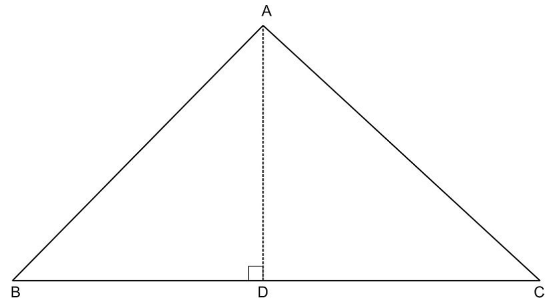
Review Questions

1. Using the table below, determine how many solutions there would be to each problem based on the given information and by calculating $b \sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

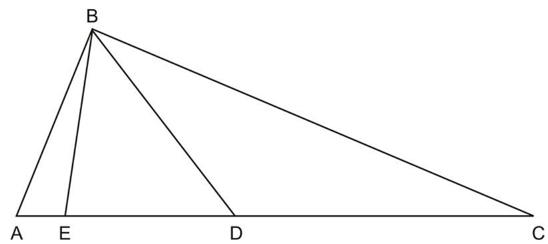
TABLE 5.6:

Given	$a >, =,$ or $< b \sin A$	Diagram	Number of solutions
a. $A = 32.5°, a = 26, b = 37$			
b. $A = 42.3°, a = 16, b = 26$			
c. $A = 47.8°, a = 13.48, b = 18.2$			
d. $A = 51.5°, a = 3.4, b = 4.2$			

2. Using the information in the table above, find all possible measures of angle B if any exist.
3. Prove using the Law of Sines: $\frac{a-c}{c} = \frac{\sin A - \sin C}{\sin C}$
4. Give the measure of a non-included angle and the lengths of two sides so that two triangles exist. Explain why two triangles exist for the measures you came up with.
5. If $a = 22$ and $b = 31$, find the values of A so that:
   a. There is no solution
   b. There is one solution
   c. There are two solutions
6. In the figure below, $AB = 13.7, AD = 9.8$, and $\angle C = 42.6°$. Find $\angle A, \angle B, BC$, and AC.

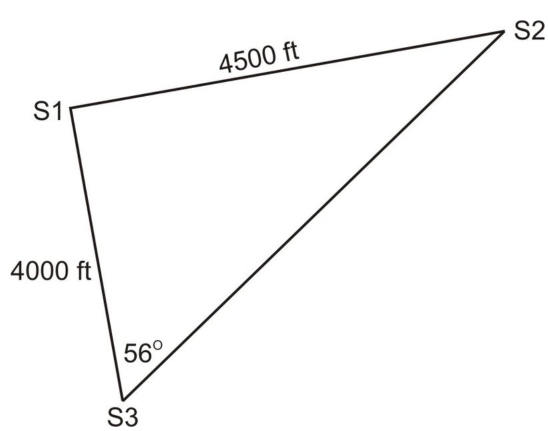


7. In the figure below, $\angle C = 21.8^\circ, BE = 9.9, BD = 10.2, ED = 7.6$, and $\angle ABC = 109.6^\circ$. Find the following:

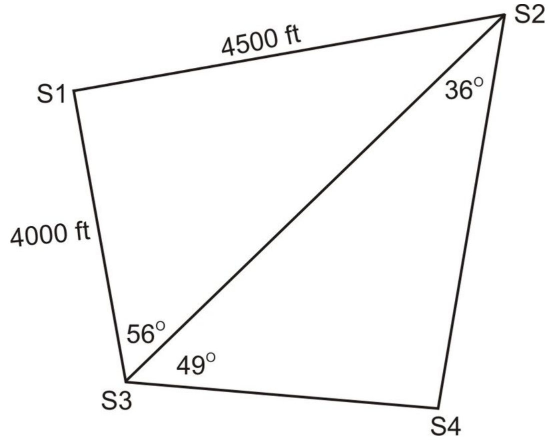


a. $\angle EBD$
b. $\angle BDE$
c. $\angle DEB$
d. $\angle BDC$
e. $\angle BEA$
f. $\angle DBC$
g. $\angle ABE$
h. $\angle BAE$
i. BC
j. AB
k. AE
l. DC
m. AC

8. Radio detection sensors for tracking animals have been placed at three different points in a wildlife preserve. The distance between Sensor 1 and Sensor 2 is 4500ft. The distance between Sensor 1 and Sensor 3 is 4000ft. The angle formed by Sensor 3 with Sensors 1 and 2 is 56°. If the range of Sensor 3 is 6000ft, will it be able to detect all movement from its location to the location of Sensor 2?



9. In problem 8 above, a fourth sensor is placed in the wildlife preserve. Sensor 2 forms a 36 angle with Sensors 3 and 4, and Sensor 3 forms a 49 angle with Sensors 2 and 4. How far away is Sensor 4 from Sensors 2 and 3?


S1
S2
S3
S4
4500 ft
4000 ft
36°
56°
49°

5.5 General Solutions of Triangles

Learning Objectives

- Use the Pythagorean Theorem, trigonometry functions, the Law of Sines, and the Law of Cosines to solve various triangles.
- Understand when it is appropriate to use each method.
- Apply the methods above in real-world and applied problems.

In the previous sections we have discussed a number of methods for finding a missing side or angle in a triangle. Previously, we only knew how to do this in right triangles, but now we know how to find missing sides and angles in oblique triangles as well. By combining all of the methods we've learned up until this point, it is possible for us to find all missing sides and angles in any triangle we are given.

Summary of Triangle Techniques

Below is a chart summarizing the triangle techniques that we have learned up to this point. This chart describes the type of triangle (either right or oblique), the given information, the appropriate technique to use, and what we can find using each technique.

TABLE 5.7:

Type of Triangle:	**Given Information:**	**Technique:**	**What we can find:**
Right	Two sides	Pythagorean Theorem	Third side
Right	One angle and one side	Trigonometric ratios	Either of the other two sides
Right	Two sides	Trigonometric ratios	Either of the other two angles
Oblique	2 angles and a non-included side (AAS)	Law of Sines	The other non-included side
Oblique	2 angles and the included side (ASA)	Law of Sines	Either of the non-included sides
Oblique	2 sides and the angle opposite one of those sides (SSA) –Ambiguous case	Law of Sines	The angle opposite the other side (can yield no, one, or two solutions)
Oblique	2 sides and the included angle (SAS)	Law of Cosines	The third side
Oblique	3 sides	Law of Cosines	Any of the three angles

Using the Law of Cosines

It is possible for us to completely solve a triangle using the Law of Cosines. In order to do this, we will need to apply the Law of Cosines multiple times to find all of the sides and/or angles we are missing.

Example 1: In triangle $ABC, a = 12, b = 13, c = 8$. Solve the triangle.

Solution: Since we are given all three sides in the triangle, we can use the Law of Cosines. Before we can solve the triangle, it is important to know what information we are missing. In this case, we do not know any of the angles, so we are solving for $\angle A, \angle B$, and $\angle C$. We will begin by finding $\angle A$.

$$\begin{aligned} 12^2 &= 8^2 + 13^2 - 2(8)(13)\cos A \\ 144 &= 233 - 208\cos A \\ -89 &= -208\cos A \\ 0.4278846154 &= \cos A \\ 64.7 &\approx \angle A \end{aligned}$$

Now, we will find $\angle B$ by using the Law of Cosines. Keep in mind that you can now also use the Law of Sines to find $\angle B$. Use whatever method you feel more comfortable with.

$$\begin{aligned} 13^2 &= 8^2 + 12^2 - 2(8)(12)\cos B \\ 169 &= 208 - 192\cos B \\ -39 &= -192\cos B \\ 0.2031 &= \cos B \\ 78.3^\circ &\approx \angle B \end{aligned}$$

We can now quickly find $\angle C$ by using the Triangle Sum Theorem, $180^\circ - 64.7^\circ - 78.3^\circ = 37^\circ$

Example 2: In triangle $DEF, d = 43, e = 37$, and $\angle F = 124^\circ$. Solve the triangle.

Solution: In this triangle, we have the SAS case because we know two sides and the included angle. This means that we can use the Law of Cosines to solve the triangle. In order to solve this triangle, we need to find side $f, \angle D$, and $\angle E$. First, we will need to find side f using the Law of Cosines.

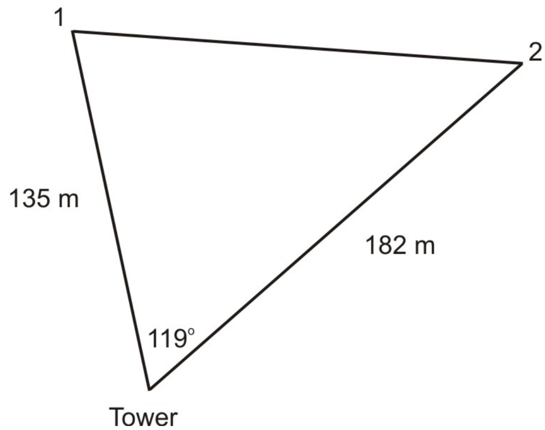
$$\begin{aligned} f^2 &= 43^2 + 37^2 - 2(43)(37)\cos 124 \\ f^2 &= 4997.351819 \\ f &\approx 70.7 \end{aligned}$$

Now that we know f, we know all three sides of the triangle. This means that we can use the Law of Cosines to find either angle D or angle E. We will find angle D first.

$$\begin{aligned} 43^2 &= 70.7^2 + 37^2 - 2(70.7)(37)\cos D \\ 1849 &= 6367.49 - 5231.8\cos D \\ -4518.49 &= -5231.8\cos D \\ 0.863658779 &= \cos D \\ 30.3^\circ &\approx \angle D \end{aligned}$$

To find angle E, we need only to use the Triangle Sum Theorem, $\angle E = 180 - (124 + 30.3) = 25.7°$.

Example 3: A control tower is receiving signals from two microchips implanted in wild tigers. Microchip 1 is 135 miles from the control tower and microchip 2 is 182 miles from the control tower. If the control tower forms a 119° angle with both microchips, how far apart are the two tigers?



Solution: To find the distance between the two tigers, we need to find the distance between the two microchips. We will call this distance x. Since we know two sides and the included angle, we can use the Law of Cosines to find x.

$$\begin{aligned} x^2 &= 135^2 + 182^2 - 2(135)(182)\cos 119 \\ x^2 &= 75172.54474 \\ x &= 274.2 \ miles \end{aligned}$$

The two tigers are 274.2 miles apart.

Using the Law of Sines

It is also possible for us to completely solve a triangle using the Law of Sines if we begin with the ASA case, the AAS case, or the SSA case. We must remember that when given the SSA case, it is possible that we may encounter the Ambiguous Case.

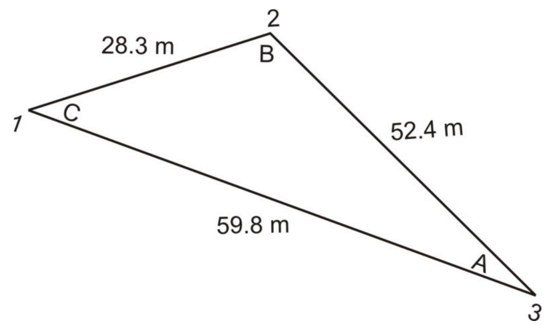
Example 4: In triangle $ABC, A = 43°, B = 82°$, and $c = 10.3$. Solve the triangle.

Solution: This is an example of the ASA case, which means that we can use the Law of Sines to solve the triangle. In order to use the Law of Sines, we must first know angle C, which we can find using the Triangle Sum Theorem, $\angle C = 180° - (43° + 82°) = 55°$.

Now that we know $\angle C$, we can use the Law of Sines to find either side a or side b.

$$\begin{aligned} \frac{\sin 55}{10.3} &= \frac{\sin 43}{a} \\ a &= \frac{10.3 \sin 43}{\sin 55} \\ a &= 8.6 \end{aligned} \qquad \begin{aligned} \frac{\sin 55}{10.3} &= \frac{\sin 82}{b} \\ b &= \frac{10.3 \sin 82}{\sin 55} \\ b &= 12.5 \end{aligned}$$

Example 5: A cruise ship is based at Island 1, but makes trips to Island 2 and Island 3 during the day. Island 3 lies directly east of Island 1. If the distance from Island 1 to Island 2 is 28.3 miles, from Island 2 to 3 is 52.4 miles, and Island 3 to 1 is 59.8 miles, what heading (angle) must the captain:



a. Leave Island 1

b. Leave Island 2

c. Leave Island 3

Solution: In order to find all three angles in the triangle, we must use the Law of Cosines because we are dealing with the SSS case. Once we find one angle using the Law of Cosines, we can use the Law of Sines to find a second angle. Then, we can use the Triangle Sum Theorem to find the third angle.

We will begin by finding $\angle B$ because it is the largest angle.

$$\begin{aligned}59.8^2 &= 52.4^2 + 28.3^2 - 2(52.4)(28.3)\cos B\\3576.04 &= 3546.65 - 2965.84\cos B\\29.39 &= -2965.84\cos B\\-0.0099095029 &= \cos B\\B &= 90.6^\circ\end{aligned}$$

Now that we know $\angle B$, we can find either $\angle A$ or $\angle C$. We will find $\angle C$ first since it is the second largest angle.

$$\begin{aligned}\frac{\sin 90.6}{59.8} &= \frac{\sin C}{52.4}\\\frac{52.4\sin 90.6}{59.8} &= \sin C\\0.876203135 &= \sin C\\\angle C &= 61.2^\circ\end{aligned}$$

Now that we know $\angle B$ and $\angle C$, we can use the Triangle Sum Theorem to find $\angle A = 180^\circ - (61.2^\circ + 90.6^\circ) = 28.2^\circ$.

Now, we must convert our angles into headings.

When going from Island 1 to Island 2, 61.2° would be a heading of $N28.8^\circ E$. Also, since $\angle A = 28.2^\circ$, if we were to travel from Island 3 to Island 2, our heading would be $W28.2^\circ N$. This means that when going from Island 2 to Island 3, the heading would be in the exact opposite direction, or $E28.2^\circ S$. When going from Island 3 back to Island 1, since we know that Island 3 is directly east of Island 1, the captain must now sail in the direction opposite of east, or directly west, which is $N90^\circ W$.

Points to Consider

- Is there ever a situation where you would need to use the Law of Sines *before* using the Law of Cosines?
- In what situation might you consider using the Law of Cosines instead of Law of Sines if both were applicable?
- Why do we only have to use the Law of Cosines one time before we can switch to using the Law of Sines?

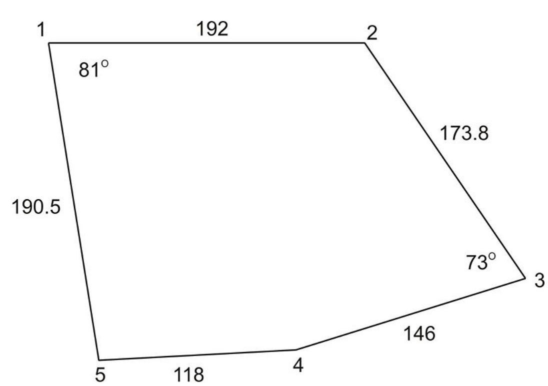
Review Questions

1. Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of each triangle and label the given information. Also, state how many solutions (if any) each triangle would have. If a triangle has no solution or two solutions, explain why.

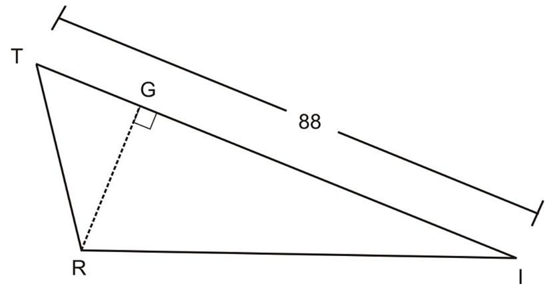
TABLE 5.8:

Given	Drawing	Case	Law	Number of Solutions
a. $A = 69°, B = 12°, a = 22.3$, find b.				
b. $a = 1.4, b = 2.3, C = 58°$, find c.				
c. $a = 3.3, b = 6.1, c = 4.8$, find A.				
d. $a = 15, b = 25, A = 58°$, find B.				
e. $a = 45, b = 60, A = 47°$, find B.				

2. Using the information in the chart above, solve for the requested side or angle.
3. Using the information in the chart in question 1 and your answers from question 2, determine what information you are still missing from each triangle. Then, solve for each piece, solving each triangle.
4. The side of a rhombus is 12 cm and the longer diagonal is 21.5cm. Find the area of the rhombus and the measures of the angles in the rhombus.
5. Find the area of the pentagon below.



6. In the figure drawn below, angle T is 56.8° and $RT = 38$. Using the figure below, find the length of the altitude drawn to the longest side, the area of the two triangles formed by this altitude, RI and angle I.



7. Refer back to Example 5, the island hopping problem. Suppose the cruise ship were based on Island 3 and traveled to Island 2 and then to Island 1, before returning to Island 3.
   a. What would be its heading when going from Island 3 to Island 2?
   b. What would be its heading when going from Island 2 to Island 1?
   c. What would be its heading when going from Island 1 back to Island 3?
8. A golfer is standing on the tee of a golf hole that has a 115° bend to the left. The distance from the tee to the bend is 218 yards. The distance from the bend to the green is 187 yards.
   a. How far would the golfer need to hit the ball if he wanted to make it to the green in one shot?
   b. At what angle would he need to hit the ball?
9. A golfer is standing on the tee, which is 320 yards from the cup on the green. After he hits his first shot, which is sliced to the right, his ball forms a 162.2° angle with the tee and the cup, and the cup forms a 14.2° angle with his ball and the tee.
   a. What is the degree of his slice?
   b. How far was his first shot?
   c. How far away from the cup is he?

5.6 Vectors

Learning Objectives

- Understand directed line segments, equal vectors, and absolute value in relation to vectors.
- Perform vector addition and subtraction.
- Find the resultant vector of two displacements.

In previous examples, we could simply use triangles to represent direction and distance. In real-life, there are typically other factors involved, such as the speed of the object (that is moving in the given direction and distance) and wind. We need another tool to represent not only direction but also magnitude (length) or force. This is why we need vectors. Vectors capture the interactions of real world velocities, forces and distance changes.

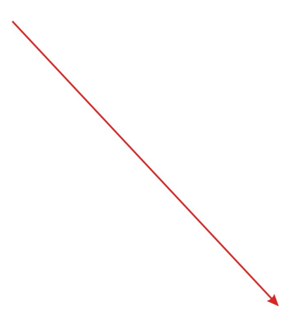
Any application in which direction is specified requires the use of vectors. A **vector** is any quantity having **direction** and **magnitude**. Vectors are very common in science, particularly physics, engineering, electronics, and chemistry in which one must consider an object's motion (either velocity or acceleration) and the direction of that motion.

In this section, we will look at how and when to use vectors. We will also explore vector addition, subtraction, and the resultant of two displacements. In addition we will look at real-world problems and application involving vectors.

Directed Line Segments, Equal Vectors, and Absolute Value

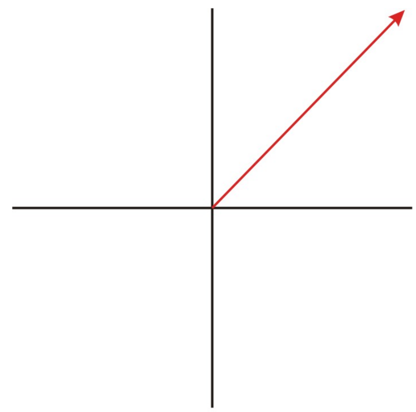
A vector is represented diagrammatically by a directed line segment or arrow. A **directed line segment** has both **magnitude** and **direction**. **Magnitude** refers to the length of the directed line segment and is usually based on a scale. The vector quantity represented, such as influence of the wind or water current may be completely invisible.

A 25 mph wind is blowing from the northwest. If 1 $cm = 5\ mph$, then the vector would look like this:



An object affected by this wind would travel in a southeast direction at 25 mph.

A vector is said to be in **standard position** if its **initial point** is at the origin. The initial point is where the vector begins and the **terminal point** is where it ends. The axes are arbitrary. They just give a place to draw the vector.



vector in standard position

If we know the coordinates of a vector's initial point and terminal point, we can use these coordinates to find the magnitude and direction of the vector.

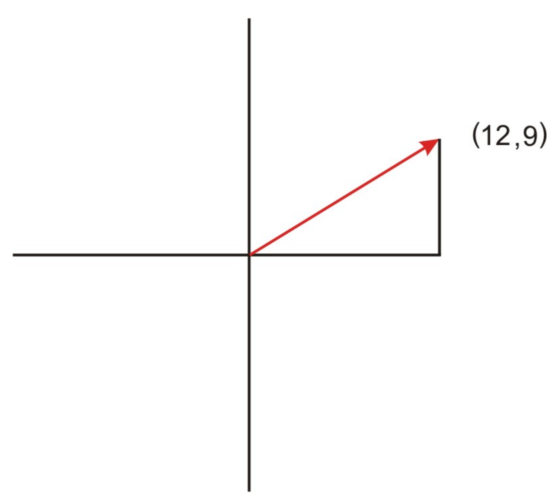
All vectors have **magnitude**. This measures the total distance moved, total velocity, force or acceleration. "Distance" here applies to the magnitude of the vector even though the vector is a measure of velocity, force, or acceleration. In order to find the magnitude of a vector, we use the distance formula. A vector can have a negative magnitude. A force acting on a block pushing it at 20 lbs north can be also written as vector acting on the block from the south with a magnitude of -20 lbs. Such negative magnitudes can be confusing; making a diagram helps. The -20 lbs south can be re-written as +20 lbs north without changing the vector. Magnitude is also called the **absolute value** of a vector.

Example 1: If we know the coordinates of the initial point and the terminal point, we can find the magnitude by using the distance formula. Initial point (0,0) and terminal point (3,5).

Solution: $|\vec{v}| = \sqrt{(3-0)^2+(5-0)^2} = \sqrt{9+25} = 5.8$ The magnitude of $\vec{v}$ is 5.8.

If we don't know the coordinates of the vector, we must use a ruler and the given scale to find the magnitude. Also notice the notation of a vector, which is usually a lower case letter (typically u, v, or w) in italics, with an arrow over it, which indicates direction. If a vector is in standard position, we can use trigonometric ratios such as sine, cosine and tangent to find the **direction** of that vector.

Example 2: If a vector is in standard position and its terminal point has coordinates of (12, 9) what is the direction?



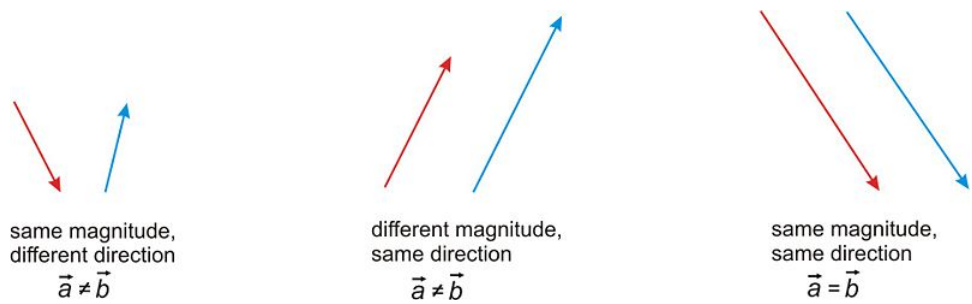
Solution: The horizontal distance is 12 while the vertical distance is 9. We can use the tangent function since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{9}{12}$$
$$\tan^{-1}\frac{9}{12} = 36.9^\circ$$

So, the direction of the vector is 36.9°.

If the vector isn't in standard position and we don't know the coordinates of the terminal point, we must a protractor to find the direction.

Two vectors are **equal** if they have the same magnitude and direction. Look at the figures below for a visual understanding of **equal vectors**.



Example 3: Determine if the two vectors are equal.

$\vec{a}$ is in standard position with terminal point (-4, 12)

$\vec{b}$ has an initial point of (7, -6) and terminal point (3, 6)

Solution: You need to determine if both the magnitude and the direction are the same.

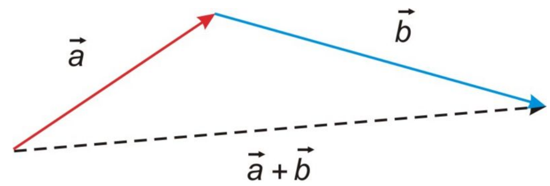
$$\text{Magnitude}: \quad |\vec{a}| = \sqrt{(0-(-4))^2+(0-12)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}$$
$$|\vec{b}| = \sqrt{(7-3)^2+(-6-6)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}$$
$$\text{Direction}: \quad \vec{a} \rightarrow \tan\theta = \frac{12}{-4} \rightarrow \theta = 108.43^\circ$$
$$\vec{b} \rightarrow \tan\theta = \frac{-6-6}{7-3} = \frac{-12}{4} \rightarrow \theta = 108.43^\circ$$

Because the magnitude and the direction are the same, we can conclude that the two vectors are equal.

