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MATH S121
A Foundation In Pure Mathematics
(Free Courseware)

香港公開大學
THE OPEN UNIVERSITY
OF HONG KONG

Contents

Chapter 1 Counting and Basic Probability

1.1 About this module

Welcome to this free courseware module 'Counting and Basic Probability'!

This module is taken from the OUHK course *MATH S121 A Foundation in Pure Mathematics (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&lang=eng&ncode=MATH%20S121&shopping=Y&TYPE=CI&CODE=M121)*,a ten-credit, Foundation level course that is compulsory for a large number of Science and Technology programmes and an option in a number of other programmes, offered by the School of Science and Technology (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcSubWeb&l=C_ST&lid=191133000200&lang=eng) of the OUHK. MATH S121 aims to help students develop their skills in handling mathematical activities as well as their understanding of the rigour of mathematics for pursuing middle and higher-level courses in mathematics, computing, and engineering.

MATH S121 is mainly presented in printed format and comprises 13 study units. Each unit contains study content, activities, examples, exercises, self-tests, assigned readings, etc for students' self-learning. This module (The materials for this module, taken from the print-based course MATH S121, have been specially adapted to make them more suitable for studying online, and multimedia elements have been added where appropriate. In addition to this topic on 'Counting' and 'The probability of one event', which is an extract from Block III Unit 2 of the course, the original Unit 2 also includes the topic 'The probability of events', and 'Conditional probability'.) retains most of these elements, so you can have a taste of what an OUHK course is like. Please note that no credits can be earned on completion of this module. If you would like to pursue it further, you are welcome to enrol in *MATH S121 A Foundation in Pure Mathematics (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&lang=eng&ncode=MATH%20S121&shopping=Y&TYPE=CI&CODE=M121)*.

This module will take you about **twelve hours** to complete, including the time for completing the activities and exercises.

Good luck, and enjoy your study!

1.2 Introduction

Everyone in Hong Kong has heard of the Mark Six!

Players select six different numbers from 49 possibilities in the hope of winning a small fortune - or possibly even a large one.

So how many different choices of six numbers out of 49 are there? If you just sat down to count out all the possible combination, it would take you a very long time indeed. But in mathematics we have other counting tools that can assist us in this counting task.

Learning to use these tools is the basis for learning about counting and probability.

The first section in this module introduces basic counting principles, and helps you work through some simple counting problems. You will learn how the Addition and Multiplication Principles can be applied to determine the number of possible combinations. You will then look at two important counting tools, namely permutations and combinations, and compare the different counting principle between them.

The second section in this module introduces you to probability as a measure of chance. It gives you a numerical means of comparing different degrees of chance. You learn by looking at common games of chance: tossing a coin, selecting a playing card, or throwing dice.

1.3 Counting

Before you can study the properties of probability, you need to understand the basic counting principles which is a fundamental mathematical idea and an essential part of probability.

In this section, you'll develop the concept of two counting principles and use them to determine the number of different outcomes of a certain event, without having to list all of the elements.

1.3.1 Number of outcomes of an event

In general, we use a letter "*E*" to represent "an event", and *n(E)* to represent "*the number of outcomes of an event E*".

Let's look at a simple example. We may have an event *E* defined as

$$E = "day\ of\ the\ week"$$

Since there are 7 days in a week, in this example *the number of outcomes of an event E*" is

$$n(E) = 7.$$

1.3.2 Addition Principle

Let's look at a simple counting problem.

Example 1

Given a small set of numbers denoted by *A* = {-4, -2, 1, 3, 5, 6, 7, 8, 9, 10}, it is easy to see that there are a total of 10 numbers in *A*.

Now, if we are interested in determining the number of elements of the events E_1, E_2, and *E*, which are defined as:

- E_1 = choosing a negative number from *A*
- E_2 = choosing an odd number from *A*
- *E* = choosing a negative or an odd number from *A*.

Apply the Addition Principle to determine the number of outcomes of events *E* using E_1 and E_2.

Solution

E_1 = {-4, -2}, and the number of outcomes for E_1 is $n(E_1)$ = 2.

E_2 = {1, 3, 5, 7, 9}, and the number of outcomes for *E2* is $n(E_2)$ = 5.