Vector Addition

The sum of two or more vectors is called the **resultant** of the vectors. There are two methods we can use to find the resultant: the triangle method and the parallelogram method.

The Triangle Method: To use the triangle method, we draw the vectors one after another and place the initial point of the second vector at the terminal point of the first vector. Then, we draw the resultant vector from the initial point of the first vector to the terminal point of the second vector. *This method is also referred to as the tip-to-tail method.*



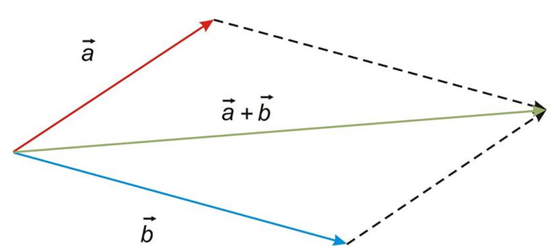
To find the sum of the resultant vector we would use a ruler and a protractor to find the magnitude and direction.

The resultant vector can be much longer than either $\vec{a}$ or $\vec{b}$, or it can be shorter. Below are some more examples of the triangle method.

Example 4:

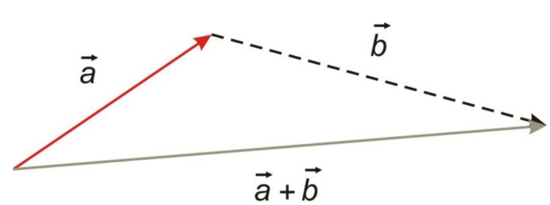
Example 5:

The Parallelogram Method: Another method we could use is the parallelogram method. To use the parallelogram method, we draw the vectors so that their initial points meet. Then, we draw in lines to form a parallelogram. The resultant is the diagonal from the initial point to the opposite vertex of the parallelogram. *It is important to note that we cannot use the parallelogram method to find the sum of a vector and itself.*



To find the sum of the resultant vector, we would again use a ruler and a protractor to find the magnitude and direction.

If you look closely, you'll notice that the parallelogram method is really a version of the triangle or tip-to-tail method. If you look at the top portion of the figure above, you can see that one side of our parallelogram is really vector *b* translated.

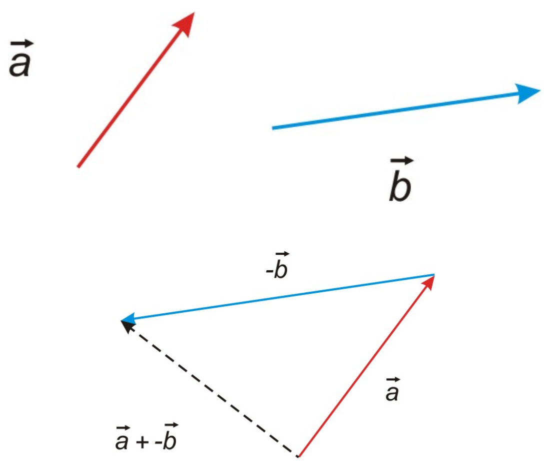


Vector Subtraction

As you know from Algebra, $A - B = A + (-B)$. When we think of vector subtraction, we must think about it in terms of adding a negative vector. A **negative** vector is the same magnitude of the original vector, but its direction is opposite.

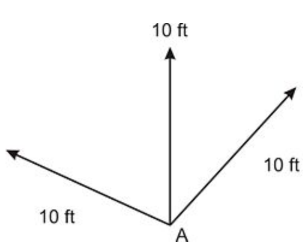
In order to subtract two vectors, we can use either the triangle method or the parallelogram method from above. The only difference is that instead of adding vectors A and B, we will be adding A and $-B$.

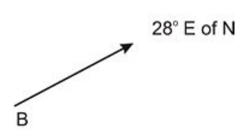
Example 6: Using the triangle method for subtraction.



Resultant of Two Displacements

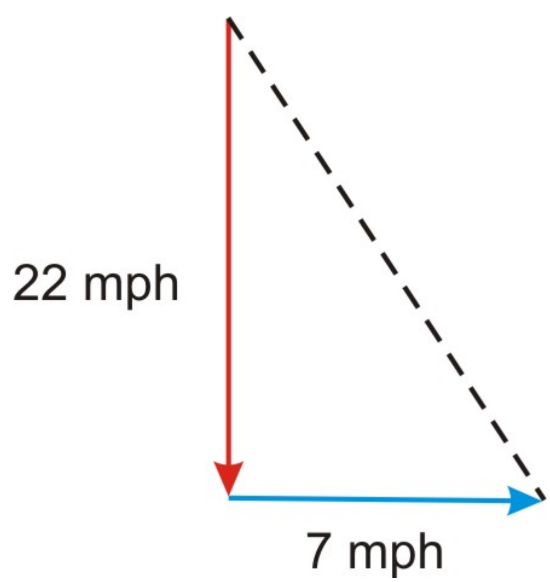
We can use vectors to find direction, velocity, and force of moving objects. In this section we will look at a few applications where we will use resultants of vectors to find speed, direction, and other quantities. A displacement is a distance considered as a vector. If one is 10 ft away from a point, then any point at a radius of 10 ft from that point satisfies the condition. If one is 28 degrees to the east of north, then only one point satisfies this.





Example 7: A cruise ship is traveling south at 22 mph. A wind is also blowing the ship eastward at 7 mph. What speed is the ship traveling at and in what direction is it moving?

Solution: In order to find the direction and the speed the boat is traveling, we must find the resultant of the two vectors representing 22 mph south and 7 mph east. Since these two vectors form a right angle, we can use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the resultant vector.



First, we will find the speed.

$$\begin{aligned} 22^2 + 7^2 &= x^2 \\ 533 &= x^2 \\ 23.1 &= x \end{aligned}$$

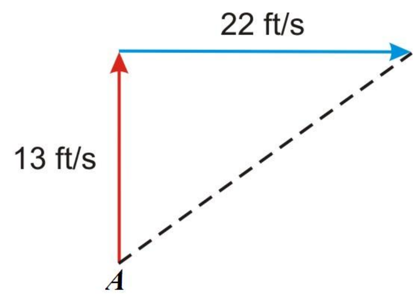
The ship is traveling at a speed of 23.1mph.

To find the direction, we will use tangent, since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{7}{22}$$
$$\tan^{-1}\frac{7}{22} = 17.7^\circ$$

The ship's direction is $S17.7^\circ E$.

Example 8: A hot air balloon is rising at a rate of 13 ft/sec, while a wind is blowing at a rate of 22 ft/sec. Find the speed at which the balloon is traveling as well as its angle of elevation.



First, we will find the speed at which our balloon is rising. Since we have a right triangle, we can use the Pythagorean Theorem to find calculate the magnitude of the resultant.

$$x^2 = 13^2 + 22^2$$
$$x^2 = 653$$
$$x = 25.6\ ft/\text{sec}$$

The balloon is traveling at rate of 25.6 feet per second.

To find the angle of elevation of the balloon, we need to find the angle it makes with the horizontal. We will find the angle A in the triangle and then we will subtract it from 90°.

$$\tan A = \frac{22}{13}$$
$$A = \tan^{-1}\frac{22}{13}$$
$$A = 59.4^\circ$$

Angle with the horizontal $= 90 - 59.4 = 30.6^\circ$.

The balloon has an angle of elevation of 30.6°.

Example 9: Continuing on with the previous example, find:

a. How far from the lift off point is the balloon in 2 hours? Assume constant rise and constant wind speed. (this is *total displacement*)

b. How far must the support crew travel on the ground to get under the balloon? (*horizontal displacement*)

c. If the balloon stops rising after 2 hours and floats for another 2 hours, how far from the initial point is it at the end of the 4 hours? How far away does the crew have to go to be under the balloon when it lands?

Solution:

a. After two hours, the balloon will be 184,320 feet from the lift off point (25.6 ft/sec multiplied by 7200 seconds in two hours).

b. After two hours, the horizontal displacement will be 158,400 feet (22ft/sec multiplied by 7200 seconds in two hours).

c. After two hours, the balloon will have risen 93,600 feet. After an additional two hours of floating (horizontally only) in the 22ft/sec wind, the balloon will have traveled 316,800 feet horizontally (22ft/second times 14,400 seconds in four hours).

We must recalculate our resultant vector using Pythagorean Theorem.

$$x = \sqrt{93600^2 + 316800^2} = 330338\ ft.$$

The balloon is 330,338 feet from its initial point. The crew will have to travel 316,800 feet or 90 miles (horizontal displacement) to be under the balloon when it lands.

Points to Consider

- Is it possible to find the magnitude and direction of resultants without using a protractor and ruler and without using right triangles?
- How can we use the Law of Cosines and the Law of Sines to help us find magnitude and direction of resultants?

Review Questions

1. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if:
   a. $|\vec{m}| = 29.8 |\vec{n}| = 37.7$
   b. $|\vec{m}| = 2.8 |\vec{n}| = 5.4$
   c. $|\vec{m}| = 11.9 |\vec{n}| = 9.4$
2. For $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ below, make a diagram of each addition or subtraction. $|\vec{a}| = 6cm$, direction $= 45° |\vec{b}| = 3.2cm$, direction $= 30° |\vec{c}| = 1.3cm$, direction $= 110° |\vec{d}| = 4.8cm$, direction $= 80°$
   a. $\vec{a} + \vec{b}$
   b. $\vec{a} + \vec{d}$
   c. $\vec{c} + \vec{d}$
   d. $\vec{a} - \vec{d}$
   e. $\vec{b} - \vec{a}$
   f. $\vec{d} - \vec{c}$
3. Does $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$? Explain your answer.
4. A plane is traveling north at a speed of 225 mph while an easterly wind is blowing the plane west at 18 mph. What is the direction and the speed of the plane?
5. Two workers are pulling on ropes attached to a tree stump. One worker is pulling the stump east with 330 Newtons of forces while the second working is pulling the stump north with 410 Newtons of force. Find the magnitude and direction of the resultant force on the tree stump.
6. Assume $\vec{a}$ is in standard position. For each terminal point is given, find the magnitude and direction of each vector.

a. (12, 18)
b. (-3, 6)

7. Given the initial and terminal coordinates of $\vec{a}$, find the magnitude and direction.

a. initial (2, 4) terminal (8, 6)
b. initial (5, -2) terminal (3, 1)

8. The magnitudes of vectors $\vec{a}$ and $\vec{b}$ are given, along with the angle they make with each other, theta, when positioned tip-to-tail. Find the magnitude of the resultant and the angle it makes with *a*.

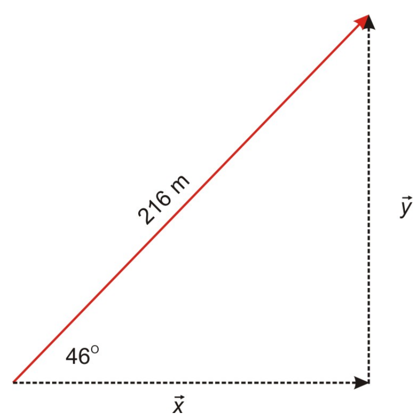
a. $|\vec{a}| = 31, |\vec{b}| = 31, \theta = 132^\circ$
b. $|\vec{a}| = 29, |\vec{b}| = 44, \theta = 26^\circ$

5.7 Component Vectors

Learning Objectives

- Perform scalar multiplication with vectors.
- Find the resultant as a sum of two components.
- Find the resultant as magnitude and direction.
- Use component vectors to solve real-world and applied problems.

A car has traveled 216 miles in a direction of 46° north of east. How far east of its initial point has it traveled? How far north has the car traveled?



The car traveled on a vector distance called a displacement. It moved in a line to a particular distance from the starting point. Having two **components** in their expression, vectors are confusing to some. A diagram helps sort out confusion. Looking at vectors by separating them into components allows us to deal with many real-world problems. The components often relate to very different elements of the problem, such as wind speed in one direction and speed supplied by a motor in another.

In order to find how far the car has traveled east and how far it has traveled north, we will need to find the horizontal and vertical components of the vector. To find $\vec{x}$, we use cosine and to find $\vec{y}$ we use sine.

$$\begin{aligned} \cos 46 &= \frac{|\vec{x}|}{216} = \frac{x}{216} \\ \cos 46 &= \frac{x}{216} \\ 216 \cos 46 &= x \\ x &= 150.0 \end{aligned} \qquad \begin{aligned} \sin 46 &= \frac{|\vec{y}|}{216} = \frac{y}{216} \\ \sin 46 &= \frac{y}{216} \\ 216 \sin 46 &= y \\ y &= 155.4 \end{aligned}$$

In this section, we will learn about component vectors and how to find them. We will also explore other ways of finding the magnitude and direction of a resultant of two or more vectors. We will be using many of the tools we learned in the previous sections dealing with right and oblique triangles.

Vector Multiplied by a Scalar

In working with vectors there are two kinds of quantities employed. The first is the vector, a quantity that has both magnitude and direction. The second quantity is a scalar. Scalars are just numbers. The magnitude of a vector is a scalar quantity. A vector can be multiplied by a real number. This real number is called a **scalar**. The product of a vector $\vec{a}$ and a scalar k is a vector, written $\vec{ka}$. It has the same direction as $\vec{a}$ with a magnitude of $k|\vec{a}|$ if $k > 0$. If $k < 0$, the vector has the opposite direction of $\vec{a}$ and a magnitude of $k|\vec{a}|$.

Example 1: The speed of the wind before a hurricane arrived was 20 mph from the SSE ($N22.5°W$). It quadrupled when the hurricane arrived. What is the current vector for wind velocity?

Solution: The wind is coming now at 80 mph from the same direction.

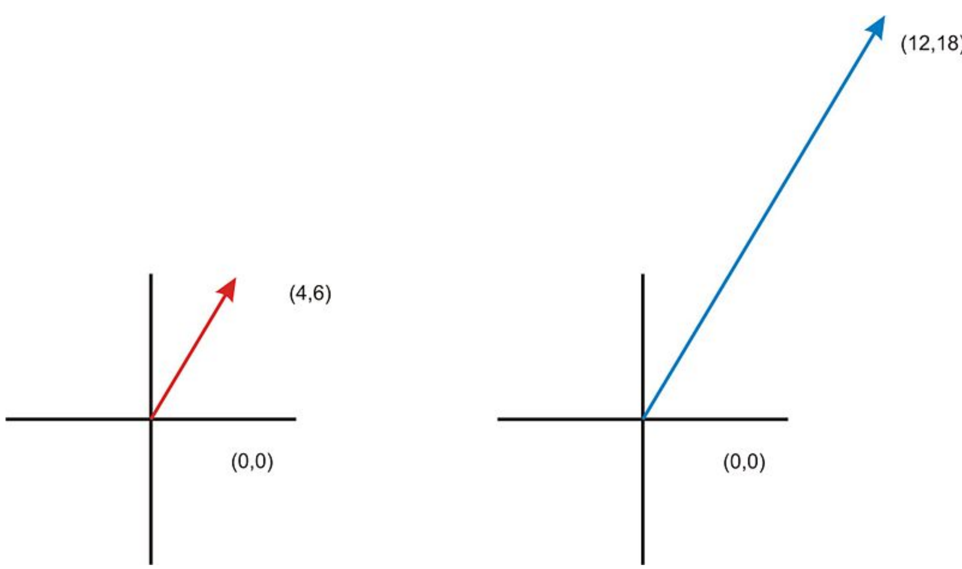
Example 2: A sailboat was traveling at 15 knots due north. After realizing he had overshot his destination, the captain turned the boat around and began traveling twice as fast due south. What is the current velocity vector of the ship?

Solution: The ship is traveling at 30 knots in the opposite direction.

If the vector is expressed in coordinates with the starting end of the vector at the origin, this is called standard form. To perform a scalar multiplication, we multiply our scalar by both the coordinates of our vector. The word scalar comes from “scale.” Multiplying by a scalar just makes the vectors longer or shorter, but doesn’t change their direction.

Example 3: Consider the vector from the origin to (4, 6). What would the representation of a vector that had three times the magnitude be?

Solution: Here $k = 3$ and $\vec{v}$ is the directed segment from (0,0) to (4, 6).



Multiply each of the components in the vector by 3.

$$\vec{kv} = (0,0)\ to\ (12,18)$$

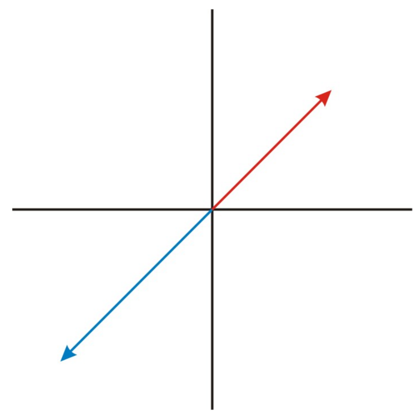
The new coordinates of the directed segment are (0, 0), (12, 18).

Example 4: Consider the vector from the origin to (3, 5). What would the representation of a vector that had -2 times the magnitude be?

Solution: Here, $k = -2$ and $\vec{v}$ is the directed segment from (0, 0) to (3, 5).

$$\vec{kv} = (-2(3), -2(5)) = (-6, -10)$$

Since $k < 0$, our result would be a directed segment that is twice and long but in the opposite direction of our original vector.



Translation of Vectors and Slope

What would happen if we performed scalar multiplication on a vector that didn't start at the origin?

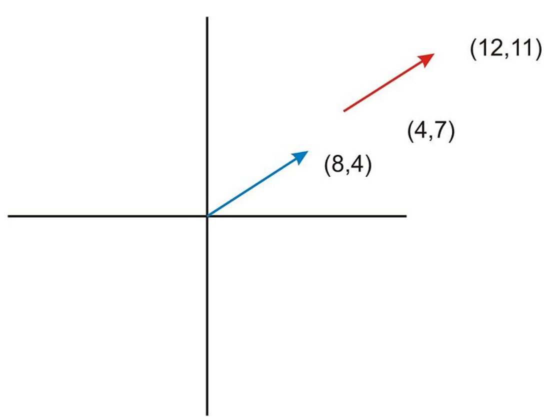
Example 5: Consider the vector from (4, 7) to (12, 11). What would the representation of a vector that had 2.5 times the magnitude be?

Solution: Here, $k = 2.5$ and $\vec{v} =$ the directed segment from (4, 7) to (12, 11).

Mathematically, two vectors are equal if their direction and magnitude are the same. The positions of the vectors do not matter. This means that if we have a vector that is not in standard position, we can translate it to the origin. The initial point of $\vec{v}$ is (4, 7). In order to **translate** this to the origin, we would need to add (-4, -7) to both the initial and terminal points of the vector.

Initial point: $(4,7) + (-4,-7) = (0,0)$

Terminal point: $(12,11) + (-4,-7) = (8,4)$



Now, to calculate $\vec{kv}$:

$$\vec{kv} = (2.5(8), 2.5(4))$$
$$\vec{kv} = (20, 10)$$

The new coordinates of the directed segment are (0, 0) and (20, 10). To translate this back to our original terminal point:

Initial point: $(0,0) + (4,7) = (4,7)$

Terminal point: $(20,10) + (4,7) = (24,17)$

The new coordinates of the directed segment are (4, 7) and (24, 17).

Vectors with the same magnitude and direction are equal. This means that the same ordered pair could represent many different vectors. For instance, the ordered pair (4, 8) can represent a vector in standard position where the initial point is at the origin and the terminal point is at (4, 8). This vector could be thought of as the resultant of a horizontal vector with a magnitude or 4 units and a vertical vector with a magnitude of 8 units. Therefore, any vector with a horizontal component of 4 and vertical component of 8 could also be represented by the ordered pair (4, 8).

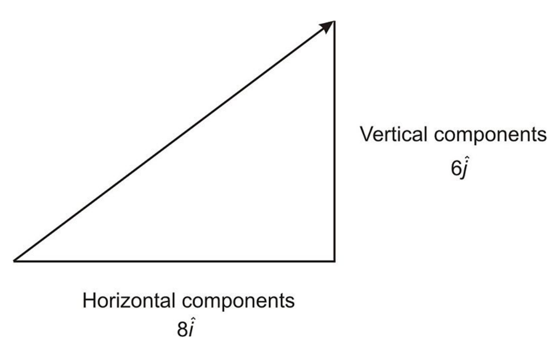
If you think back to Algebra, you know that the slope of a line is the change in y over the change in x, or the vertical change over the horizontal change. Looking at our vectors above, since they all have the same horizontal and vertical components, they all have the same slope, even though they do not all start at the origin.

Unit Vectors and Components

A **unit vector** is a vector that has a magnitude of one unit and can have any direction. Traditionally $\hat{i}$ (read "i hat") is the unit vector in the x direction and $\hat{j}$ (read "j hat") is the unit vector in the y direction. $|\hat{i}| = 1$ and $|\hat{j}| = 1$. Unit vectors on perpendicular axes can be used to express all vectors in that plane. Vectors are used to express position and motion in three dimensions with $\hat{k}$ ("k hat") as the unit vector in the z direction. We are not studying 3D space in this course. The unit vector notation may seem burdensome but one must distinguish between a vector and the components of that vector in the direction of the $x-$ or $y-$axis. The unit vectors carry the meaning for the direction of the vector in each of the coordinate directions. The number in front of the unit vector shows its magnitude or length. Unit vectors are convenient if one wishes to express a 2D or 3D vector as a sum of two or three orthogonal components, such as $x-$ and $y-$axes, or the $z-$axis. (Orthogonal components are those that intersect at right angles.)

Component vectors of a given vector are two or more vectors whose sum is the given vector. The sum is viewed as equivalent to the original vector. Since component vectors can have any direction, it is useful to have them perpendicular to one another. Commonly one chooses the x and y axis as the basis for the unit vectors. Component vectors do not have to be orthogonal.

A vector from the origin (0, 0) to the point (8, 0) is written as $8\hat{i}$. A vector from the origin to the point (0, 6) is written as $6\hat{j}$.



The reason for having the component vectors perpendicular to one another is that this condition allows us to use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the components. One can solve vector problems without use of unit vectors if specific information about orientation or direction in space such as N, E, S or W is part of the problem.

Resultant as the Sum of Two Components

We can look at any vector as the resultant of two perpendicular components. If we generalize the figure above, $|\vec{r}|\hat{\imath}$ is the horizontal component of a vector $\vec{q}$ and $|\vec{s}|\hat{\jmath}$ is the vertical component of $\vec{q}$. Therefore $\vec{r}$ is a magnitude, $|\vec{r}|$, times the unit vector in the x direction and $\vec{s}$ is its magnitude, $|\vec{s}|$, times the unit vector in the y direction. The sum of $\vec{r}$ plus $\vec{s}$ is: $\vec{r}+\vec{s}=\vec{q}$. This addition can also be written as $|\vec{r}|\hat{\imath}+|\vec{s}|\hat{\jmath}=\vec{q}$.

If we are given the vector $\vec{q}$, we can find the components of $\vec{q},\vec{r}$, and $\vec{s}$ using trigonometric ratios if we know the magnitude and direction of $\vec{q}$.

Example 6: If $|\vec{q}|= 19.6$ and its direction is 73°, find the horizontal and vertical components.

Solution: If we know an angle and a side of a right triangle, we can find the other remaining sides using trigonometric ratios. In this case, $\vec{q}$ is the hypotenuse of our triangle, $\vec{r}$ is the side adjacent to our 73° angle, $\vec{s}$ is the side opposite our 73° angle, and $\vec{r}$ is directed along the $x-$axis.

To find $\vec{r}$, we will use cosine and to find $\vec{s}$ we will use sine. Notice this is a scalar equation so all quantities are just numbers. It is written as the quotient of the magnitudes, not the vectors.

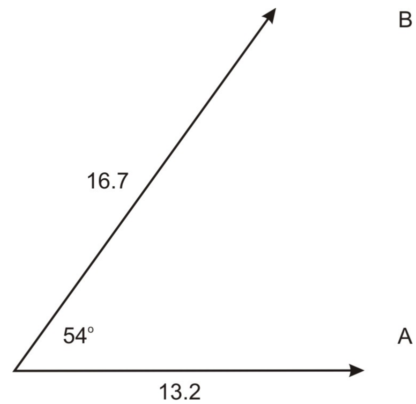
$$\begin{aligned}\cos 73 &= \frac{|\vec{r}|}{|\vec{q}|}=\frac{r}{q}\\ \cos 73 &= \frac{r}{19.6}\\ r &= 19.6\cos 73\\ r &= 5.7\end{aligned}\qquad\qquad\begin{aligned}\sin 73 &= \frac{|\vec{s}|}{|\vec{q}|}=\frac{s}{q}\\ \sin 73 &= \frac{s}{19.6}\\ s &= 19.6\sin 73\\ s &= 18.7\end{aligned}$$

The horizontal component is 5.7 and the vertical component is 18.7. One can rewrite this in vector notation as $5.7\hat{\imath}+18.7\hat{\jmath}=\vec{q}$. The components can also be written $\vec{q}=\langle 5.7, 18.7\rangle$, with the horizontal component first, followed by the vertical component. Be careful not to confuse this with the notation for plotted points.

Resultant as Magnitude and Direction

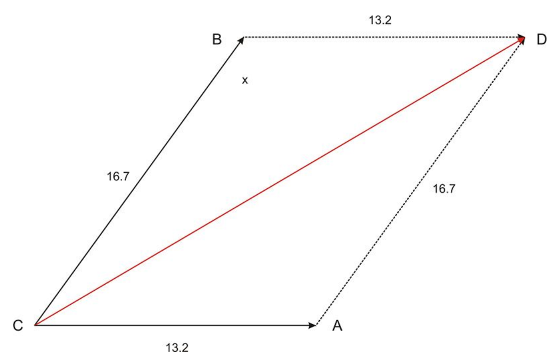
If we don't have two perpendicular vectors, we can still find the magnitude and direction of the resultant without a graphic estimate with a construction using a compass and ruler. This can be accomplished using both the Law of Sines and the Law of Cosines.

Example 7: $\vec{A}$ makes a 54° angle with $\vec{B}$. The magnitude of $\vec{A}$ is 13.2. The magnitude of $\vec{B}$ is 16.7. Find the magnitude and direction the resultant makes with the smaller vector.



There is no preferred orientation such as a compass direction or any necessary use of x and y coordinates. The problem can be solved without the use of unit vectors.

Solution: In order to solve this problem, we will need to use the parallelogram method. Since vectors only have magnitude and direction, one can move them on the plane to any position one wishes, as long as the magnitude and direction remain the same. First, we will complete the parallelogram: Label the vectors. Move $\vec{b}$ so its tail is on the tip of $\vec{a}$. Move $\vec{a}$ so its tail is on the tip of $\vec{b}$. This makes a parallelogram because the angles did not change during the translation. Put in labels for the vertices of the parallelogram.



Since opposite angles in a parallelogram are congruent, we can find angle A.

$$\begin{aligned}
\angle CBD + \angle CAD + \angle ACB + \angle BDA &= 360 \\
2\angle CBD + 2\angle ACB &= 360 \\
\angle ACB &= 54^\circ \\
2\angle CBD &= 360 - 2(54) \\
\angle CBD &= \frac{360 - 2(54)}{2} = 126
\end{aligned}$$

Now, we know two sides and the included angle in an oblique triangle. This means we can use the Law of Cosines to find the magnitude of our resultant.

$$\begin{aligned}
x^2 &= 13.2^2 + 16.7^2 - 2(13.2)(16.7)\cos 126 \\
x^2 &= 712.272762 \\
x &= 26.7
\end{aligned}$$

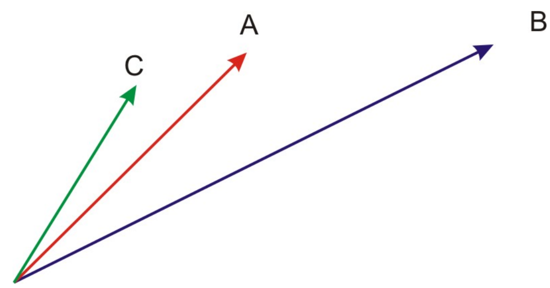
To find the direction, we can use the Law of Sines since we now know an angle and a side across from it. We choose the Law of Sines because it is a proportion and less computationally intense than the Law of Cosines.

$$\frac{\sin\theta}{16.7} = \frac{\sin 126}{26.7}$$
$$\sin\theta = \frac{16.7\sin 126}{26.7}$$
$$\sin\theta = 0.5060143748$$
$$\theta = \sin^{-1}\ 0.5060 = 30.4^\circ$$

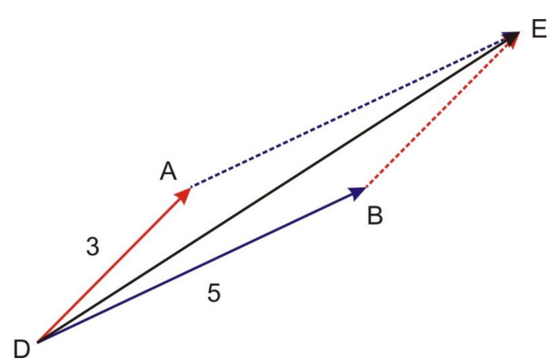
The magnitude of the resultant is 26.7 and the direction it makes with the smaller vector is 30.4° counterclockwise.

We can use a similar method to add three or more vectors.

Example 8: Vector A makes a 45° angle with the horizontal and has a magnitude of 3. Vector B makes a 25° angle with the horizontal and has a magnitude of 5. Vector *C* makes a 65° angle with the horizontal and has a magnitude of 2. Find the magnitude and direction (with the horizontal) of the resultant of all three vectors.



Solution: To begin this problem, we will find the resultant using Vector *A* and Vector *B*. We will do this using the parallelogram method like we did above.



Since Vector *A* makes a 45° angle with the horizontal and Vector *B* makes a 25° angle with the horizontal, we know that the angle between the two ($\angle ADB$) is 20°.

To find $\angle DBE$:

$$2\angle ADB + 2\angle DBE = 360$$
$$\angle ADB = 20^\circ$$
$$2\angle DBE = 360 - 2(20)$$
$$\angle DBE = \frac{360-2(20)}{2} = 160$$

Now, we will use the Law of Cosines to find the magnitude of DE.

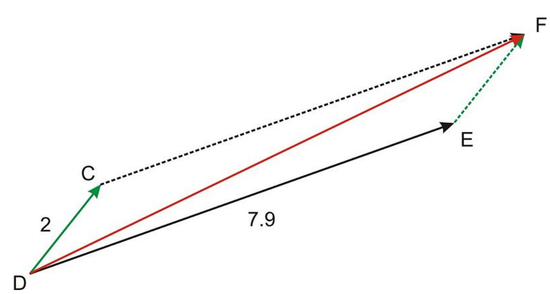
$$\begin{aligned}
DE^2 &= 3^2 + 5^2 - 2(3)(5)\cos 160 \\
DE^2 &= 62 \\
DE &= 7.9
\end{aligned}$$

Next, we will use the Law of Sines to find the measure of angle EDB.

$$\begin{aligned}
\frac{\sin 160}{7.9} &= \frac{\sin \angle EDB}{3} \\
\sin \angle EDB &= \frac{3 \sin 160}{7.9} \\
\sin \angle EDB &= .1299 \\
\angle EDB &= \sin^{-1}\ 0.1299 = 7.46^\circ
\end{aligned}$$

We know that Vector B forms a 25° angle with the horizontal so we add that value to the measure of $\angle EDB$ to find the angle DE makes with the horizontal. Therefore, DE makes a 32.46° angle with the horizontal.

Next, we will take DE, and we will find the resultant vector of DE and Vector C from above. We will repeat the same process we used above.



Vector C makes a 65° angle with the horizontal and DE makes a 32° angle with the horizontal. This means that the angle between the two $(\angle CDE)$ is 33°. We will use this information to find the measure of $\angle DEF$.

$$\begin{aligned}
2\angle CDE + 2\angle DEF &= 360 \\
\angle CDE &= 33^\circ \\
2\angle DEF &= 360 - 2(33) \\
\angle DEF &= \frac{360 - 2(33)}{2} = 147
\end{aligned}$$

Now we will use the Law of Cosines to find the magnitude of DF.

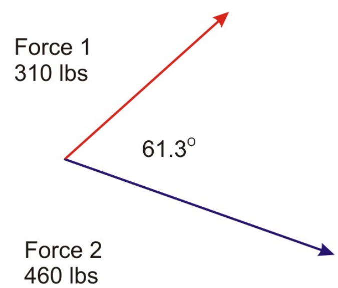
$$\begin{aligned}
DF^2 &= 7.9^2 + 2^2 - 2(7.9)(2)\cos 147 \\
DF^2 &= 92.9 \\
DF &= 9.6
\end{aligned}$$

Next, we will use the Law of Sines to find $\angle FDE$.

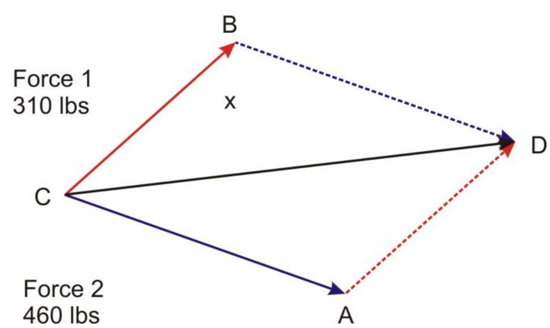
$$\begin{aligned}\frac{\sin 147}{9.6} &= \frac{\sin \angle FDE}{2}\\ \sin \angle FDE &= \frac{2\sin 147}{9.6}\\ \sin \angle FDE &= .1135\\ \angle FDE &= \sin^{-1} 0.1135 = 6.5^\circ = 7^\circ\end{aligned}$$

Finally, we will take the measure of $\angle FDE$ and add it to the 32° angle that DE forms with the horizontal. Therefore, DF forms a 39° angle with the horizontal.

Example 9: Two forces of 310 lbs and 460 lbs are acting on an object. The angle between the two forces is 61.3°. What is the magnitude of the resultant? What angle does the resultant make with the smaller force?



Solution: We do not need unit vectors here as there is no preferred direction like a compass direction or a specific axis. First, to find the magnitude we will need to figure out the other angle in our parallelogram.



$$\begin{aligned}2\angle ACB + 2\angle CAD &= 360\\ \angle ACB &= 61.3^\circ\\ 2\angle CAD &= 360 - 2(61.3)\\ \angle CAD &= \frac{360 - 2(61.3)}{2} = 118.7\end{aligned}$$

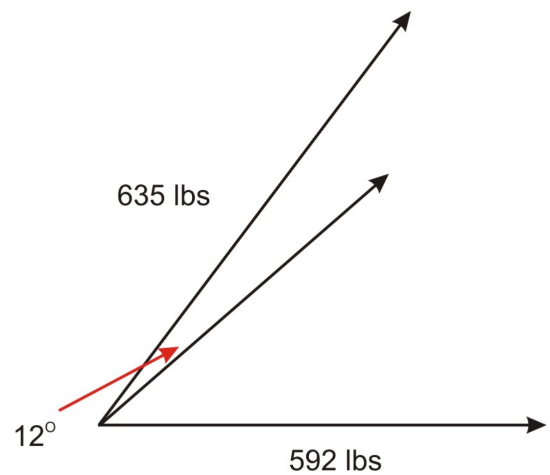
Now that we know the other angle, we can find the magnitude using the Law of Cosines.

$$\begin{aligned}x^2 &= 460^2 + 310^2 - 2(460)(310)\cos 118.7^\circ\\ x^2 &= 444659.7415\\ x &= 667\end{aligned}$$

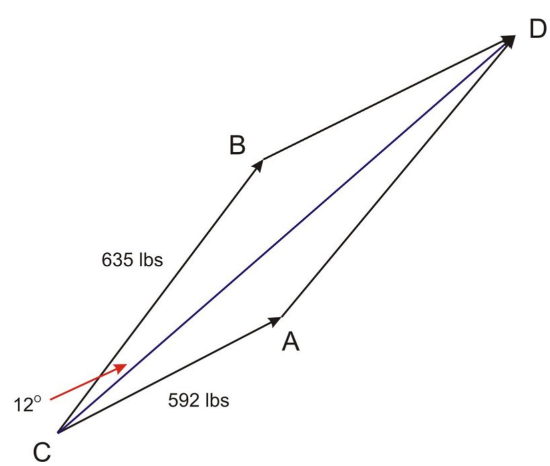
To find the angle the resultant makes with the smaller force, we will use the Law of Sines.

$$\frac{\sin\theta}{460} = \frac{\sin 118.7}{666.8}$$
$$\sin\theta = \frac{460\sin 118.7}{666.8}$$
$$\sin\theta = .6049283888$$
$$\theta = \sin^{-1}\ 0.6049 = 37.2^\circ$$

Example 10: Two trucks are pulling a large chunk of stone. Truck 1 is pulling with a force of 635 lbs at a 53° angle from the horizontal while Truck 2 is pulling with a force of 592 lbs at a 41° angle from the horizontal. What is the magnitude and direction of the resultant force?



Solution: Since Truck 1 has a direction of 53° and Truck 2 has a direction of 41°, we can see that the angle between the two forces is 12°. We need this angle measurement in order to figure out the other angles in our parallelogram.



$$2\angle ACB + 2\angle CAD = 360$$
$$\angle ACB = 12^\circ$$
$$2\angle CAD = 360 - 2(12)$$
$$\angle CAD = \frac{360 - 2(12)}{2} = 168$$

Now, use the Law of Cosines to find the magnitude of the resultant.

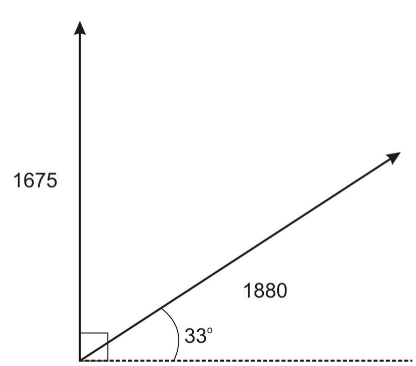
$$\begin{aligned} x^2 &= 635^2 + 592^2 - 2(635)(592)\cos 168^\circ \\ x^2 &= 1489099 \\ x &= 1220.3\ lbs \end{aligned}$$

Now to find the direction we will use the Law of Sines.

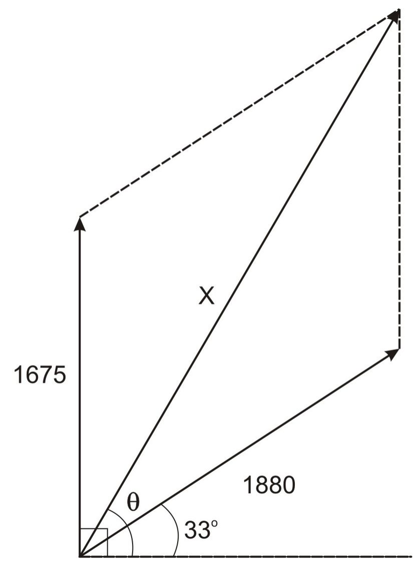
$$\begin{aligned} \frac{\sin\theta}{635} &= \frac{\sin 168}{1220.3} \\ \sin\theta &= \frac{635\sin 168}{1220.3} \\ \sin\theta &= 0.1082 \\ \theta &= \sin^{-1}\ 0.1082 = 6^\circ \end{aligned}$$

Since we want the direction we need to add the 6° to the 41° from the smaller force. The magnitude is 1220 lbs and 47° counterclockwise from the horizontal.

Example 11: Two tractors are being used to pull down the framework of an old building. One tractor is pulling on the frame with a force of 1675 pounds and is headed directly north. The second tractor is pulling on the frame with a force of 1880 pounds and is headed 33° north of east. What is the magnitude of the resultant force on the building? What is the direction of the result force?

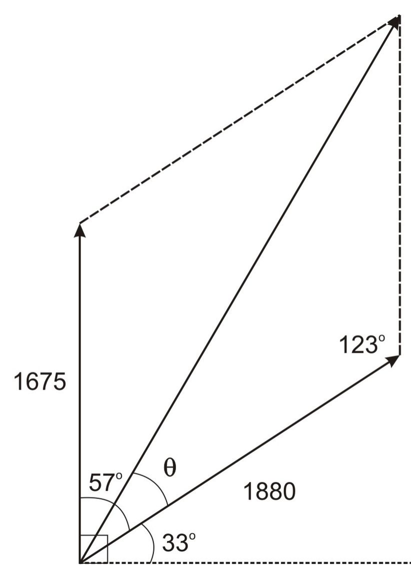


Solution: We are asked to find the resultant force and direction, which means we are dealing with vectors. In order to complete our diagram, we will need to connect our two vectors and draw in our resultant. We will refer to the magnitude of our resultant as x and the direction of our resultant as θ.



When finding the resultant of two vectors, we can choose from either the triangle method or the parallelogram method. We will solve this problem using the parallelogram method. Looking at the diagram, we can see that the two vectors form an angle of 57, (90 - 33). This means that the angle opposite the angle formed by our two vectors is also 57. To find the other two angles in our parallelogram, we know that the sum of all the angles must add up to 360 and that opposite angles must be congruent, $\frac{360-(57+57)}{2} = 123$.

Now, we can use two sides of our parallelogram and our resultant to form a triangle in which we know two sides and the included angle (SAS).



This means that we can use the Law of Cosines to find the magnitude (x) of the resultant.

$$x^2 = 1675^2 + 1880^2 - 2(1675)(1880)\cos 123$$
$$x^2 = 9770161.643$$
$$x = 3125.7$$

To find the direction (θ), we can use the Law of Sines since we now know an angle and the side opposite it.

$$\frac{\sin 123}{3125.7} = \frac{\sin\theta}{1675}$$
$$\frac{1675\sin 123}{3125.7} = \sin\theta$$
$$0.449427 = \sin\theta$$
$$26.71 = \theta$$

Now that we know θ, in order to find the angle of the resultant, we must add the 33° from the $x-$axis to $\theta, 33^\circ + 26.71^\circ = 59.71^\circ$.

Points to Consider

- How you can verify if your answers to problems involving vectors that are not perpendicular are correct?
- In what ways are solving problems with oblique triangles and solving problems involving vectors similar?
- In what ways are they different?
- When is it appropriate to use vectors instead of oblique triangles to solve problems?
- When is it helpful to use unit vectors? When can one solve a problem without explicitly using them?

Review Questions

1. Find the resulting ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (0,0)\ to\ (5,4)$ and $\vec{c} = (0,0)\ to\ (-3,7)$.
   a. $\vec{a} = 2\vec{b}$
   b. $\vec{a} = -\frac{1}{2}\vec{c}$
   c. $\vec{a} = 0.6\vec{b}$
   d. $\vec{a} = -3\vec{b}$
2. Find the magnitude of the horizontal and vertical components of the following vectors given that the coordinates of their initial and terminal points.
   a. initial $= (-3,8)$ terminal $= (2,-1)$
   b. initial $= (7,13)$ terminal $= (11,19)$
   c. initial $= (4.2,-6.8)$ terminal $= (-1.3,-9.4)$
3. Find the magnitude of the horizontal and vertical components if the resultant vector's magnitude and direction are given.
   a. magnitude $= 75$ direction $= 35^\circ$
   b. magnitude $= 3.4$ direction $= 162^\circ$
   c. magnitude $= 15.9$ direction $= 12^\circ$
4. Two forces of 8.50 Newtons and 32.1 Newtons act on an object at right angles. Find the magnitude of the resultant and the angle that it makes with the smaller force.
5. Forces of 140 Newtons and 186 Newtons act on an object. The angle between the forces is 43°. Find the magnitude of the resultant and the angle it makes with the larger force.
6. An incline ramp is 12 feet long and forms an angle of 28.2° with the ground. Find the horizontal and vertical components of the ramp.

7. An airplane is traveling at a speed of 155 km/h. It's heading is set at 83° while there is a 42.0 km/h wind from 305°. What is the airplane's actual heading?
8. A speedboat is capable of traveling at 10.0 mph, but is in a river that has a current of 2.00 mph. In order to cross the river at right angle, in what direction should the boat be heading?
9. If $\overrightarrow{AB}$ is any vector, what is $\overrightarrow{AB} + \overrightarrow{BA}$?

Chapter Summary

This chapter has taught us how to solve any kind of triangle, using the Law of Cosines, Law of Sines and vectors. We also discovered two additional formulas for finding the area, Heron's Formula and $\frac{1}{2}\,bc\sin A$. Then, the ambiguous case for the Law of Sines was introduced. This is when you are given two sides and the non-included angle and have to solve for another angle in the triangle. There can be no, one or two solutions and you need to compare the two given sides to determine which option it is. Finally, vectors were discussed. We learned how to add, subtract and multiply them by a scalar. Vectors are very useful for representing speed, wind velocity and force.

Vocabulary

Ambiguous case
A situation that occurs when applying the Law of Sines in an oblique triangle when two sides and a non-included angle are known. The ambiguous case can yield no solution, one solution, or two solutions.

component vectors
Two or more vectors whose vector sum, the resultant, is the given vector. Components can be on axes or more generally in space.

directed line segment
A line segment having both magnitude and direction, often used to represent a vector.

displacement
When an object moves a certain distance in a certain direction.

equal vectors
Vectors with the same magnitude and direction.

force
When an object is pushed or pulled in a certain direction.

Heron's Formula
A formula used to calculate the area of a triangle when all three sides are known. $K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$ or half of the perimeter of the triangle.

included angle
The angle in between two known sides of a triangle.

included side
The side in between two known sides of a triangle.

initial point
The starting point of a vector

Law of Cosines
A general statement relating the lengths of the sides of a general triangle to the cosine of one of its angles. $a^2 = b^2 + c^2 - 2bc\cos A$ or $b^2 = a^2 + c^2 - 2ac\cos B$ or $c^2 = a^2 + b^2 - 2ab\cos C$

Law of Sines
A statement about the relationship between the sides and the angles in any triangle. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

magnitude
Length of a vector.

negative vector
A vector with the same magnitude as the original vector but with the opposite direction.

non-included angle
An angle that is not in between two known sides of a triangle.

non-included side
A side that is not in between two known sides of a triangle.

oblique triangle
A non-right triangle.

resultant
The sum of two or more vectors

scalar
A real number by which a vector can be multiplied. The magnitude of a vector is always a scalar.

standard position
A vector with its initial point at the origin of a coordinate plane.

terminal point
The ending point of a vector.

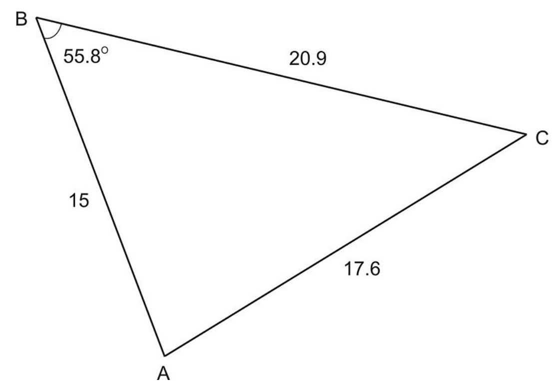
unit vector
A vector that has a magnitude of one unit. These generally point on coordinate axes.

vector
Any quantity having magnitude and direction, often represented by an arrow.

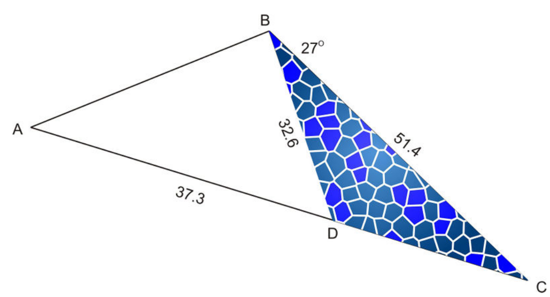
velocity
When an object travels at a certain speed in a certain direction.

Review Questions

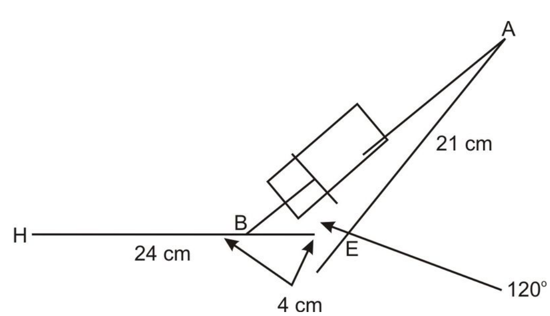
1. Use the Law of Cosines to determine whether or not the following triangle is drawn accurately. If not, determine how much $\angle B$ is off by.



2. An artist is making a large sculpture for the lobby of a new building. She has drawn out what she wants the sculpture to look like at the left. If she wants $BC = 51.4\ feet, BD = 32.6\ feet, AD = 37.3\ feet$ and $\angle DBC = 27^\circ$, verify that the length of AB would be 34.3 feet. If not, figure out the correct measure.

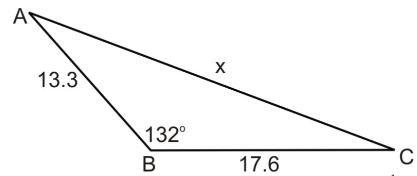


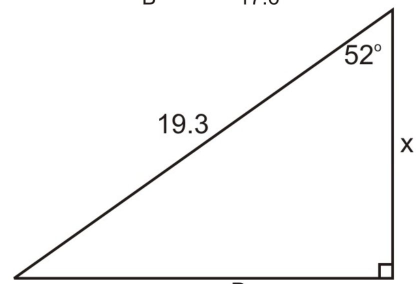
3. A family's farm plot is a trapezoid with dimensions: the longer leg is 3,000 ft and the shortest side is 2,100 ft. The other leg is 2,400 ft. The shorter diagonal is 2,200 ft. What is the area of the land in square feet?
4. A biomechanics class is designing a functioning artificial arm for an adult. They are using a hydraulic cylinder (fluid filled) to be the bicep's muscle. The elbow is at point E. The forearm dimension EH is 24 cm. The upper arm dimension EA is 21 cm. The cylinder attaches from the top of the upper arm at point A and to a point on the lower arm 4 cm from the mechanical elbow at point B. When fluid is pumped out of the cylinder the distance AB is shortened. The forearm goes up raising the hand at point H.

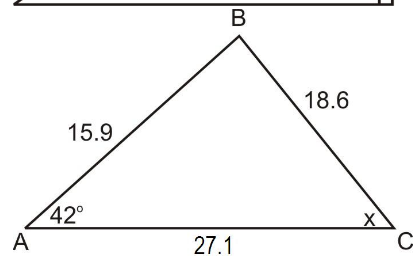


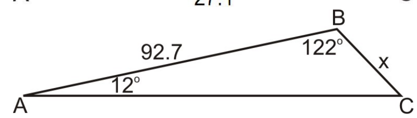
Some fluid is pumped out of the cylinder to make the distance AB 5 cm shorter. What is the new angle of the arm, $\angle AEH$?

5. For each figure below, use the Law of Sines, the Law of Cosines, or the the trig functions to solve for x.

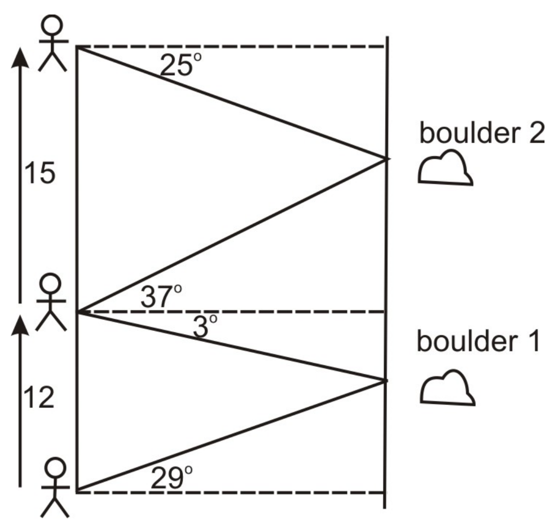
a. 

b. 

c. 

d. 

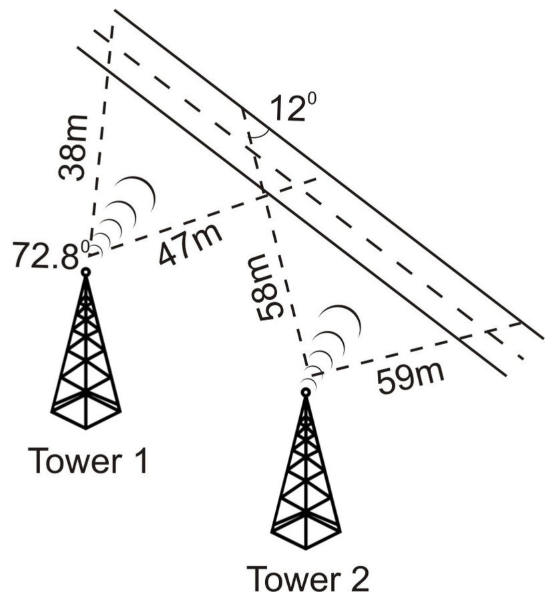
6. A surveyor has the job of determining the distance across the Palo Duro Canyon in Amarillo, Texas, the second largest canyon in the United States. Standing on one side of the canyon, he measures the angle formed by the edge of the canyon and the line of sight to a large boulder on the other side of the canyon. He then drives 12 km and measures the angle formed by the edge of the canyon and the new line of sight to the boulder.



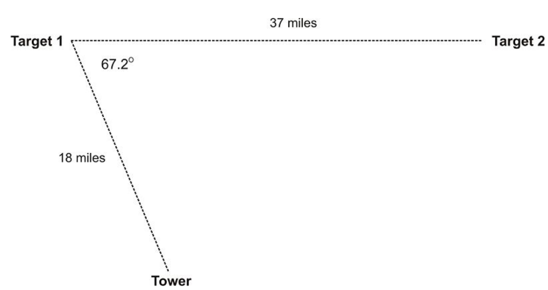
   a. If the first angle formed is 29° and the second angle formed is 3°, find the distance across the canyon.
   b. The surveyor spots another boulder while he is at his second spot, and finds that it forms a 37° angle with his line of sight. He then drives 15 km further and finds that the boulder forms a 25° angle with the line of sight. What is the distance between the two boulders?

7. Two cell phone companies have towers along Route 47. Company 1's tower is 38 miles from one point on Route 47 and 47 miles from another point. This tower's signal forms a 72.8° angle. Company 2's tower is 58 miles from one point of Route 47 and 59 miles from another. Company 2's signal forms a 12° angle with the road at the point that is 58 miles from the tower. For how many miles would a person driving along Route 47

have service with Company A? Company B? If the signals start to overlap 32 miles from Tower 1 (along the same line as the 47 mile), how long does the two coverages overlap over the highway?



8. Find the magnitude and direction of each resultant vector (addition). $\vec{m}$ and $\vec{n}$ are perpendicular.
   a. $|\vec{m}| = 48.3, |\vec{n}| = 47.6$
   b. $|\vec{m}| = 18.6, |\vec{n}| = 17.5$
9. Given the initial and terminal coordinates of $\vec{a}$, find the magnitude and direction.
   a. initial (-4, 19) terminal (12, 1)
   b. initial (11, -21) terminal (21, -11)
10. A golfer tees off at the 16^{th} hole. The pin is 425 yards from tee-off and his ball is 16° off of the straight line to the hole. If his ball is 137 yards from the hole, how far did he hit the ball?
11. Street A runs north and south and intersects with Street B, which runs east and west. Street C intersects both A and B, and it intersects Street A at a 36° angle. There are stoplights at each of these intersections. If the distance between the two stoplights on Street C is 0.5miles, what is the distance between the two stoplights on Street A?
12. During a baseball game, a ball is hit into right field. The angle from the ball to home to 2^{nd} base is 18°. The angle from the ball to 2^{nd} to home is 127°. The distance from home to 2^{nd} base is 127.3 ft. How far was the ball hit? How far is second base from the ball?
13. The military is testing out a new infrared sensor that can detect movement up to thirty miles away. Will the sensor be able to detect the second target? If not, how far out of the range of the sensor is Target 2?



14. An environmentalist is sampling the water in a local lake and finds a strain of bacteria that lives on the surface of the lake. In a one square foot area, he found 5.2×10^{13} bacteria. There are three docks in a certain section of the lake. If Dock 3 is 396 ft from Dock 1, how many bacteria are living on the surface of the water between the three docks?

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9703 .

CHAPTER 6 The Polar System

Chapter Outline

6.1 POLAR COORDINATES

6.2 GRAPHING BASIC POLAR EQUATIONS

6.3 CONVERTING BETWEEN SYSTEMS

6.4 MORE WITH POLAR CURVES

6.5 THE TRIGONOMETRIC FORM OF COMPLEX NUMBERS

6.6 THE PRODUCT & QUOTIENT THEOREMS

6.7 DE MOIVRE'S AND THE NTH ROOT THEOREMS

6.1 Polar Coordinates

Introduction

This chapter introduces and explores the polar coordinate system, which is based on a radius and theta. Students will learn how to plot points and basic graphs in this system as well as convert x and y coordinates into polar coordinates and vice versa. We will explore the different graphs that can be generated in the polar system and also use polar coordinates to better understand different aspects of complex numbers.

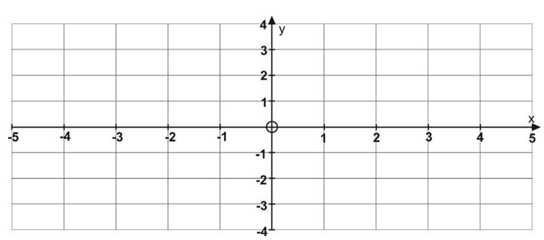
Learning Objectives

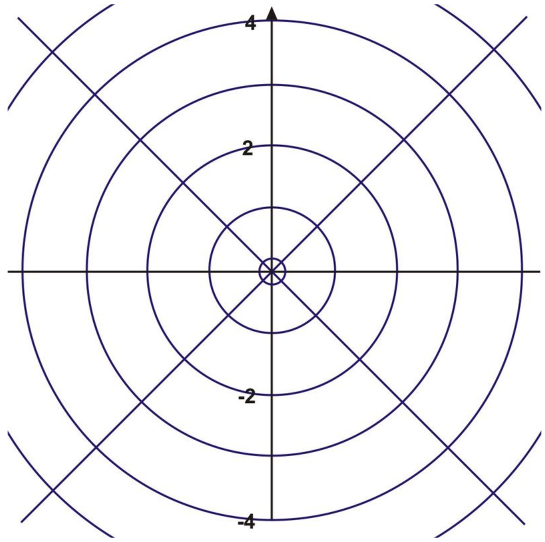
- Distinguish between and understand the difference between a rectangular coordinate system and a polar coordinate system.
- Plot points with polar coordinates on a polar plane.

Plotting Polar Coordinates

The graph paper that you have used for plotting points and sketching graphs has been rectangular grid paper. All points were plotted in a rectangular form (x,y) by referring to a perpendicular $x-$ and $y-$axis. In this section you will discover an alternative to graphing on rectangular grid paper –graphing on circular grid paper.

Look at the two options below:



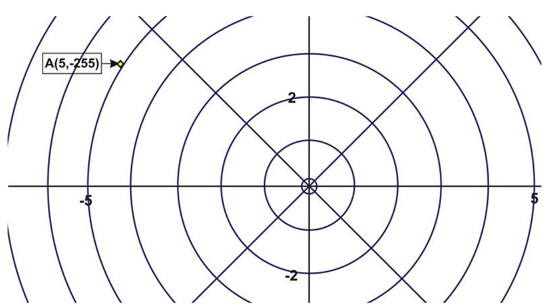


You are all familiar with the rectangular grid paper shown above. However, the circular paper lends itself to new discoveries. The paper consists of a series of concentric circles-circles that share a common center. The common center O, is known as the pole or origin and the polar axis is the horizontal line r that is drawn from the pole in a positive direction. The point P that is plotted is described as a directed distance r from the pole and by the angle that $\overline{OP}$ makes with the polar axis. The coordinates of P are (r, θ).

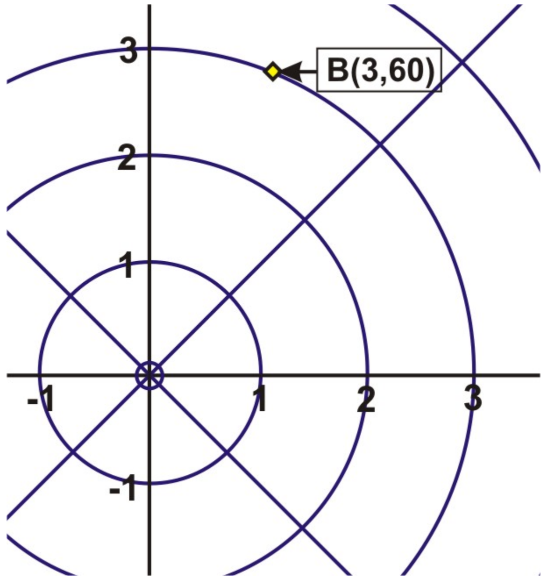
These coordinates are the result of assuming that the angle is rotated counterclockwise. If the angle were rotated clockwise then the coordinates of P would be $(r, -\theta)$. These values for P are called polar coordinates and are of the form $P(r, \theta)$ where r is the absolute value of the distance from the pole to P and θ is the angle formed by the polar axis and the terminal arm $\overline{OP}$.

Example 1: Plot the point $A(5, -255^\circ)$ and the point $B(3, 60^\circ)$

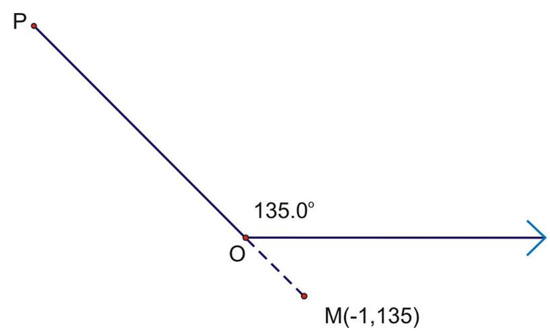
Solution, A: To plot A, move from the pole to the circle that has $r = 5$ and then rotate 255° **clockwise** from the polar axis and plot the point on the circle. Label it A.

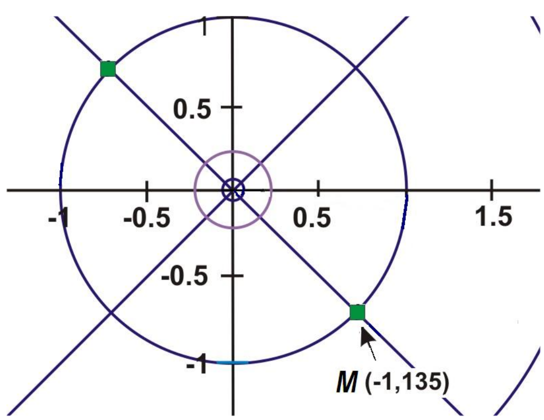


Solution, B: To plot B, move from the pole to the circle that has $r = 3$ and then rotate 60° **counter clockwise** from the polar axis and plot the point on the circle. Label it B.



These points that you have plotted have r values that are greater than zero. How would you plot a polar point in which the value of r is less than zero? How could you plot these points if you did not have polar paper? If you were asked to plot the point $(-1, 135^\circ)$ or $\left(-1, \frac{3\pi}{4}\right)$ you would rotate the terminal arm $\overline{OP}$ counterclockwise 135° or $\frac{3\pi}{4}$. (Remember that the angle can be expressed in either degrees or radians). To accommodate $r = -1$, extend the terminal arm $\overline{OP}$ in the opposite direction the number of units equal to $|r|$. Label this point M or whatever letter you choose. The point can be plotted, without polar paper, as a rotation about the pole as shown below.

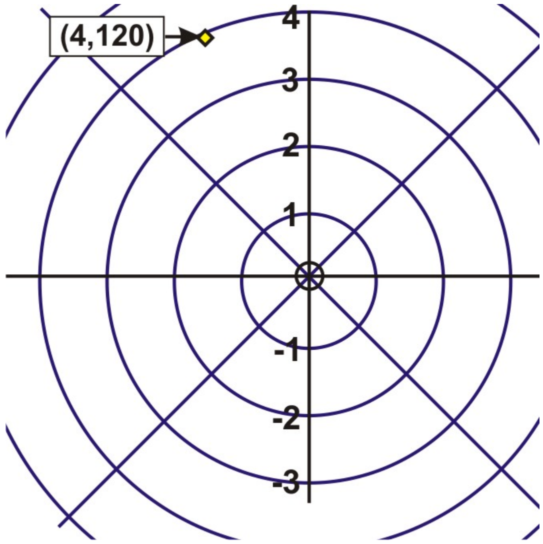




The point is reflected across the pole to point M.

There are multiple representations for the coordinates of a polar point $P(r,\theta)$. If the point P has polar coordinates (r,θ), then P can also be represented by polar coordinates $(r,\theta+360k^\circ)$ or $(-r,\theta+[2k+1]180^\circ)$ if θ is measured in degrees or by $(r,\theta+2\pi k)$ or $(-r,\theta+[2k+1]\pi)$ if θ is measured in radians. Remember that k is any integer and represents the number of rotations around the pole. Unless there is a restriction placed upon θ, there will be an infinite number of polar coordinates for $P(r,\theta)$.

Example 2: Determine four pairs of polar coordinates that represent the following point $P(r,\theta)$ such that $-360^\circ \le \theta \le 360^\circ$.



Solution: Pair 1 $\rightarrow (4,120^\circ)$. Pair 2 $\rightarrow (4,-240^\circ)$ comes from using $k=-1$ and $(r,\theta+360^\circ k),(4,120^\circ+360(-1))$. Pair 3 $\rightarrow (-4,300^\circ)$ comes from using $k=0$ and $(-r,\theta+[2k+1]180^\circ),(-4,120^\circ+[2(0)+1]180^\circ)$. Pair 4 $\rightarrow (-4,-60^\circ)$ comes from using $k=-1$ and $(-r,\theta+[2k+1]180^\circ),(-4,120^\circ+[2(-1)+1]180^\circ)$.

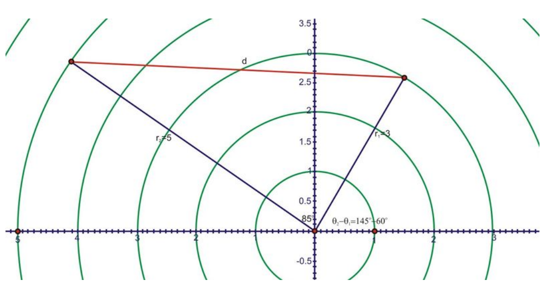
These four pairs of polar coordinates all represent the same point P. You can apply the same procedure to determine polar coordinates of points that have θ measured in radians. This will be an exercise for you to do at the end of the lesson.

The Distance between Two Polar Coordinates

Just like the Distance Formula for x and y coordinates, there is a way to find the distance between two polar coordinates. One way that we know how to find distance, or length, is the Law of Cosines, $a^2=b^2+c^2-2bc\cos A$ or $a=\sqrt{b^2+c^2-2bc\cos A}$. If we have two points (r_1,θ_1) and (r_2,θ_2), we can easily substitute r_1 for b and r_2 for c. As for A, it needs to be the angle between the two radii, or $(\theta_2-\theta_1)$. Finally, a is now distance and you have $d=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}$.

Example 3: Find the distance between $(3,60^\circ)$ and $(5,145^\circ)$.

Solution: After graphing these two points, we have a triangle. Using the new Polar Distance Formula, we have $d=\sqrt{3^2+5^2-2(3)(5)\cos 85^\circ}\approx 5.6$.



Example 4: Find the distance between $(9, -45^\circ)$ and $(-4, 70^\circ)$.

Solution: This one is a little trickier than the last example because we have negatives. The first point would be plotted in the fourth quadrant and is equivalent to $(9, 315^\circ)$. The second point would be $(4, 70^\circ)$ reflected across the pole, or $(4, 250^\circ)$. Use these two values of θ for the formula. Also, the radii should always be positive when put into the formula. That being said, the distance is $d = \sqrt{9^2 + 4^2 - 2(9)(4)\cos(315 - 250)^\circ} \approx 8.16$.

Points to Consider

- How is the polar coordinate system similar/different from the rectangular coordinate system?
- How do you plot a point on a polar coordinate grid?
- How do you determine the coordinates of a point on a polar grid?
- How do you calculate the distance between two points that have polar coordinates?

Review Questions

1. Graph each point:
   a. $M(2.5, 210^\circ)$
   b. $S\left(-3.5, \frac{5\pi}{6}\right)$
   c. $A\left(1, \frac{3\pi}{4}\right)$
   d. $Y\left(5.25, -\frac{\pi}{3}\right)$
2. For the given point $A\left(-4, \frac{\pi}{4}\right)$, list three different pairs of polar coordinates that represent this point such that $-2\pi \leq \theta \leq 2\pi$.
3. For the given point $B(2, 120^\circ)$, list three different pairs of polar coordinates that represent this point such that $-2\pi < \theta < 2\pi$.
4. Given P_1 and P_2, calculate the distance between the points.
   a. $P_1(1, 30^\circ)$ and $P_2(6, 135^\circ)$
   b. $P_1(2, -65^\circ)$ and $P_2(9, 85^\circ)$
   c. $P_1(-3, 142^\circ)$ and $P_2(10, -88^\circ)$
   d. $P_1(5, -160^\circ)$ and $P_2(16, -335^\circ)$

6.2 Graphing Basic Polar Equations

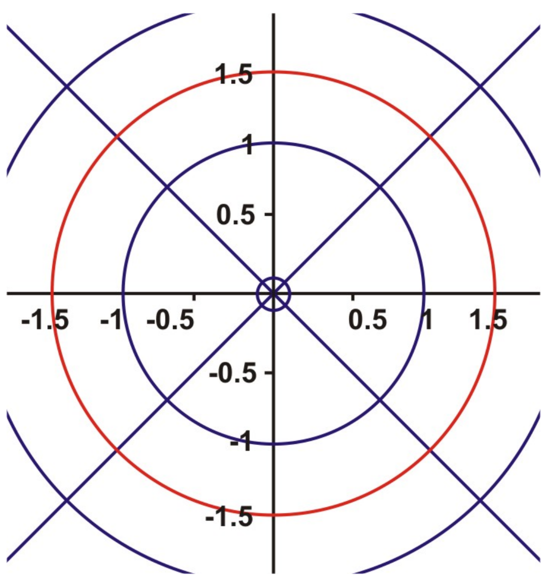
Learning Objectives

- Graph polar equations.
- Graph and recognize limaçons and cardioids.
- Determine the shape of a limaçon from the polar equation.

Just as in graphing on a rectangular grid, you can also graph polar equations on a polar grid. These equations may be simple or complex. To begin, you should try something simple like $r = k$ or $\theta = k$ where k is a constant. The solution for $r = 1.5$ is simply all ordered pairs such that $r = 1.5$ and θ is any real number. The same is true for the solution of $\theta = 30^\circ$. The ordered pairs will be any real number for r and θ will equal 30°. Here are the graphs for each of these polar equations.

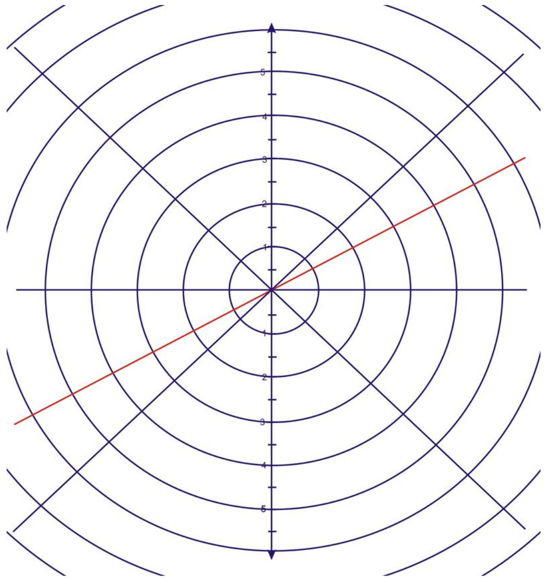
Example 1: On a polar plane, graph the equation $r = 1.5$

Solution: The solution is all ordered pairs of (r, θ) such that r is always 1.5. This means that it doesn't matter what θ is, so the graph is a circle with radius 1.5 and centered at the origin.



Example 2: On a polar plane, graph the equation $\theta = 30^\circ$

Solution: For this example, the r value, or radius, is arbitrary. θ must equal 30°, so the result is a straight line, with an angle of elevation of 30°.



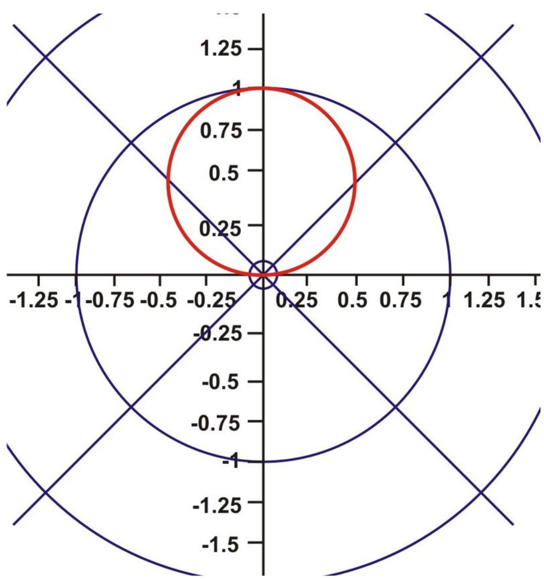
To begin graphing more complicated polar equations, we will make a table of values for $y = \sin\theta$ or in this case $r = \sin\theta$. When the table has been completed, the graph will be drawn on a polar plane by using the coordinates (r, θ).

Example 3: Create a table of values for $r = \sin\theta$ such that $0^\circ \leq \theta \leq 360^\circ$ and plot the ordered pairs. (Note: Students can be directed to use intervals of 30° or allow them to create their own tables.)

TABLE 6.1:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	**0**	**0.5**	**0.9**	**1**	**0.9**	**0.5**	**0**	**-0.5**	**-0.9**	**-1**	**-0.9**	**-0.5**	**0**

Remember that the values of $\sin\theta$ are the $r-$values.



This is a sinusoid curve of one revolution.

We will now repeat the process for $r = \cos\theta$.

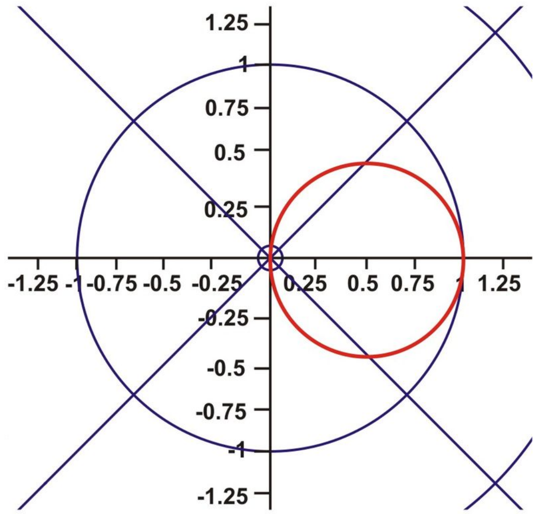
Example 4: Create a table of values for $r = \cos\theta$ such that $0^\circ \leq \theta \leq 360^\circ$ and plot the ordered pairs. (Note: Students can be directed to use intervals of 30° or allow them to create their own tables.)

TABLE 6.2:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	**1**	**0.9**	**0.5**	**0**	**-0.5**	**-0.9**	**-1**	**-0.9**	**-0.5**	**0**	**0.5**	**0.9**	**1**

Remember that the values of $\cos\theta$ are the $r-$values.

This is also a sinusoid curve of one revolution.



Notice that both graphs are circles that pass through the pole and have a diameter of one unit. These graphs can be altered by adding a number to the function or by multiplying the function by a constant or by doing both. We will explore the results of these alterations by first creating a table of values and then by graphing the resulting coordinates (r, θ).

Example 5: Create a table of values for $r = 2 + 3\sin\theta$ such that $0 \le \theta \le 2\pi$ and plot the ordered pairs. Remember that the values of $2 + 3\sin\theta$ are the $r-$values.

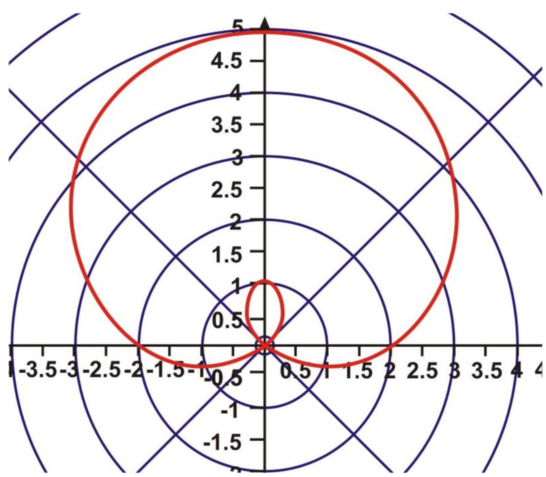


TABLE 6.3:

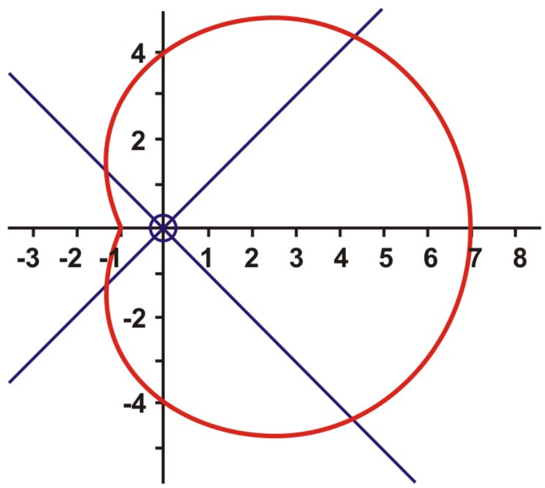
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$2 + 3\sin\theta$	**2.0**	**3.5**	**4.6**	**5.0**	**4.6**	**3.5**	**2.0**	**0.5**	**-0.6**	**-1.0**	**-0.6**	**0.5**	**2.0**

This sinusoid curve is called a limaçon. It has $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ as its polar equation. Not all limaçons have the inner loop as a part of the shape. Some may curve to a point, have a simple indentation (known as a dimple) or curve outward. The shape of the limaçon depends upon the ratio of $\frac{a}{b}$ where a is a constant and b is the coefficient of the trigonometric function. In example 5, the ratio of $\frac{a}{b} = \frac{2}{3}$ which is < 1. All limaçons that meet this criterion will have an inner loop.

Using the same format as was used in the examples above, the following limaçons were graphed. If you like, you may create the table of values for each of these functions.

i) $r = 4 + 3 \cos \theta$ such that $0 \le \theta \le 2\pi$

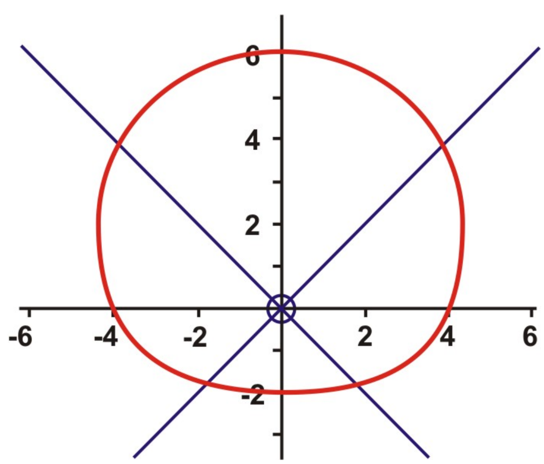
$\frac{a}{b} = \frac{4}{3}$ which is > 1 but < 2



This is an example of a dimpled limaçon.

ii) $r = 4 + 2 \sin \theta$ such that $0 \le \theta \le 2\pi$

$\frac{a}{b} = \frac{4}{2}$ which is ≥ 2



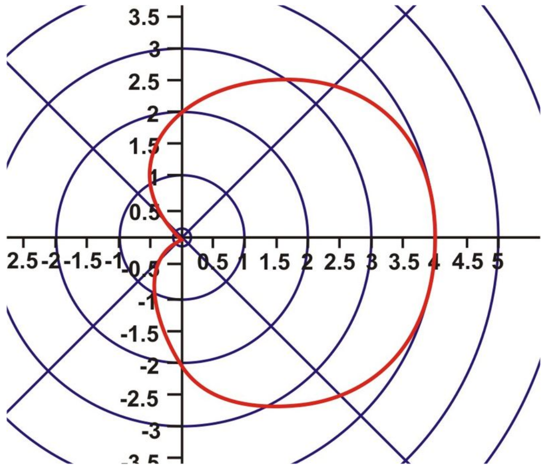
This is an example of a convex limaçon.

Example 6: Create a table of values for $r = 2 + 2 \cos \theta$ such that $0 \le \theta \le 2\pi$ and plot the ordered pairs. Remember that the values of $2 + 2 \cos \theta$ are the $r-$values.

TABLE 6.4:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$2 + 2\cos$	**4.0**	**3.7**	**3.0**	**2.0**	**1.0**	**0.27**	**0**	**.27**	**1.0**	**2.0**	**3.0**	**3.7**	**4.0**

This type of curve is called a cardioid. It is a special type of limaçon that has $r = a + a\cos\theta$ or $r = a + a\sin\theta$ as its polar equation. The ratio of $\frac{a}{b} = \frac{2}{2}$ which is equal to 1.



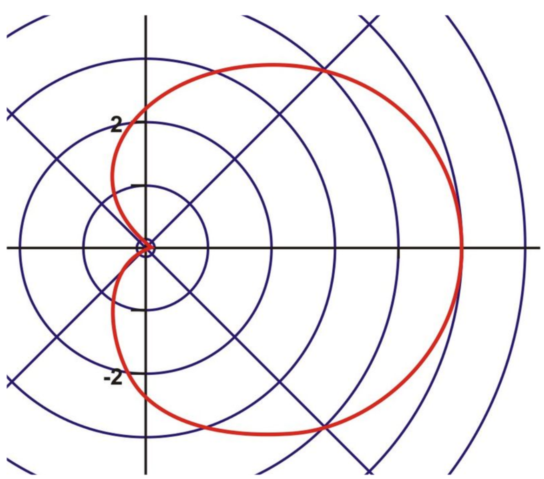
Examples 3 and 4 were shown with θ measured in degrees while examples 5 and 6 were shown with θ measured in radians. The results in the tables and the resulting graphs will be the same in both units.

Now that you are familiar with the limaçon and the cardioid, also called classical curves, it is time to examine the polar pattern of the cardioid microphone. The polar pattern is modeled by the polar equation $r = 2.5 + 2.5\cos\theta$. The values of a and b are equal which means that the ratio $\frac{a}{b} = 1$. Therefore the limaçon will be a cardioid.

Create a table of values for $r = 2.5 + 2.5\cos\theta$ such that $0^\circ \le \theta \le 360^\circ$ and graph the results.

TABLE 6.5:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$2.5 + 2.5\cos\theta$	**5.0**	**4.7**	**3.8**	**2.5**	**1.3**	**0.3**	**0**	**0.3**	**1.3**	**2.5**	**3.8**	**4.7**	**5.0**



Transformations of Polar Graphs

Equations of limaçons have two general forms:

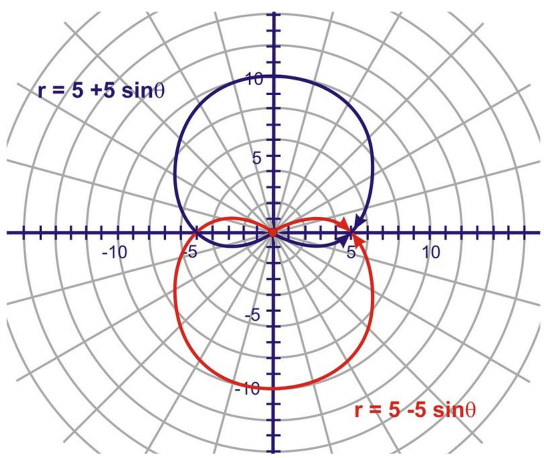
$$r = a \pm b \sin\theta \qquad \text{and} \qquad r = a \pm b \cos\theta$$

The values of "a'' and "b'' will determine the shape of the graph and whether or not it passes through the origin. When the values of "a'' and "b'' are equal, the graph will be a rounded heart-shape called a **cardioid**. The general polar equation of a cardioid can be written as $r = a(1 \pm \sin\theta)$ and $r = a(1 \pm \cos\theta)$. Note: The general polar equation of a cardioid can also be written as $r = a(-1 \pm \sin\theta)$ and $r = a(-1 \pm \cos\theta)$. This will be discussed later in the chapter.

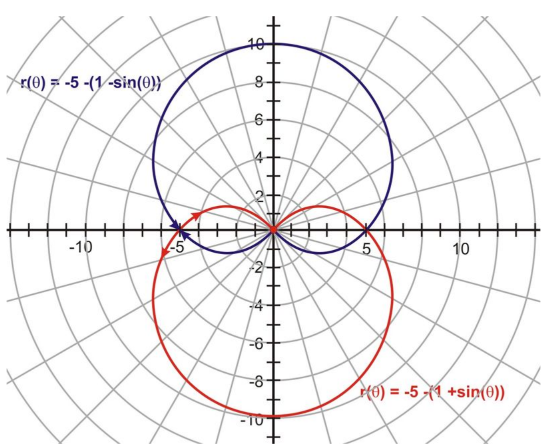
Example 7: Graph the following polar equations on the same polar grid and compare the graphs.

$$r = 5 + 5\sin\theta \qquad r = 5 - 5\sin\theta$$
$$r = 5(1 + \sin\theta) \qquad r = 5(1 - \sin\theta)$$

Solution:



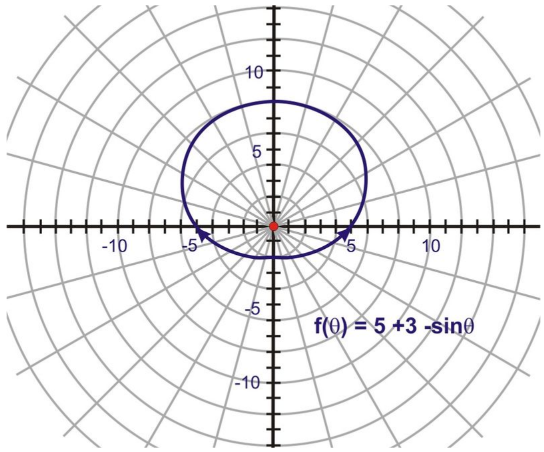
The cardioid is symmetrical about the positive $y-$axis and the point of indentation is at the pole. The result of changing + to - is a reflection in the $x-$axis. The cardioid is symmetrical about the negative $y-$axis and the point of indentation is at the pole.



Changing the value of "a'' to a negative did not change the graph of the cardioid.

Example 8: What effect will changing the values of a and b have on the cardioid if $a > b$? We can discover the answer to this question by plotting the graph of $r = 5 + 3\sin\theta$.

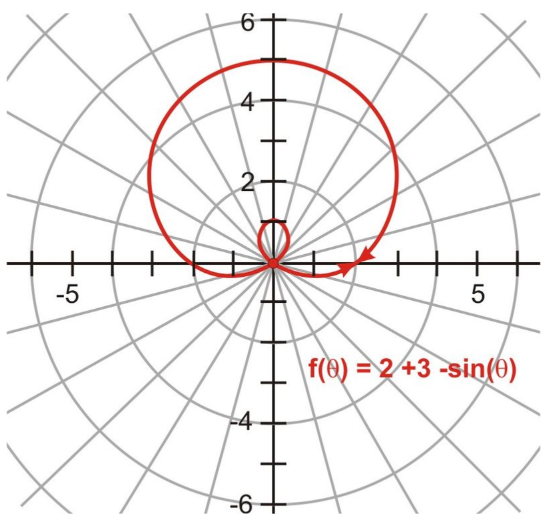
Solution:



The cardioid is symmetrical about the positive $y-$axis and the point of indentation is pulled away from the pole.

Example 9: What effect will changing the values of a and b or changing the function have on the cardioid if $a < b$? We can discover the answer to this question by plotting the graph of $r = 2 + 3\sin\theta$.

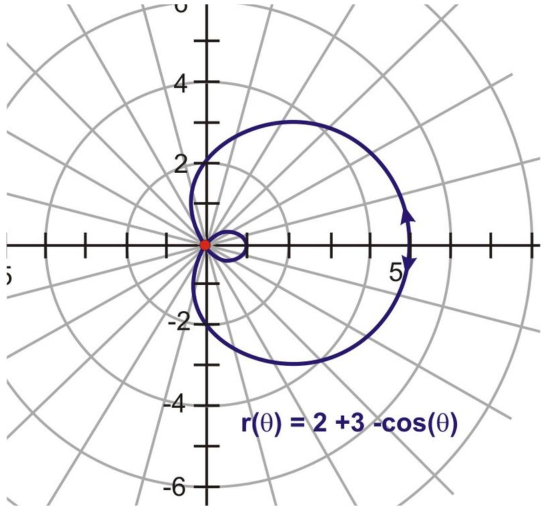
Solution:



The cardioid is now a looped limaçon symmetrical about the positive $y-$axis. The loop crosses the pole.

$$r = 2 + 3\cos\theta$$

The cardioid is now a looped limaçon symmetrical about the positive $x-$axis. The loop crosses the pole. Changing the function to cosine rotated the limaçon 90° clockwise.



As you have seen from all of the graphs, transformations can be performed by making changes in the constants and/or the functions of the polar equations.

Applications

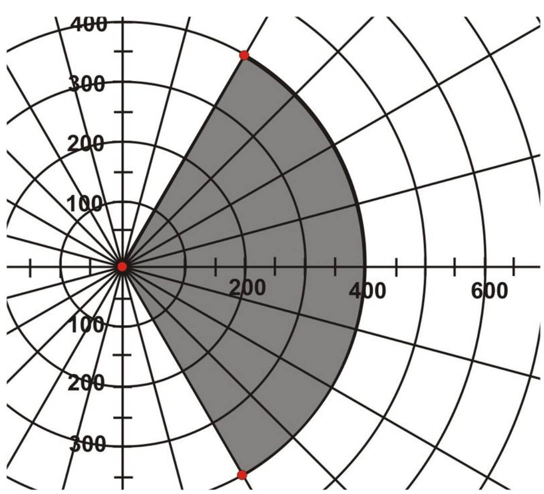
In this subsection we will explore examples of real-world problems that use polar coordinates and polar equations as their solutions.

Example 10: A local charity is sponsoring an outdoor concert to raise money for the children's hospital. To accommodate as many patrons as possible, they are importing bleachers so that all the fans will be seated during the performance. The seats will be placed in an area such that $\frac{-\pi}{3} \leq \theta \leq \frac{\pi}{3}$ and $0 \leq r \leq 4$, where r is measured in hundreds of feet. The stage will be placed at the origin (pole) and the performer will face the audience in the direction of the polar axis (r).

a. Create a polar graph of this area.

b. If all the seats are occupied and each seat takes up 5 square feet of space, how many people will be seated in the bleachers?

Solution: Now that the region has been graphed, the next step is to calculate the area of this sector. To do this, use the formula $A = \frac{1}{2}r^2\theta$.



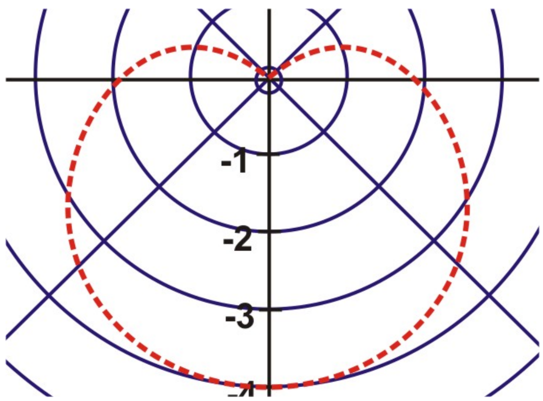
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(400)^2\left(\frac{2\pi}{3}\right)$$
$$A \approx 167552\ ft^2.$$

$$167552\ ft^2. \div 5\ ft^2. \approx 33510$$

The number of people in the bleachers is 33510.

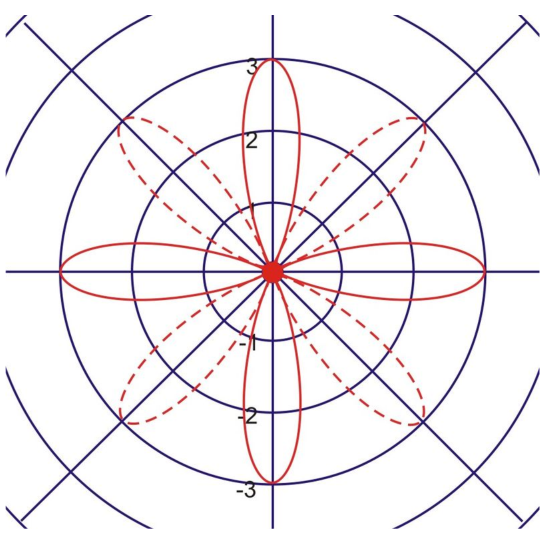
Example 11: When Valentine's Day arrives, hearts can be seen everywhere. As an alternative to purchasing a greeting card, use a computer to create a heart shape. Write an equation that could be used to create this heart and be careful to ensure that it is displayed in the correct position.

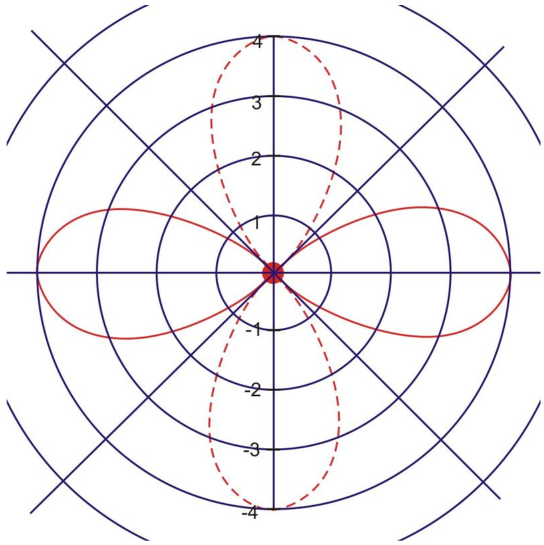
Solution: The classical curve that resembles a heart is a cordioid. You may have to experiment with the equation to create a heart shape that is displayed in the correct direction. One example of an equation that produces a proper heart shape is $r = -2 - 2\sin\theta$.

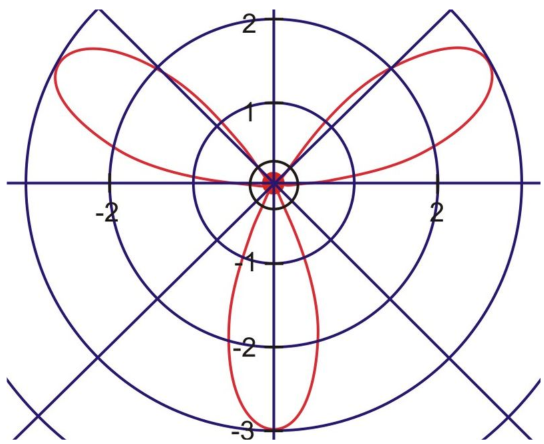


You can create other hearts by replacing the number 2 in the equation. Another equation is $r = -3 - 3\sin\theta$.

Example 12: For centuries, people have been making quilts. These are frequently created by sewing a uniform fabric pattern onto designated locations on the quilt. Using the equation that models a rose curve, create three patterns that could be used for a quilt. Write the equation for each rose and sketch its graph. Explain why the patterns have different numbers of petals. Can you create a sample quilt?

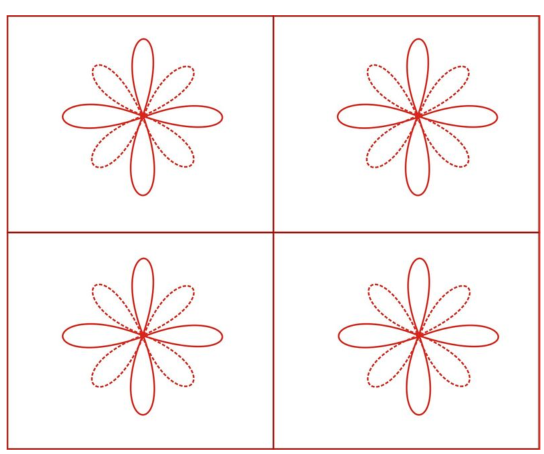






Solution: The rose curve is a graph of the polar equation of the form $r = a \cos n\theta$ or $r = a \sin n\theta$. If n is odd, then the number of petals will be equal to n. If n is even, then the number of petals will be equal to $2n$.

A Sample Quilt:



Graphing Polar Equations on the Calculator

You can use technology, the TI graphing calculator, to create these graphs. However, there are steps that must be followed in order to graph polar equations correctly on the graphing calculator. We will go through the step by step process to plot the polar equation $r = 3\cos\theta$.

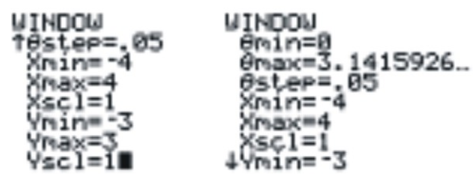
Example 13: Graph $r = 3\cos\theta$ using the TI-83 graphing calculator.

Solution: Press the [MODE] button. Scroll down to **Func** and over to highlight **Pol**. Also, while on this screen, make sure that **Radian** is highlighted. Now you must edit the axes for the graph. Press [WINDOW] 0 [ENTER] [2^{nd}] [π] [ENTER] .05

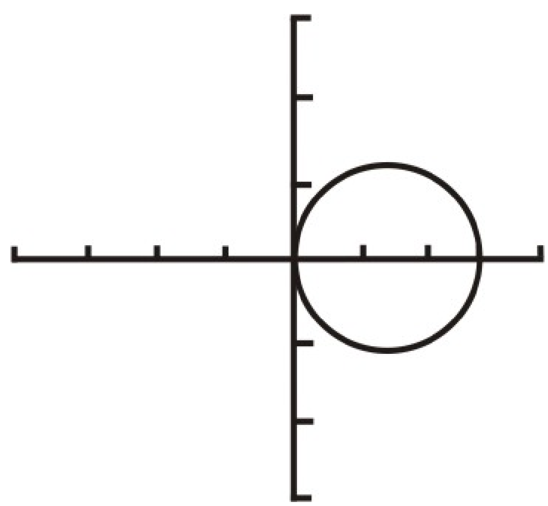
[ENTER] 1 [ENTER] $(-)$ 3 [ENTER] 3 [ENTER] 1 [ENTER].

When you have completed these steps, the screen should look like this:

The second **WINDOW** shows part of the first screen since you had to scroll down to access the remaining items.



Enter the equation. Press [$Y = 3\cos X, T, \theta, n)$] Press [GRAPH].

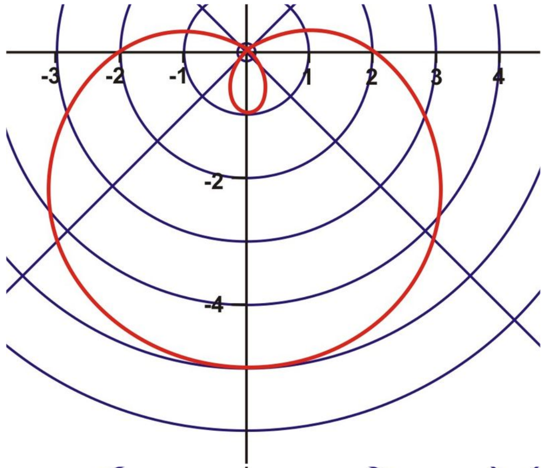


Sometimes the polar equation you graph will look more like an ellipse than a circle. If this happens, press [ZOOM] 5 to set a square viewing window. This will make the graph appear like a circle.

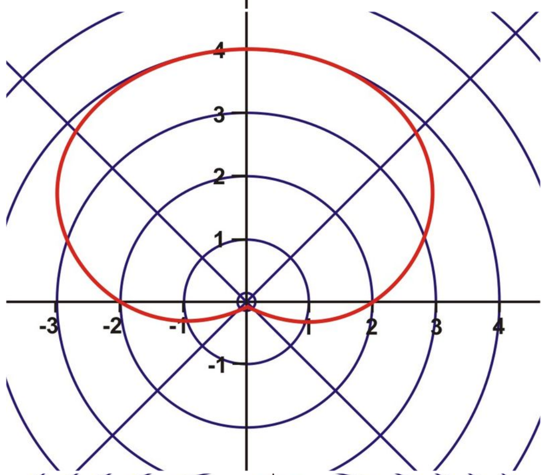
Note: If you want the calculator to graph complete rose petals when n is even, you must set $\theta\text{max} = 2\pi$.

Review Questions

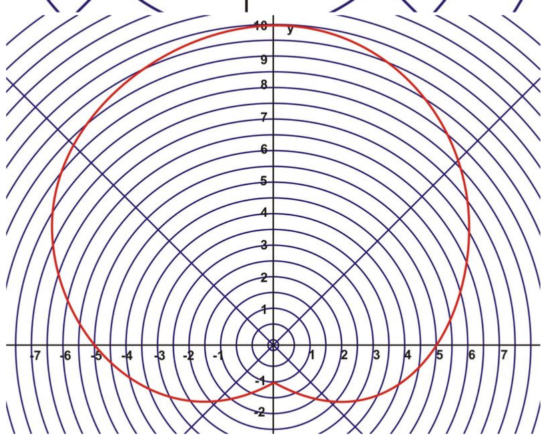
1. Name the classical curve in each of the following diagrams and explain why you feel you're your answer is correct. Also, find the equation of each curve.



a.



b.



c.

2. Graph each curve below. Comparing your answers from part one, determine if you can find a pattern for how to find the equation of a classical curve from its graph.

   a. $r = -3 - 3\cos\theta$
   b. $r = 2 + 4\sin\theta$
   c. $r = 4$
   d. $\theta = \frac{\pi}{2}$
   e. $r = 5 + 3\cos\theta$
   f. $r = -6 - 5\sin\theta$

3. Another classical curve we saw is called a rose and it is modeled by the function $r = a\cos n\theta$ or $r = a\sin n\theta$ where n is any positive integer. Graph $r = 4\cos 2\theta$ and $r = 4\cos 3\theta$. Is there a difference in the curves? Explain.
4. Graph the roses below. Determine if you can find a pattern for how to find the equation of a rose from its graph.

a. $r = 3\sin 4\theta$
b. $r = 2\sin 5\theta$
c. $r = 3\cos 3\theta$
d. $r = -4\sin 2\theta$
e. $r = 5\cos 4\theta$
f. $r = -2\cos 6\theta$

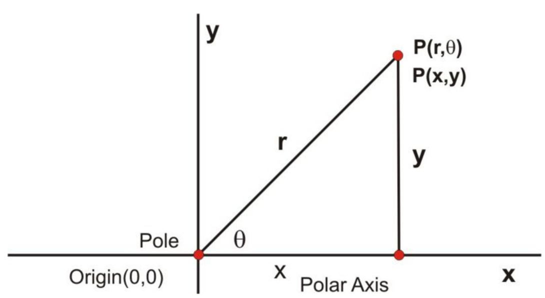
6.3 Converting Between Systems

Learning Objectives

- Convert rectangular coordinates to polar coordinates.
- Convert equations given in rectangular form to equations in polar form and vice versa.

Polar to Rectangular

Just as x and y are usually used to designate the rectangular coordinates of a point, r and θ are usually used to designate the polar coordinates of the point. r is the distance of the point to the origin. θ is the angle that the line from the origin to the point makes with the positive $x-$axis. The diagram below shows both polar and Cartesian coordinates applied to a point P. By applying trigonometry, we can obtain equations that will show the relationship between polar coordinates (r,θ) and the rectangular coordinates (x,y)



The point P has the polar coordinates (r,θ) and the rectangular coordinates (x,y).

Therefore

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

These equations, also known as coordinate conversion equations, will enable you to convert from polar to rectangular form.

Example 1: Given the following polar coordinates, find the corresponding rectangular coordinates of the points: $W(4,-200^\circ), H\left(4,\frac{\pi}{3}\right)$

Solution:

a) For $W(4,-200^\circ), r = 4$ and $\theta = -200^\circ$

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ x &= 4\cos(-200^\circ) & y &= 4\sin(-200^\circ) \\ x &= 4(-.9396) & y &= 4(.3420) \\ x &\approx -3.76 & y &\approx 1.37 \end{aligned}$$

The rectangular coordinates of W are approximately $(-3.76, 1.37)$.

b) For $H\left(4, \frac{\pi}{3}\right)$, $r = 4$ and $\theta = \frac{\pi}{3}$

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ x &= 4\cos\frac{\pi}{3} & y &= 4\sin\frac{\pi}{3} \\ x &= 4\left(\frac{1}{2}\right) & y &= 4\left(\frac{\sqrt{3}}{2}\right) \\ x &= 2 & y &= 2\sqrt{3} \end{aligned}$$

The rectangular coordinates of H are $(2, 2\sqrt{3})$ or approximately $(2, 3.46)$.

In addition to writing polar coordinates in rectangular form, the coordinate conversion equations can also be used to write polar equations in rectangular form.

Example 2: Write the polar equation $r = 4\cos\theta$ in rectangular form.

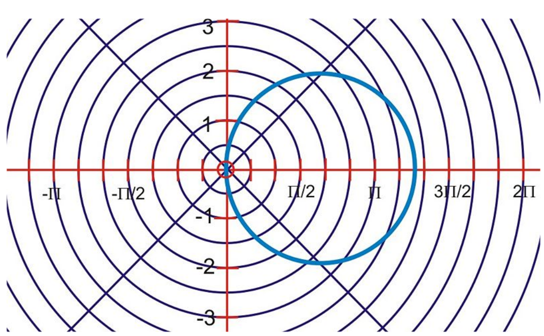
Solution:

$$\begin{aligned} r &= 4\cos\theta & & \\ r^2 &= 4r\cos\theta & & \textit{Multiply both sides by r.} \\ x^2 + y^2 &= 4x & & r^2 = x^2 + y^2 \textit{ and } x = r\cos\theta \end{aligned}$$

The equation is now in rectangular form. The r^2 and θ have been replaced. However, the equation, as it appears, does not model any shape with which we are familiar. Therefore, we must continue with the conversion.

$$\begin{aligned} x^2 - 4x + y^2 &= 0 & & \\ x^2 - 4x + 4 + y^2 &= 4 & & \textit{Complete the square for } x^2 - 4x. \\ (x-2)^2 + y^2 &= 4 & & \textit{Factor } x^2 - 4x + 4. \end{aligned}$$

The rectangular form of the polar equation represents a circle with its centre at (2, 0) and a radius of 2 units.



This is the graph represented by the polar equation $r = 4\cos\theta$ for $0 \le \theta \le 2\pi$ or the rectangular form $(x-2)^2 + y^2 = 4$.

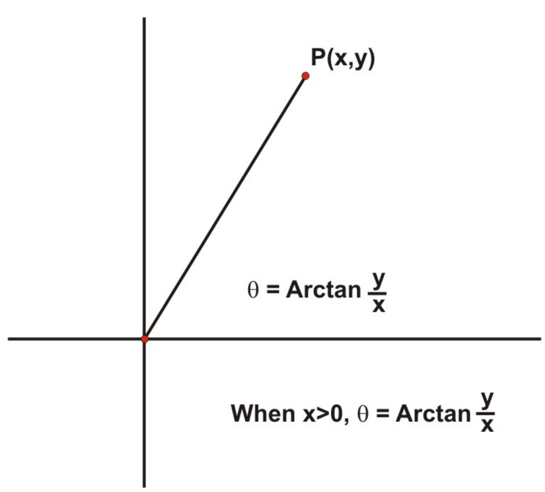
Example 3: Write the polar equation $r = 3\csc\theta$ in rectangular form.

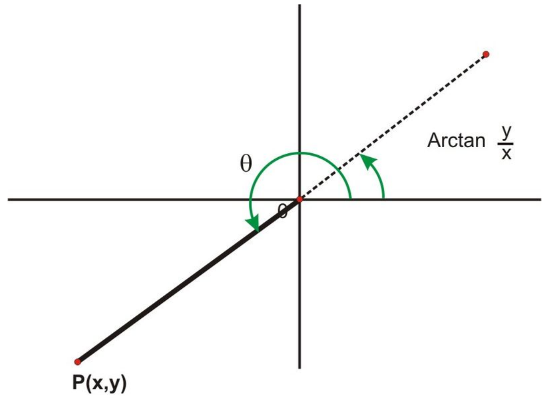
Solution:

$$\begin{aligned} r &= 3\csc\theta \\ \frac{r}{\csc\theta} &= 3 && \textit{divide by} \csc\theta \\ r\cdot\frac{1}{\csc\theta} &= 3 \\ r\sin\theta &= 3 && \sin\theta = \frac{1}{\csc\theta} \\ y &= 3 && y = r\sin\theta \end{aligned}$$

Rectangular to Polar

When converting rectangular coordinates to polar coordinates, we must remember that there are many possible polar coordinates. We will agree that when converting from rectangular coordinates to polar coordinates, one set of polar coordinates will be sufficient for each set of rectangular coordinates. Most graphing calculators are programmed to complete the conversions and they too provide one set of coordinates for each conversion. The conversion of rectangular coordinates to polar coordinates is done using the Pythagorean Theorem and the Arctangent function. The Arctangent function only calculates angles in the first and fourth quadrants so π radians must be added to the value of θ for all points with rectangular coordinates in the second and third quadrants.





In addition to these formulas, $r = \sqrt{x^2+y^2}$ is also used in converting rectangular coordinates to polar form.

Example 4: Convert the following rectangular coordinates to polar form.

$P(3,-5)$ and $Q(-9,-12)$

Solution: For $P(3,-5)$ $x = 3$ and $y = -5$. The point is located in the fourth quadrant and $x > 0$.

$$\begin{aligned} r &= \sqrt{x^2+y^2} & \theta &= Arc\ \tan\frac{y}{x} \\ r &= \sqrt{(3)^2+(-5)^2} & \theta &= \tan^{-1}\left(-\frac{5}{3}\right) \\ r &= \sqrt{34} & \theta &\approx -1.03 \\ r &\approx 5.83 \end{aligned}$$

The polar coordinates of $P(3,-5)$ are $P(5.83,-1.03)$.

For $Q(-9,-12)$ $x = -9$ and $y = -5$. The point is located in the third quadrant and $x < 0$.

$$\begin{aligned} r &= \sqrt{x^2+y^2} & \theta &= Arc\tan\frac{y}{x}+\pi \\ r &= \sqrt{(-9)^2+(-12)^2} & \theta &= \tan^{-1}\left(\frac{-12}{-9}\right)+\pi \\ r &= \sqrt{225} & \theta &\approx 4.07 \\ r &= 15 \end{aligned}$$

The polar coordinates of $Q(-9,-12)$ are $Q(15,4.07)$

Converting Equations

To write a rectangular equation in polar form, the conversion equations of $x = r\cos\theta$ and $y = r\sin\theta$ are used.

Example 5: Write the rectangular equation $x^2+y^2 = 2x$ in polar form.

Solution: Remember $r = \sqrt{x^2+y^2}, r^2 = x^2+y^2$ and $x = r\cos\theta$.

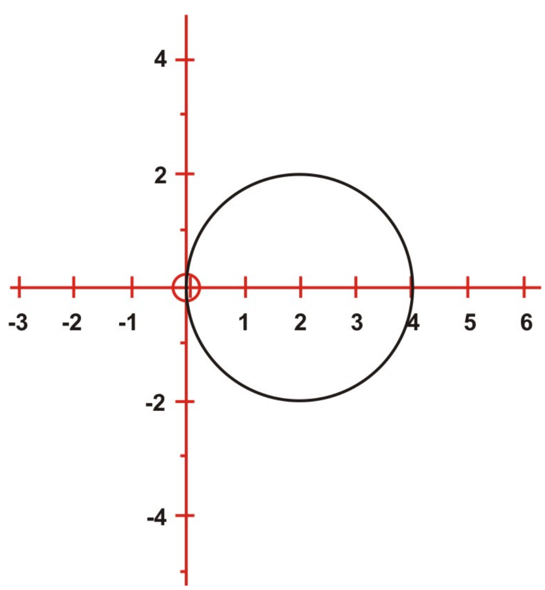
$$\begin{aligned} x^2+y^2 &= 2x \\ r^2 &= 2(r\cos\theta) && \text{Pythagorean Theorem and } x = r\cos\theta \\ r^2 &= 2r\cos\theta \\ r &= 2\cos\theta && \text{Divide each side by } r \end{aligned}$$

Example 6: Write the rectangular equation $(x-2)^2+y^2=4$ in polar form.

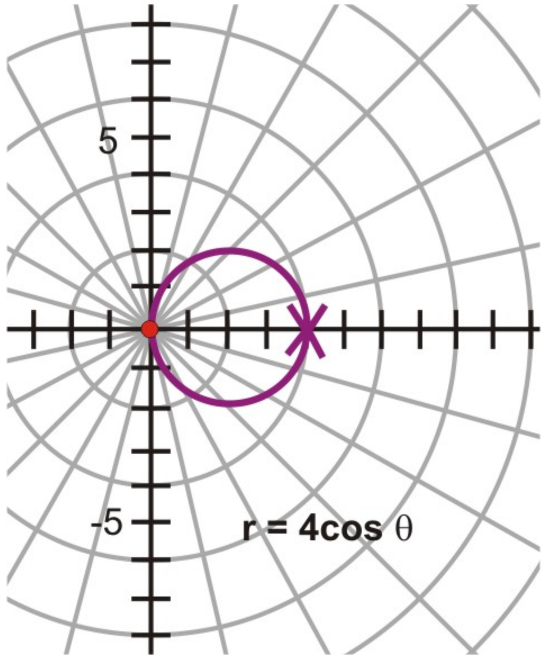
Solution: Remember $x=r\cos\theta$ and $y=r\sin\theta$.

$$\begin{aligned} &(x-2)^2+y^2=4 \\ &(r\cos\theta-2)^2+(r\sin\theta)^2=4 && x=r\cos\theta \text{ and } y=r\sin\theta \\ &r^2\cos^2\theta-4r\cos\theta+4+r^2\sin^2\theta=4 && \text{expand the terms} \\ &r^2\cos^2\theta-4r\cos\theta+r^2\sin^2\theta=0 && \text{subtract 4 from each side} \\ &r^2\cos^2\theta+r^2\sin^2\theta=4r\cos\theta && \text{isolate the squared terms} \\ &r^2(\cos^2\theta+\sin^2\theta)=4r\cos\theta && \text{factor } r^2 - \text{a common factor} \\ &r^2=4r\cos\theta && \text{Pythagorean Identity} \\ &r=4\cos\theta && \text{Divide each side by } r \end{aligned}$$

If the graph of the polar equation is the same as the graph of the rectangular equation, then the conversion has been determined correctly.



$(x-2)^2+y^2=4$



The rectangular equation $(x-2)^2+y^2=4$ represents a circle with center (2, 0) and a radius of 2 units. The polar equation $r=4\cos\theta$ is a circle with center (2, 0) and a radius of 2 units.

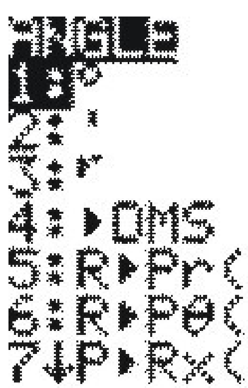
Converting Using the Graphing Calculator

You have learned how to convert back and forth between polar coordinates and rectangular coordinates by using the various formulae presented in this lesson. The TI graphing calculator allows you to use the angle function to convert coordinates quickly from one form to the other. The calculator will provide you with only one pair of polar coordinates for each pair of rectangular coordinates.

Example 7: Express the rectangular coordinates of $A(-3,7)$ as polar coordinates.

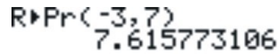
Polar coordinates are expressed in the form (r,θ). An angle can be measured in either degrees or radians, and the calculator will express the result in the form selected in the MODE menu of the calculator.

Press MODE and cursor down to Radian Degree. Highlight **Degree.** Press 2nd mode to return to home screen. To access the angle menu of the calculator press 2nd APPS and this screen will appear:



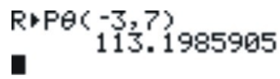
Cursor down to **5** and press ENTER. The following screen will appear

. Press **-3, 7)** ENTER and the value of *r* will appear



. Access the angle menu again by pressing 2^{nd} APPS. When the angle menu screen appears, cursor down to **6** and pres ENTER or press **6** on the calculator. The screen

will appear. Press **-3, 7)** ENTER and the value of θ will appear.



This procedure can be repeated to determine the rectangular coordinates in radians. Before starting, press MODE and cursor down to Radian Degree and highlight **Radian**.

Example 8: Express the polar coordinates of $(300, 70°)$ in rectangular form.

The angle θ is given in degrees so the mode menu of the calculator should also be set in degree. Therefore, press MODE and cursor down to Radian Degree and highlight **degree.** Press 2^{nd} mode to return to home screen. To access the angle menu of the calculator press 2^{nd} APPS and this screen will appear:

Cursor down to 7 and press ENTER or press 7 on the calculator. The following screen will screen will appear:

Press **300, 70)** and the value of *x* will appear

Access the angle menu again by pressing 2^{nd} APPS. When the angle menu screen appears, cursor down to **8** and pres ENTER or press **8** on the calculator. The screen

will appear. Press **300,70)** ENTER and the value of y will appear

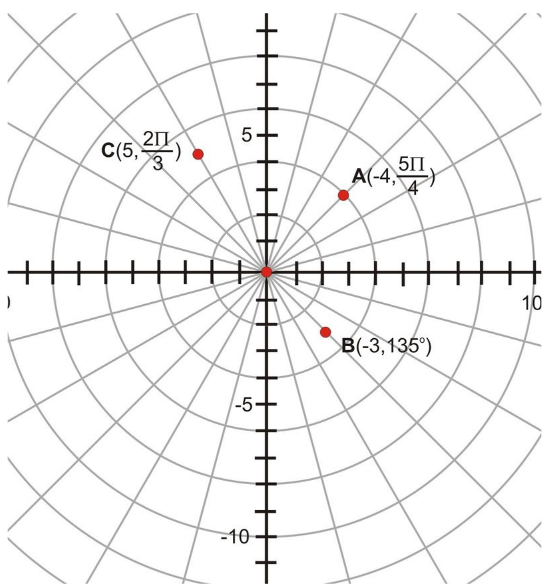
.

Points to Consider

- When we convert coordinates from polar form to rectangular form, the process is very straightforward. However, when converting a coordinate from rectangular form to polar form there are some choices to make. For example the point 0,1 could translate to $(1, 2\pi)$ or to $(1, -4\pi)$, and so on.
- Are there any advantages to using polar coordinates instead of rectangular coordinates? List any situations in which this is the case. What types of curves are easier to draw with polar coordinates?
- List situations in which rectangular coordinates are preferable.

Review Questions

1. For the following polar coordinates that are shown on the graph, determine the rectangular coordinates for each point.



2. Write the following polar equations in rectangular form.
   a. $r = 6\cos\theta$
   b. $r\sin\theta = -3$
   c. $r = 2\sin\theta$
   d. $r\sin^2\theta = 3\cos\theta$
3. Write the following rectangular points in polar form.
   a. $A(-2, 5)$ using radians
   b. $B(5, -4)$ using radians
   c. $C(1, 9)$ using degrees

d. $D(-12,-5)$ using degrees

4. Write the rectangular equations in polar form.

a. $(x-4)^2+(y-3)^2=25$
b. $3x-2y=1$
c. $x^2+y^2-4x+2y=0$
d. $x^3=4y^2$

6.4 More with Polar Curves

Learning Objectives

- Graph polar curves to see the points of intersection of the curves.
- Graph equivalent polar curves.
- Recognize equivalent polar curves from their equations.

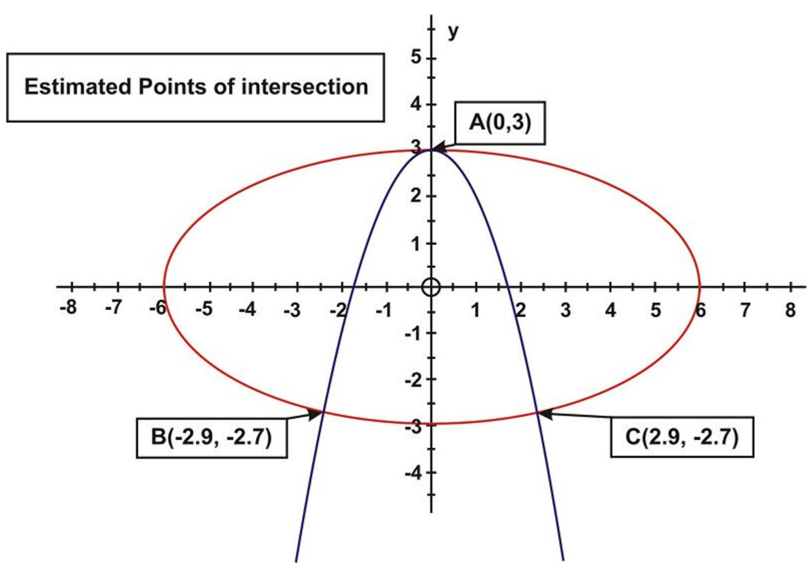
Intersections of Polar Curves

When you worked with a system of linear equations with two unknowns, finding the point of intersection of the equations meant finding the coordinates of the point that satisfied both equations. If the equations are rectangular equations for curves, determining the point(s) of intersection of the curves involves solving the equations algebraically since each point will have one ordered pair of coordinates associated with it.

Example 1: Solve the following system of equations algebraically:

$$x^2+4y^2-36=0$$
$$x^2+y=3$$

Solution: Before solving the system, graph the equations to determine the number of points of intersection.



The graph of $x^2+4y^2-36=0$ is an ellipse and the graph represented by $x^2+y=3$ is a parabola. There are three points of intersection. To determine the exact values of these points, algebra must be used.

$$x^2+4y^2-36=0 \to x^2+4y^2=36 \qquad x^2+4y^2+0y=36 \qquad x^2+4y^2+0y=36$$
$$x^2+y=3 \to x^2+0y^2+y=3 \qquad -1(x^2+0y^2+y=3) \qquad -x^2-0y^2-y=-3$$
$$4y^2-y=33$$

Using the quadratic formula, $a=4$ $b=-1$ $c=-33$

$$y=\frac{-(-1)\pm\sqrt{(-1)^2-4(4)(-33)}}{2(4)}$$
$$y=\frac{1+23}{8}=3 \qquad y=\frac{1-23}{8}=-2.75$$

These values must be substituted into one of the original equations.

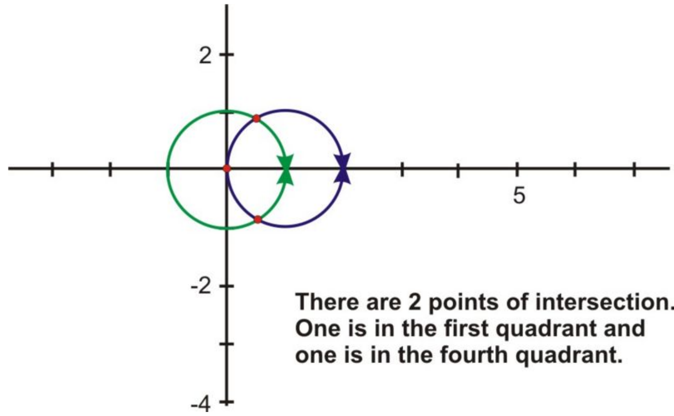
$$\begin{aligned} x^2+y&=3 \\ x^2+3&=3 \\ x^2&=0 \\ x&=0 \end{aligned} \qquad \begin{aligned} x^2+y&=3 \\ x^2+(-2.75)&=3 \\ x^2&=5.75 \\ x&=\pm\sqrt{5.75}\approx 2.4 \end{aligned}$$

The three points of intersection as determined algebraically in Cartesian representation are $\mathrm{A}(0,3), \mathrm{B}(2.4,-2.75)$ and $\mathrm{C}(2.4,2.75)$.

If we are working with polar equations to determine the polar coordinates of a point of intersection, we must remember that there are many polar coordinates that represent the same point. Remember that switching to polar form changes a great deal more than the notation. Unlike the Cartesian system which has one name for each point, the polar system has an infinite number of names for each point. One option would be to convert the polar coordinates to rectangular form and then to convert the coordinates for the intersection points back to polar form. Perhaps the best option would be to explore some examples. As these examples are presented, be sure to use your graphing calculator to create your own visual representations of the equations presented.

Example 2: Determine the polar coordinates for the intersection point(s) of the following polar equations: $r=1$ and $r=2\cos\theta$.

Solution: Begin with the graph. Using the process described in the technology section in this chapter; create the graph of these polar equations on your graphing calculator. Once the graphs are on the screen, use the **trace** function and the arrow keys to move the cursor around each graph. As the cursor is moved, you will notice that the equation of the curve is shown in the upper left corner and the values of θ, x, y are shown (in decimal form) at the bottom of the screen. The values change as the cursor is moved.



$$r = 1 \qquad 2\cos\theta = 1$$
$$r = 2\cos\theta \qquad \cos\theta = \frac{1}{2}$$
$$\cos^{-1}(\cos\theta) = \cos^{-1}\frac{1}{2}$$

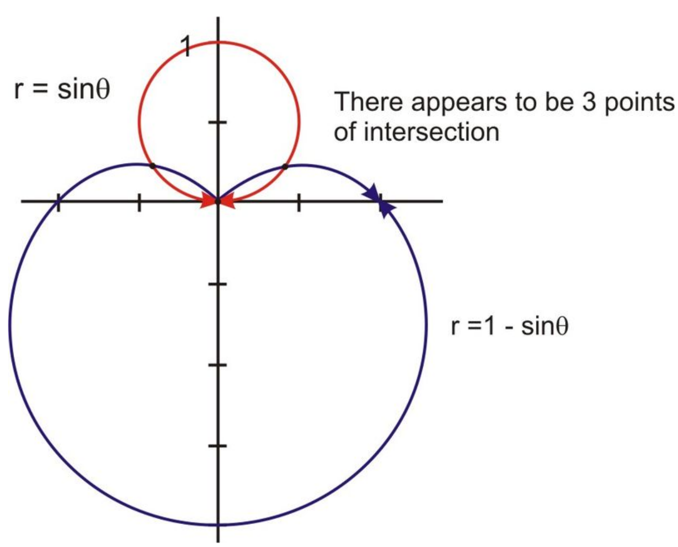
$\theta = \frac{\pi}{3}$ in the first quadrant and $\theta = \frac{5\pi}{3}$ in the fourth quadrant.

The points of intersection are $\left(1, \frac{\pi}{3}\right)$ and $\left(1, \frac{5\pi}{3}\right)$. However, these two solutions only cover the possible values $0 \le \theta \le 2\pi$. If you consider that $\cos\theta = \frac{1}{2}$ is true for an infinite number of theta these solutions must be extended to include $\left(1, \frac{\pi}{3}\right)$ and $\left(1, \frac{5\pi}{3}\right) + 2\pi k, k\varepsilon Z$. Now the solutions include all possible rotations.

This example was solved as any system of rectangular equations would be solved. Does this approach work all the time?

Example 3: Find the intersection of the graphs of $r = \sin\theta$ and $r = 1 - \sin\theta$

Solution: Begin with the graph. You can create these graphs using your graphing calculator.



$$\begin{aligned} &r = \sin\theta && \sin\theta = 1 - \sin\theta \\ &r = 1 - \sin\theta && 2\sin\theta = 1 \\ & && \sin\theta = \frac{1}{2} \\ &r = \sin\theta && \theta = \frac{\pi}{6} \text{ in the first quadrant and } \theta = \frac{5\pi}{6} \text{ in the second quadrant.} \\ &r = \sin\frac{\pi}{6} && \text{The intersection points are } \left(\frac{1}{2}, \frac{\pi}{6}\right) \text{ and } \left(\frac{1}{2}, \frac{5\pi}{6}\right) \\ &r = \frac{1}{2} && \text{Another intersection point seems to be the origin } (0,0). \end{aligned}$$

If you consider that $\sin\theta = \frac{1}{2}$ is true for an infinite number of theta as was $\cos\theta = \frac{1}{2}$ in the previous example, the same consideration must be applied to include all possible solutions. To prove if the origin is indeed an intersection point, we must determine whether or not both curves pass through (0, 0).

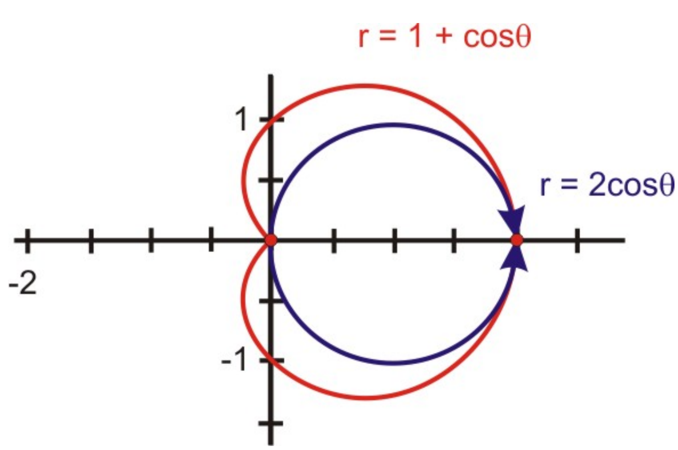
$$\begin{aligned} &r = \sin\theta && r = 1 - \sin\theta \\ &0 = \sin\theta && 0 = 1 - \sin\theta \\ &r = 0 && 1 = \sin\theta \\ & && \frac{\pi}{2} = 0 \end{aligned}$$

From this investigation, the point (0, 0) was on the curve $r = \sin\theta$ and the point $\left(0, \frac{\pi}{2}\right)$ was on the curve $r = 1 - \sin\theta$. Because the second coordinates are different, it seems that they are two different points. However, the coordinates represent the same point (0,0). The intersection points are $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and (0,0).

Sometimes it is helpful to convert the equations to rectangular form, solve the system and then convert the polar coordinates back to polar form.

Example 4: Find the intersection of the graphs of $r = 2\cos\theta$ and $r = 1 + \cos\theta$

Solution: Begin with the graph:



$r = 2\cos\theta$ expressed in rectangular form

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta && \textit{Multiply by } r \\ x^2+y^2 &= 2x && \textit{Substitution} \end{aligned}$$

$r = 1 + cos\theta$ expressed in rectangular form

$$\begin{aligned} r &= 1+\cos\theta \\ r^2 &= r + r\cos\theta && \textit{Multiply by } r \\ x^2+y^2 &= \sqrt{x^2+y^2}+x && \textit{Substitution} \end{aligned}$$

The equations are now in rectangular form. Solve the system of equations.

$$\begin{aligned} x^2+y^2 &= 2x \\ x^2+y^2 &= \sqrt{x^2+y^2}+x \\ 2x &= \sqrt{2x}+x \\ x &= \sqrt{2x} \\ x^2 &= 2x \\ x^2-2x &= 0 \\ x(x-2) &= 0 \\ x &= 0 && x-2=0 \\ & && x=2 \end{aligned}$$

Substituting these values into the first equation:

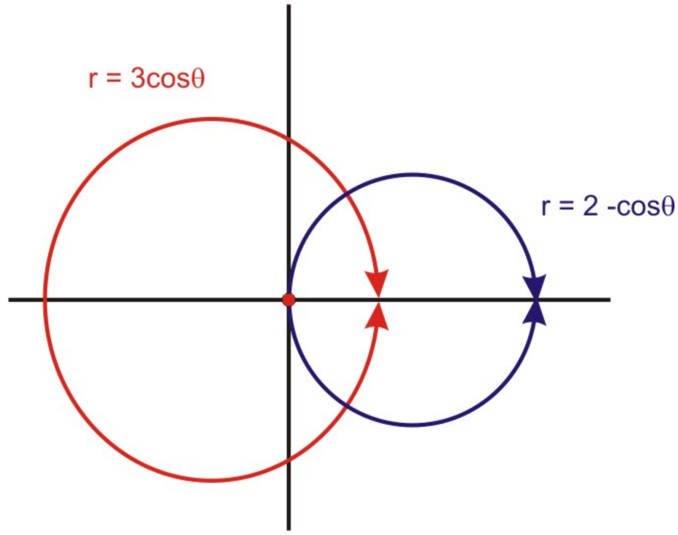
$$\begin{aligned} x^2+y^2 &= 2x & x^2+y^2 &= 2x \\ (0)^2+y^2 &= 2(0) & (2)^2+y^2 &= 2(2) \\ y^2 &= 0 & 4+y^2 &= 4 \\ y &= 0 & y^2 &= 0 \\ & & y &= 0 \end{aligned}$$

The points of intersection are $(0,0)$ and $(2,0)$.

The rectangular coordinates are $(0,0)$ and $(2,0)$. Converting these coordinates to polar coordinates give the same coordinates in polar form. The points can be converted by using the angle menu of the TI calculator.

Example 5: Josie is drawing a mural with polar equations. One mural is represented by the equation $r = 3\cos\theta$ and the other by $r = 2 - \cos\theta$. She wants to see where they will intersect before she transfers her image onto the wall where she is painting.

Solution: To determine where they will intersect, we will begin with a graph.



$$r = 3\cos\theta$$
$$r = 2 - \cos\theta$$
$$3\cos\theta = 2 - \cos\theta$$
$$3\cos\theta + \cos\theta = 2$$
$$4\cos\theta = 2$$
$$\cos\theta = \frac{2}{4} = \frac{1}{2}$$
$$\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}$$

$$r = 3\cos\theta \qquad r = 3\cos\theta$$
$$r = 3\cos\frac{\pi}{3} \qquad r = 3\cos\frac{5\pi}{3}$$
$$r = 3\cdot\frac{1}{2} = \frac{3}{2} \qquad r = 3\cdot\frac{1}{2} = \frac{3}{2}$$

Josie's murals would intersect and two points $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

Equivalent Polar Curves

The expression "same only different" comes into play in this lesson. We will graph two distinct polar equations that will produce two equivalent graphs. Use your graphing calculator and create these curves as the equations are presented.

Previously, graphs were generated of a limaçon, a dimpled limaçon, a looped limaçon and a cardioid. All of these were of the form $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$. The easiest way to see what polar equations produce equivalent curves is to use either a graphing calculator or a software program to generate the graphs of various polar equations.

Example 6: Plot the following polar equations and compare the graphs.

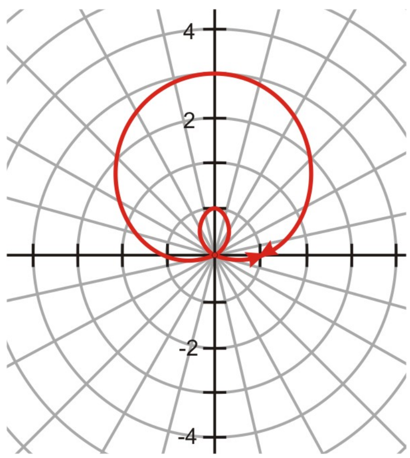
a)

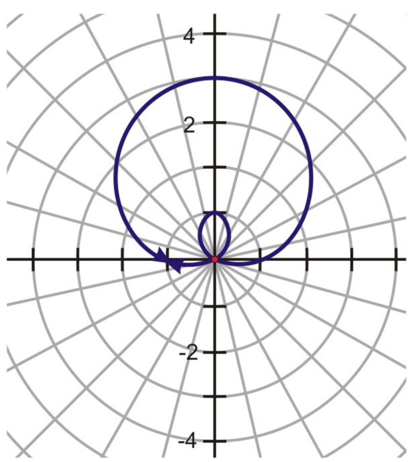
$$r = 1 + 2\sin\theta$$
$$r = -1 + 2\sin\theta$$

b)

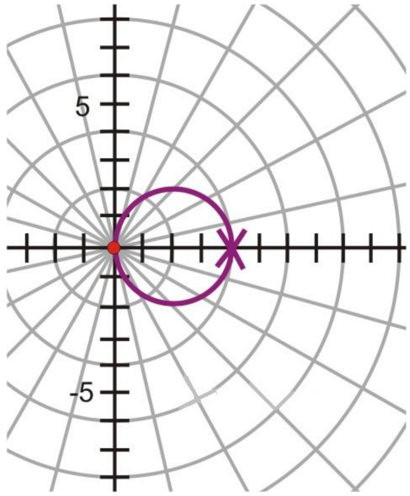
$$r = 4\cos\theta$$
$$r = 4\cos(-\theta)$$

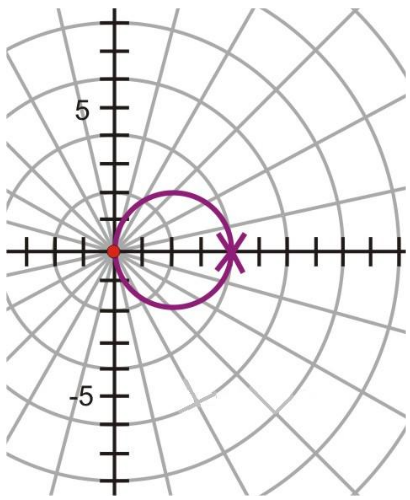
Solution: By looking at the graphs, the result is the same. So, even though *a* is different in both, they have the same graph. We can assume that the sign of *a* does not matter.





b) These functions also result in the same graph. Here, θ differed by a negative. So we can assume that the sign of θ does not change the appearance of the graph.

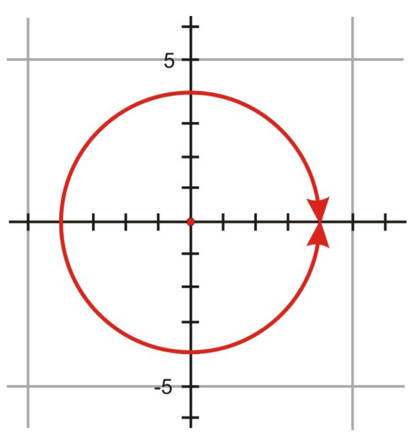


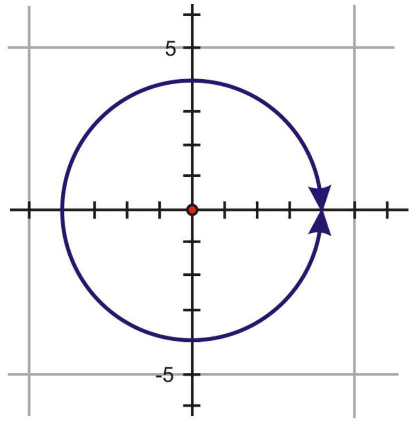


Example 7:

Graph the equations $x^2 + y^2 = 16$ and $r = 4$. Describe the graphs.

Solution:





Both equations, one in rectangular form and one in polar form, are circles with a radius of 4 and center at the origin.

Example 8: Graph the equations $(x-2)^2+(y+2)^2=8$ and $r=4\cos\theta-4\sin\theta$. Describe the graphs.

Solution: There is not a visual representation shown here, but on your calculator you should see that the graphs are circles centered at (2, -2) with a radius $2\sqrt{2}\approx 2.8$.

Points to Consider

- When looking for intersections, which representation is easier to work with? Look over the examples and find some in which doing the algebra in polar coordinates is more direct than finding intersections in Cartesian form.
- Will polar curves always intersect?
- If not, when will intersection not occur?
- If two polar curves have different equations, can they be the same curve?

Review Questions

1. Find the intersection of the graphs of $r=\sin 3\theta$ and $r=3\sin\theta$.
2. Find the intersection of the graphs of $r=2+2\sin\theta$ and $r=2-2\cos\theta$
3. Write the rectangular equation $x^2+y^2=6x$ in polar form and graph both equations. Should they be equivalent?
4. Determine if $r=-2+\sin\theta$ and $r=2-\sin\theta$ are equivalent *without* graphing.
5. Determine if $r=-3+4\cos(-\pi)$ and $r=3+4\cos\pi$ are equivalent *without* graphing.
6. Graph the equations $r=7-3\cos\frac{\pi}{3}$ and $r=7-3\cos\left(-\frac{\pi}{3}\right)$. Are they equivalent?
7. Formulate a theorem about equivalent polar curves resulting from the equations $r=a\pm b\cos\theta$ or $r=a\pm b\sin\theta$. What can be different to yield the same graph? What must be the same? Explain your answer and show graphs to support your conclusions.
8. Determine two polar curves that will never intersect.

6.5 The Trigonometric Form of Complex Numbers

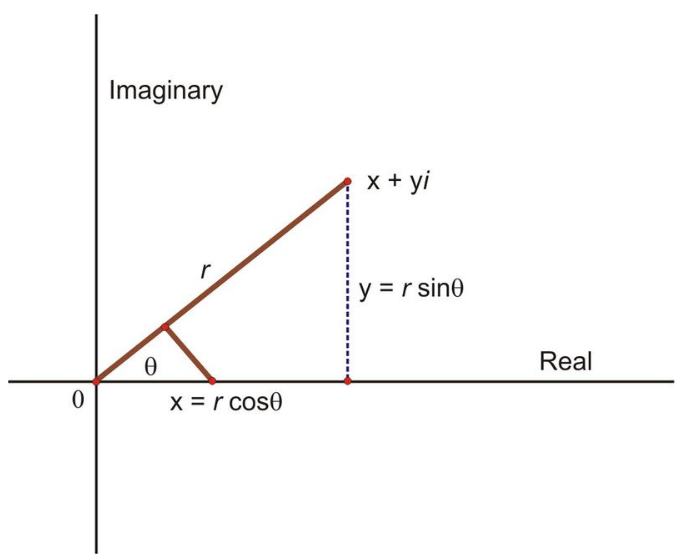
Learning Objectives

- Understand the relationship between the rectangular form of complex numbers and their corresponding polar form.
- Convert complex numbers from standard form to polar form and vice versa.

A number in the form $a+bi$, where a and b are real numbers, and i is the imaginary unit, or $\sqrt{-1}$, is called a complex number. Despite their names, complex numbers and imaginary numbers have very real and significant applications in both mathematics and in the real world. Complex numbers are useful in pure mathematics, providing a more consistent and flexible number system that helps solve algebra and calculus problems. We will see some of these applications in the examples throughout this lesson.

The Trigonometric or Polar Form of a Complex Number

The following diagram will introduce you to the relationship between complex numbers and polar coordinates.



In the figure above, the point that represents the number $x+yi$ was plotted and a vector was drawn from the origin to this point. As a result, an angle in standard position, θ, has been formed. In addition to this, the point that represents $x+yi$ is r units from the origin. Therefore, any point in the complex plane can be found if the angle θ and the $r-$ value are known. The following equations relate x, y, r and θ.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2+y^2 \qquad \tan\theta = \frac{y}{x}$$

If we apply the first two equations to the point $x+yi$ the result would be:

$$x+yi = r\cos\theta + ri\sin\theta \rightarrow r(\cos\theta + i\sin\theta)$$

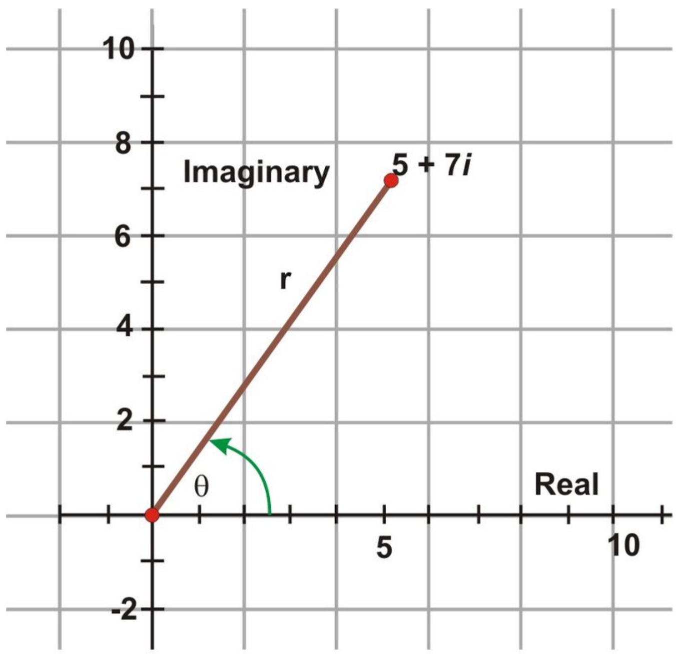
The right side of this equation $r(\cos\theta + i\sin\theta)$ is called the **polar** or **trigonometric** form of a complex number. A shortened version of this polar form is written as $r\ cis\ \theta$. The length r is called the **absolute value** or the **modulus**, and the angle θ is called the **argument** of the complex number. Therefore, the following equations define the polar form of a complex number:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad x+yi = r(\cos\theta + i\sin\theta)$$

It is now time to implement these equations perform the operation of converting complex numbers in standard form to complex numbers in polar form. You will use the above equations to do this.

Example 1: Represent the complex number $5+7i$ graphically and express it in its polar form.

Solution: As discussed in the Prerequisite Chapter, here is the graph of $5+7i$.



Graphic Representation

Converting to polar from rectangular, $x = 5$ and $y = 7$.

$$r = \sqrt{5^2 + 7^2} = 8.6$$

$$\tan\theta = \frac{7}{5}$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}\frac{7}{5}$$
$$\theta = 54.5^\circ$$

So, the polar form is $8.6(\cos 54.5^\circ + i\sin 54.5^\circ)$.

Another widely used notation for the polar form of a complex number is $r\angle\theta = r(\cos\theta + i\sin\theta)$. Now there are three ways to write the polar form of a complex number.

$$x + yi = r(\cos\theta + i\sin\theta) \qquad x + yi = rcis\theta \qquad x + yi = r\angle\theta$$

Example 2: Express the following polar form of each complex number using the shorthand representations.

a) $4.92(\cos 214.6^\circ + i\sin 214.6^\circ)$

b) $15.6(\cos 37^\circ + i\sin 37^\circ)$

Solution:

a) $4.92\angle 214.6^\circ$

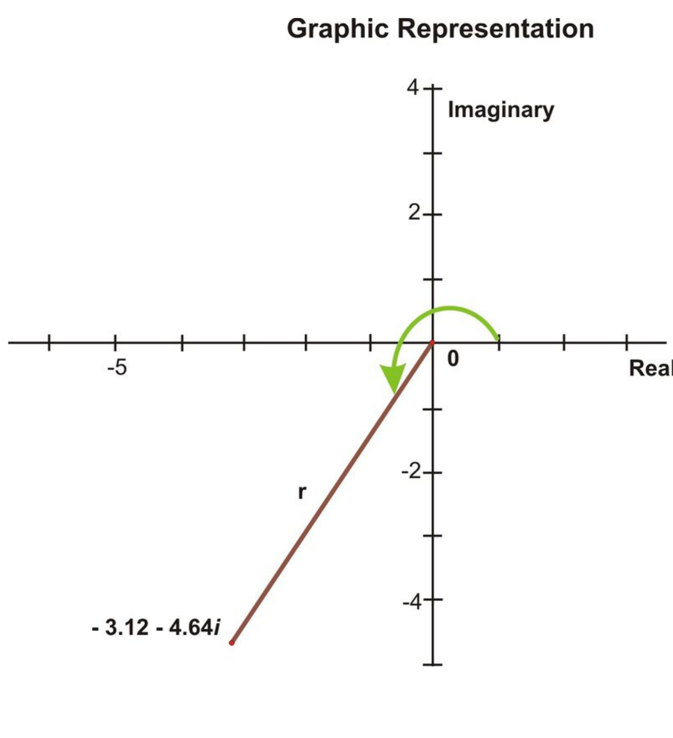
4.92 *cis* 214.6°

b) $15.6\angle 37^\circ$

15.6 *cis* 37°

Example 3: Represent the complex number $-3.12 - 4.64i$ graphically and give two notations of its polar form.

Solution: From the rectangular form of $-3.12 - 4.64i$ $x = -3.12$ and $y = -4.64$



$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-3.12)^2 + (-4.64)^2}$$
$$r = 5.59$$

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{-4.64}{-3.12}$$
$$\theta = 56.1^\circ$$

This is the reference angle so now we must determine the measure of the angle in the third quadrant. $56.1^\circ + 180^\circ = 236.1^\circ$

One polar notation of the point $-3.12-4.64i$ is $5.59(\cos 236.1^\circ + i\sin 236.1^\circ)$. Another polar notation of the point is $5.59\angle 236.1^\circ$

So far we have expressed all values of theta in degrees. Polar form of a complex number can also have theta expressed in radian measure. This would be beneficial when plotting the polar form of complex numbers in the polar plane.

The answer to the above example $-3.12-4.64i$ with theta expressed in radian measure would be:

$$\tan\theta = \frac{-4.64}{-3.12}$$

$$\tan\theta = .9788(\text{reference angle})$$

$$0.9788 + 3.14 = 4.12 \text{ rad.}$$

$$5.59(\cos 4.12 + i\sin 4.12)$$

Now that we have explored the polar form of complex numbers and the steps for performing these conversions, we will look at an example in circuit analysis that requires a complex number given in polar form to be expressed in standard form.

Example 4: The impedance Z, in ohms, in an alternating circuit is given by $Z = 4650\angle -35.2^\circ$. Express the value for Z in standard form. (In electricity, negative angles are often used.)

Solution: The value for Z is given in polar form. From this notation, we know that $r = 4650$ and $\theta = -35.2^\circ$ Using these values, we can write:

$$Z = 4650(\cos(-35.2^\circ) + i\sin(-35.2^\circ))$$
$$x = 4650\cos(-35.2^\circ) \to 3800$$
$$y = 4650\sin(-35.2^\circ) \to -2680$$

Therefore the standard form is $Z = 3800 - 2680i$ ohms.

Points to Consider

- Is it possible to perform basic operations on complex numbers in polar form?
- If operations can be performed, do the processes change for polar form or remain the same as for standard form?

Review Questions

1. Express the following polar forms of complex numbers in the two other possible ways.

a. $5\ cis\frac{\pi}{6}$
b. $3\angle 135^\circ$
c. $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

2. Express the complex number $6-8i$ graphically and write it in its polar form.
3. Express the following complex numbers in their polar form.

a. $4+3i$
b. $-2+9i$
c. $7-i$
d. $-5-2i$

4. Graph the complex number $3(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$ and express it in standard form.
5. Find the standard form of each of the complex numbers below.

a. $2\ cis\frac{\pi}{2}$
b. $4\angle\frac{5\pi}{6}$
c. $8\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$

6.6 The Product & Quotient Theorems

Learning Objectives

- Determine the quotient theorem of complex numbers in polar form.
- Determine the product theorem of complex numbers in polar form.
- Solve everyday problems that require you to use the product and/or quotient theorem of complex numbers in polar form to obtain the correct solution.

The Product Theorem

Multiplication of complex numbers in polar form is similar to the multiplication of complex numbers in standard form. However, to determine a general rule for multiplication, the trigonometric functions will be simplified by applying the sum/difference identities for cosine and sine. To obtain a general rule for the multiplication of complex numbers in polar from, let the first number be $r_1(\cos\theta_1 + i\sin\theta_1)$ and the second number be $r_2(\cos\theta_2 + i\sin\theta_2)$. The product can then be simplified by use of three facts: the definition $i^2 = -1$, the sum identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$, and the sum identity $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$.

Now that the numbers have been designated, proceed with the multiplication of these binomials.

$$
\begin{aligned}
& r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\
& r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\
& r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\
& r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]
\end{aligned}
$$

Therefore:

$$r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

Quotient Theorem

Division of complex numbers in polar form is similar to the division of complex numbers in standard form. However, to determine a general rule for division, the denominator must be rationalized by multiplying the fraction by the complex conjugate of the denominator. In addition, the trigonometric functions must be simplified by applying the sum/difference identities for cosine and sine as well as one of the Pythagorean identities. To obtain a general rule for the division of complex numbers in polar from, let the first number be $r_1(\cos\theta_1 + i\sin\theta_1)$ and the second number be $r_2(\cos\theta_2 + i\sin\theta_2)$. The product can then be simplified by use of five facts: the definition $i^2 = -1$, the difference identity $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$, the difference identity $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$, the Pythagorean identity, and the fact that the **conjugate** of $\cos\theta_2 + i\sin\theta_2$ **is** $\cos\theta_2 - i\sin\theta_2$.

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$$
$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}\cdot\frac{(\cos\theta_2-i\sin\theta_2)}{(\cos\theta_2-i\sin\theta_2)}$$
$$\frac{r_1}{r_2}\cdot\frac{\cos\theta_1\cos\theta_2-i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2-i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2-i^2\sin^2\theta_2}$$
$$\frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{\cos^2\theta_2+\sin^2\theta_2}$$
$$\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$

In general:

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$

Using the Product and Quotient Theorems

The following examples illustrate the use of the product and quotient theorems.

Example 1: Find the product of the complex numbers $3.61(\cos 56.3^\circ+i\sin 56.3^\circ)$ and $1.41(\cos 315^\circ+i\sin 315^\circ)$

Solution: Use the Product Theorem, $r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$.

$$\begin{aligned}&3.61(\cos 56.3^\circ+i\sin 56.3^\circ)\cdot 1.41(\cos 315^\circ+i\sin 315^\circ)\\&=(3.61)(1.41)[\cos(56.3^\circ+315^\circ)+i\sin(56.3^\circ+315^\circ)]\\&=5.09(\cos 371.3^\circ+i\sin 371.3^\circ)\\&=5.09(\cos 11.3^\circ+i\sin 11.3^\circ)\end{aligned}$$

*Note: Angles are expressed $0^\circ\le\theta\le 360^\circ$ unless otherwise stated.

Example 2: Find the product of $5\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\cdot\sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

Solution: First, calculate $r_1r_2=5\cdot\sqrt{3}=5\sqrt{3}$ and $\theta=\theta_1+\theta_2=\frac{3\pi}{4}+\frac{\pi}{2}=\frac{5\pi}{4}$

$$5\sqrt{3}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$

Example 3: Find the quotient of $(\sqrt{3}-i)\div(2-i2\sqrt{3})$

Solution: Express each number in polar form.

$$\begin{aligned}&\sqrt{3}-i\\&r_1=\sqrt{x^2+y^2}\\&r_1=\sqrt{(\sqrt{3})^2+(-1)^2}\\&r_1=\sqrt{4}=2\end{aligned}$$

$$\begin{aligned}&2-i2\sqrt{3}\\&r_2=\sqrt{x^2+y^2}\\&r_2=\sqrt{(2)^2+(-2\sqrt{3})^2}\\&r_2=\sqrt{16}=4\end{aligned}$$

$\frac{r_1}{r_2} = .5$

$\theta_1 = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ $\theta_2 = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right)$ $\theta = \theta_1 - \theta_2$

$\theta_1 = 5.75959 \ rad.$ $\theta_2 = 5.23599 \ rad.$ $\theta = 5.75959 - 5.23599$

$\theta = 0.5236$

Now, plug in what we found to the Quotient Theorem.

$$\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = .5(\cos 0.5236 + i\sin 0.5236)$$

Example 4: Find the quotient of the two complex numbers $28\angle 35^\circ$ and $14\angle 24^\circ$

Solution:

For $28\ \angle 35^\circ$ For $14\ \angle 24^\circ$ $\frac{r_1}{r_2} = \frac{28}{14} = 2$

$r_1 = 28$ $r_2 = 14$ $\theta = \theta_1 - \theta_2$

$\theta_1 = 35^\circ$ $\theta_2 = 24^\circ$ $\theta = 35^\circ - 24^\circ = 11^\circ$

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$
$$= 2\angle 11^\circ$$

Points to Consider

- We have performed the basic operations of arithmetic on complex numbers, but we have not dealt with any exponents or any roots of complex numbers. How might you calculate $(x+yi)^2$ or $\sqrt{r\angle\theta}$?
- How might you calculate the n^{th} power or root of a complex number?

Review Questions

1. Multiply together the following complex numbers. If they are not in polar form, change them before multiplying.
   a. $2\angle 56^\circ, 7\angle 113^\circ$
   b. $3(\cos\pi + i\sin\pi), 10\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$
   c. $2+3i, -5+11i$
   d. $6-i, -20i$
2. Part c from #1 was not in polar form. Multiply the two complex numbers together without changing them into polar form. Which method do you think is easier?

3. Use the Product Theorem to find $4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^2$.
4. The electric power (in watts) supplied to an element in a circuit is the product of the voltage e and the current i (in amps). Find the expression for the power supplied if $e = 6.80\angle 56.3^\circ$ volts and $i = 7.05\angle -15.8^\circ$ amps. *Note: Use the formula* $P = ei$.
5. Divide the following complex numbers. If they are not in polar form, change them before dividing. In
   a. $\frac{2\angle 56^\circ}{7\angle 113^\circ}$
   b. $\frac{10\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)}{5(\cos\pi+i\sin\pi)}$
   c. $\frac{2+3i}{-5+11i}$
   d. $\frac{6-i}{1-20i}$
6. Part c from #5 was not in polar form. Divide the two complex numbers without changing them into polar form. Which method do you think is easier?
7. Use the Product Theorem to find $4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^3$. Hint: use #3 to help you.
8. Using the Quotient Theorem determine $\frac{1}{4cis\frac{\pi}{6}}$.

6.7 De Moivre's and the nth Root Theorems

Learning Objectives

- Use De Moivre's Theorem to find the powers of complex numbers in polar form.
- Find the n^{th} roots of complex numbers in polar form.

De Moivre's Theorem

The basic operations of addition, subtraction, multiplication and division of complex numbers have all been explored in this chapter. The addition and subtraction of complex numbers lent themselves best to numbers expressed in standard form. However multiplication and division were easily performed when the complex numbers were in polar form. Another operation that is performed using the polar form of complex numbers is the process of raising a complex number to a power.

The polar form of a complex number is $r(\cos\theta + i\sin\theta)$. If we allow z to equal the polar form of a complex number, it is very easy to see the development of a pattern when raising a complex number in polar form to a power. To discover this pattern, it is necessary to perform some basic multiplication of complex numbers in polar form. Recall #3 and #7 from the Review Questions in the previous section.

If $z = r(\cos\theta + i\sin\theta)$ and $z^2 = z \cdot z$ then:

$$z^2 = r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta)$$
$$z^2 = r^2[\cos(\theta+\theta) + i\sin(\theta+\theta)]$$
$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

Likewise, if $z = r(\cos\theta + i\sin\theta)$ and $z^3 = z^2 \cdot z$ then:

$$z^3 = r^2(\cos 2\theta + i\sin 2\theta) \cdot r(\cos\theta + i\sin\theta)$$
$$z^3 = r^3[\cos(2\theta+\theta) + i\sin(2\theta+\theta)]$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

Again, if $z = r(\cos\theta + i\sin\theta)$ and $z^4 = z^3 \cdot z$ then

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

These examples suggest a general rule valid for all powers of z, or n. We offer this rule and assume its validity for all n without formal proof, leaving that for later studies. The general rule for raising a complex number in polar form to a power is called De Moivre's Theorem, and has important applications in engineering, particularly circuit analysis. The rule is as follows:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Where $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer.

Notice what this rule looks like geometrically. A complex number taken to the nth power has two motions: First, its distance from the origin is taken to the nth power; second, its angle is multiplied by n. Conversely, the roots of a number have angles that are evenly spaced about the origin.

Example 1: Find. $[2(\cos 120^\circ + i\sin 120^\circ)]^5$

Solution: $\theta = 120^\circ = \frac{2\pi}{3}$ rad, using De Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\begin{aligned}(\cos 120^\circ + i\sin 120^\circ)^5 &= 2^5\left[\cos 5\tfrac{2\pi}{3} + i\sin 5\tfrac{2\pi}{3}\right]\\ &= 32\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)\\ &= 32\left(-\frac{1}{2} + -\frac{i\sqrt{3}}{2}\right)\\ &= -16 - 16i\sqrt{3}\end{aligned}$$

Example 2: Find $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{10}$

Solution: Change into polar form.

$$\begin{aligned} r &= \sqrt{x^2+y^2}\\ r &= \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\\ r &= \sqrt{\frac{1}{4}+\frac{3}{4}}\\ r &= \sqrt{1} = 1\end{aligned}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\cdot -\frac{2}{1}\right) = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

The polar form of $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ is $1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Now use De Moivre's Theorem:

$$\begin{aligned} z^n &= [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)\\ \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{10} &= 1^{10}\left[\cos 10\left(\frac{2\pi}{3}\right) + i\sin 10\left(\frac{2\pi}{3}\right)\right]\\ \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{10} &= 1\left(\cos\frac{20\pi}{3} + i\sin\frac{20\pi}{3}\right)\\ \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{10} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \rightarrow \text{ Standard Form}\end{aligned}$$

n

We have explored all of the basic operations of arithmetic as they apply to complex numbers in standard form and in polar form. The last discovery is that of taking roots of complex numbers in polar form. Using De Moivre's Theorem we can develop another general rule –one for finding the n^{th} root of a complex number written in polar form.

As before, let $z = r(\cos\theta + i\sin\theta)$ and let the n^{th} root of z be $v = s(\cos\alpha + i\sin\alpha)$. So, in general, $\sqrt[n]{z} = v$ and $v^n = z$.

$$\sqrt[n]{z} = v$$
$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = s(\cos\alpha + i\sin\alpha)$$

r$(\cos\theta + i\sin\theta)^{\frac{1}{n}} = s(\cos\alpha + i\sin\alpha)$

$$r^{\frac{1}{n}}\left(\cos\frac{1}{n}\theta + i\sin\frac{1}{n}\theta\right) = s(\cos\alpha + i\sin\alpha)$$
$$r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) = s(\cos\alpha + i\sin\alpha)$$

From this derivation, we can conclude that $r^{\frac{1}{n}} = s$ or $s^n = r$ and $\alpha = \frac{\theta}{n}$. Therefore, for any integer $k(0,1,2,\ldots n-1)$, v is an n^{th} root of z if $s = \sqrt[n]{r}$ and $\alpha = \frac{\theta+2\pi k}{n}$. Therefore, the general rule for finding the n^{th} roots of a complex number if $z = r(\cos\theta + i\sin\theta)$ is: $\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)$. Let's begin with a simple example and we will leave θ in degrees.

Example 3: Find the two square roots of $2i$.

Solution: Express $2i$ in polar form.

$$r = \sqrt{x^2+y^2} \qquad \cos\theta = 0$$
$$r = \sqrt{(0)^2+(2)^2} \qquad \theta = 90^\circ$$
$$r = \sqrt{4} = 2$$

$$(2i)^{\frac{1}{2}} = 2^{\frac{1}{2}}\left(\cos\frac{90^\circ}{2} + i\sin\frac{90^\circ}{2}\right) = \sqrt{2}(\cos 45^\circ + i\sin 45^\circ) = 1+i$$

To find the other root, add 360° to θ.

$$(2i)^{\frac{1}{2}} = 2^{\frac{1}{2}}\left(\cos\frac{450^\circ}{2} + i\sin\frac{450^\circ}{2}\right) = \sqrt{2}(\cos 225^\circ + i\sin 225^\circ) = -1-i$$

Example 4: Find the three cube roots of $-2-2i\sqrt{3}$

Solution: Express $-2-2i\sqrt{3}$ in polar form:

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{(-2)^2+(-2\sqrt{3})^2}$$
$$r = \sqrt{16} = 4 \qquad \theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) = \frac{4\pi}{3}$$

$$\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)$$
$$\sqrt[3]{-2-2i\sqrt{3}} = \sqrt[3]{4}\left(\cos\frac{\frac{4\pi}{3}+2\pi k}{3} + i\sin\frac{\frac{4\pi}{3}+2\pi k}{3}\right) k = 0,1,2$$

$$z_1 = \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9}+\frac{0}{3}\right)+i\sin\left(\frac{4\pi}{9}+\frac{0}{3}\right)\right] \qquad k=0$$
$$= \sqrt[3]{4}\left[\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}\right]$$
$$z_2 = \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9}+\frac{2\pi}{3}\right)+i\sin\left(\frac{4\pi}{9}+\frac{2\pi}{3}\right)\right] \qquad k=1$$
$$= \sqrt[3]{4}\left[\cos\frac{10\pi}{9}+i\sin\frac{10\pi}{9}\right]$$
$$z_3 = \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9}+\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{9}+\frac{4\pi}{3}\right)\right] \qquad k=2$$
$$= \sqrt[3]{4}\left[\cos\frac{16\pi}{9}+i\sin\frac{16\pi}{9}\right]$$

In standard form: $z_1 = 0.276+1.563i, z_2 = -1.492-0.543i, z_3 = 1.216-1.02i$.

Solve Equations

The roots of a complex number are cyclic in nature. This means that when the roots are plotted on the complex plane, the n^{th} roots are equally spaced on the circumference of a circle.

Since you began Algebra, solving equations has been an extensive topic. Now we will extend the rules to include complex numbers. The easiest way to explore the process is to actually solve an equation. The solution can be obtained by using De Moivre's Theorem.

Example 5:

Consider the equation $x^5 - 32 = 0$. The solution is the same as the solution of $x^5 = 32$. In other words, we must determine the fifth roots of 32.

Solution:

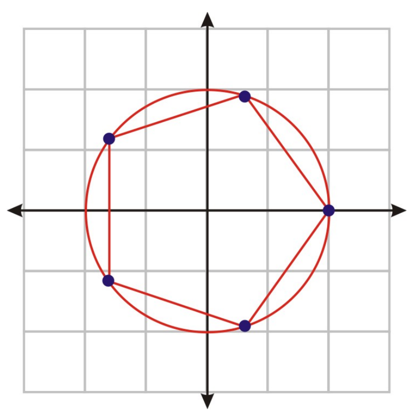
$$x^5 - 32 = 0 \text{ and } x^5 = 32.$$
$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{(32)^2+(0)^2}$$
$$r = 32$$
$$\theta = \tan^{-1}\left(\frac{0}{32}\right) = 0$$

Write an expression for determining the fifth roots of $32 = 32 + 0i$

$$\begin{aligned}
32^{\frac{1}{5}} &= [32(\cos(0+2\pi k)+i\sin(0+2\pi k)]^{\frac{1}{5}} \\
&= 2\left(\cos\frac{2\pi k}{5}+i\sin\frac{2\pi k}{5}\right) k = 0,1,2,3,4 \\
x_1 &= 2\left(\cos\frac{0}{5}+i\sin\frac{0}{5}\right) \to 2(\cos 0+i\sin 0) = 2 && for\ k=0 \\
x_2 &= 2\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right) \approx 0.62+1.9i && for\ k=1 \\
x_3 &= 2\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right) \approx -1.62+1.18i && for\ k=2 \\
x_4 &= 2\left(\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right) \approx -1.62-1.18i && for\ k=3 \\
x_5 &= 2\left(\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}\right) \approx 0.62-1.9i && for\ k=4
\end{aligned}$$

The Geometry of Complex Roots

In the previous example, we have one real and four complex roots. Plot these in the complex plane.



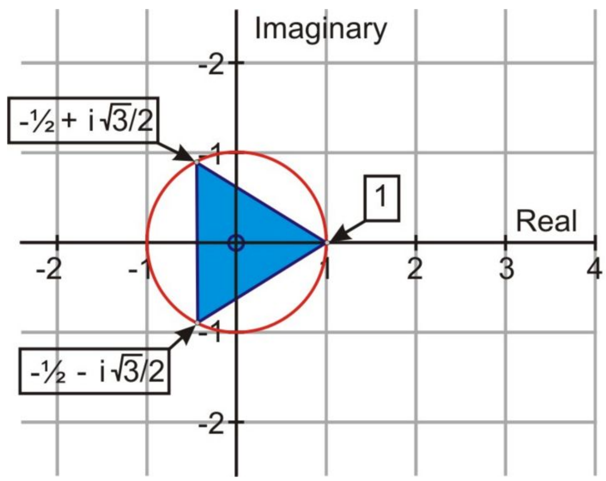
The n^{th} roots of a complex number, when graphed on the complex plane, are equally spaced around a circle. So, instead of having all the roots, all that is necessary to graph the roots is one of them and the radius of the circle. For this particular example, the roots are $\frac{2\pi}{5}$ or 72° apart (look in the root equation in the example, θ increased by $\frac{2\pi}{5}$). This goes along with what we know about regular pentagons. The roots are $\frac{2\pi}{n}$ degrees apart.

Example 6: Calculate the three cube roots of 1 and represent them graphically.

Solution: In standard form, $1 = 1+0i$ $r = 1$ and $\theta = 0$. The polar form is $1+0i = 1[\cos(0+2\pi k)+i\sin(0+2\pi k)]$. The expression for determining the cube roots of $1+0i$ is:

$$(1+0i)^{\frac{1}{3}} = 1^{\frac{1}{3}}\left(\cos\frac{0+2\pi k}{3}+i\sin\frac{0+2\pi k}{3}\right)$$

When $k = 0, k = 1$ and $k = 2$ the three cube roots of 1 are $1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$. When these three roots are represented graphically, the three points, on the circle with a radius of 1 (the cubed root of 1 is 1), form a triangle.



Points to Consider

- If the roots can be determined, will some form of De Moivre's Theorem be used?
- If the root of a complex number in polar form can be determined, can the solution to an exponential equation be found in the same way?
- Does the number of roots have anything to do with the shape of the graph?

Review Questions

1. Show that $z^3 = 1$, if $z = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$
2. Evaluate:
   a. $\left[\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8$
   b. $\left[3\left(\sqrt{3}-i\sqrt{3}\right)\right]^4$
   c. $(\sqrt{5}-i)^7$
   d. $\left[3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^{12}$
3. Rewrite the following in rectangular form: $[2(\cos 315^\circ+i\sin 315^\circ)]^3$
4. Find $\sqrt[3]{27i}$.

5. Find the principal root of $(1+i)^{\frac{1}{5}}$. Remember the principal root is the positive root i.e. $\sqrt{9} = \pm 3$ so the principal root is +3.
6. Find the fourth roots of $81i$.
7. Solve the equation $x^4 + 1 = 0$. What shape do the roots make?
8. Solve the equation $x^3 - 64 = 0$. What shape do the roots make?

Chapter Summary

In this chapter we made the connection between complex numbers and trigonometry. First, we started with the polar system, by graphing and converting equations into polar coordinates. This allowed us to compare the complex plane with the polar plane and we realized that there are many similarities. Because of this, we are able to convert complex numbers into polar, or trigonometric, form. Converting complex numbers to polar form makes it easier to multiply and divide complex numbers by using the Product and Quotient theorems. These theorems lead to De Moivre's Theorem, which is a shortcut for raising complex numbers to different powers. Finally, we were able manipulate De Moivre's Theorem to find all the complex solutions to different equations.

Vocabulary

Argument
In the complex number $r(\cos\theta + i\sin\theta)$, the argument is the angle θ.

Modulus
In the complex number $r(\cos\theta + i\sin\theta)$, the modulus is r. It is the distance from the origin to the point (x, y) in the complex plane.

Polar coordinate system
A method of recording the position of an object by using the distance from a fixed point and an angle consisting of a fixed ray from that point. Also called a polar plane.

Pole
In a polar coordinate system, it is the fixed point or origin.

Polar axis
In a polar coordinate system, it is the horizontal ray that begins at the pole and extends in a positive direction.

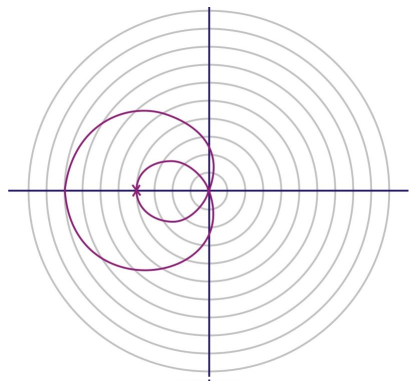
Polar coordinates
The coordinates of a point plotted on a polar plane (r, θ).

Polar Equation
An equation which uses polar coordinates.

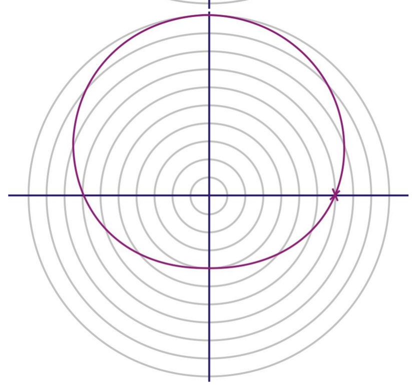
Polar Form
Also called trigonometric form is the complex number $x + yi$ written as $r(\cos\theta + i\sin\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$.

Review Questions

1. Plot $A\left(-3, \frac{3\pi}{4}\right)$ and find three other equivalent coordinates.
2. Find the distance between $(2, 94^\circ)$ and $(7, -73^\circ)$.
3. Graph the following polar curves.
    a. $r = 3\sin 5\theta$
    b. $r = 6 - 3\cos\theta$
    c. $r = 2 + 5\cos 9\theta$
4. Determine the equations of the curves below.



a.



b.

5. Convert each equation or point into polar form.
    a. $A(-6, 11)$
    b. $B(15, -8)$
    c. $C(9, 40)$
    d. $x^2 + (y-6)^2 = 36$
6. Convert each equation or point into rectangular form.
    a. $D\left(4, -\frac{\pi}{3}\right)$
    b. $E(-2, 135^\circ)$
    c. $r = 7$
    d. $r = 8\sin\theta$
7. Determine where $r = 6 + 5\sin\theta$ and $r = 3 - 4\cos\theta$ intersect.
8. Change $-3 + 8i$ into polar form.
9. Change $15\angle 240^\circ$ into rectangular form.
10. Multiply or divide the following complex numbers using polar form.
    a. $\left(7cis\frac{7\pi}{4}\right) \cdot \left(3cis\frac{\pi}{3}\right)$
    b. $\frac{8\angle 80^\circ}{2\angle -155^\circ}$

11. Expand $[4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)]^6$
12. Find the 6^{th} roots of -64 and graph them in the complex plane.
13. Find all the solutions of $x^4+32=0$.

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9704 .