Events E_1 and E_2 are mutually exclusive because there is no common outcomes in the list of E_1 and E_2. So the number of outcomes of event *E* is

$$\begin{aligned} n(E) &= n(E_1) + n(E_2) \\ &= 2 + 5 \\ &= 7 \end{aligned}$$

This answer can be confirmed if we can list out all the elements of *E*. In this example the outcomes of *E* is

$$E = \{-4, -2, 1, 3, 5, 7, 9\}$$

This gives *n(E)* = 7, which is the same answer as above.

Addition Principle

For a series of mutually exclusive events $E_1, E_2, E_3, \ldots, E_n$, and each event, say E_i has $n(E_j)$ outcomes regardless of the process made on the previous events.

Then, the total number of possible outcomes is given by

$$n(E_1) + n(E_2) + \cdots + n(E_n)$$

Example 2

In how many ways can a number be chosen from the set

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22\}$$

such that

a. it is a multiple of 3 or 8?

b. it is a multiple of 2 or 3?

Solution

a. Let E_1 = multiples of 3:

E_1 = {3, 6, 9, 12, 15, 18, 21}, so $n(E_1) = 7$.

Let E_2 = multiples of 8:

E_2 = {8, 16}, so $n(E_2) = 2$.

Events E_1 and E_2 are mutually exclusive, so

$n(E) = n(E_1) + n(E_2) = 7+2 = 9$

b. Let E_1 = multiples of 2:

E_1 = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22}, so $n(E_1) = 11$.

Let E_2 = multiples of 3:

E_2 = {3, 6, 9, 12, 15, 18, 21}, so $n(E_2) = 7$.

The events E_1 and E_2 are not mutually exclusive since they contain the same elements {6, 12, 18}. If the problem involves events that are not mutually exclusive, we can handle the counting as follows:

$$\begin{aligned} n(E) &= n(E_1) + n(E_2) - n(E_1 \cap E_2) \\ &= 11 + 7 - 3 \\ &= 15 \end{aligned}$$

where E1 ∩ E2 means 'the intersection of the sets E1 and E2'.

1.3.3 Multiplication Principle

Suppose that two independent events E_1 and E_1 can result in $n(E_1)$ and $n(E_2)$ possible outcomes, respectively, and that there are no restrictions on choice.

The total number of different combinations of possible outcomes from E1 and E1 can then be determined by using the Multiplication Principle, that is

$$n(E_1) \times n(E_2).$$

Multiplication Principle

If a process can be performed in a series of independent successive events E_1, E_2, E_3,. . . , E_n and each event, say E_i, can be completed in $n(E_j)$ ways regardless of the process made on the previous events.

Then the total number of different combinations of completing the entire process is given by

$$n(E_1) \times n(E_2) \times \cdots \times n(En)$$

Example 3

What is the total number of possible outcomes when you flip a fair coin twice?

Solution

Let T be the 'tail' and H be the 'head' of a fair coin. The events are:

- E_1 = first flip (2 possible outcomes: H or T, so $n(E_1)$= 2)
- E_2 = second flip (2 outcomes: H or T, so $n(E_2)$ = 2)

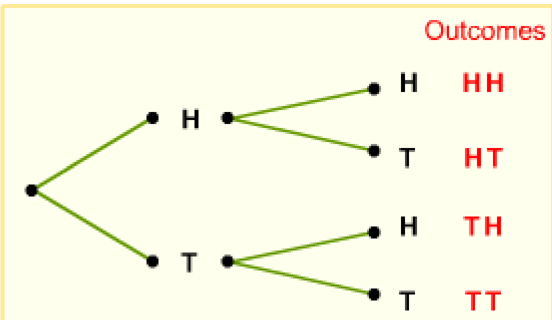
The events are independent and that the flip does not affect the outcome of the other flip.

Hence the total number of possible outcomes is

$$\begin{aligned} n(E_1) \times n(E_2) &= 2 \times 2 \\ &= 4 \end{aligned}$$

[We could also list the outcomes as: HH HT TH TT].

In the previous example, you can also model the 4 possible outcomes using a tree diagram or an outcome table, as shown below:



1st Flip	2nd Flip	Outcomes
H	H	HH
H	T	HT
T	H	TH

Table 1.1: Tree diagram

1st Flip	2nd Flip	Outcomes
T	T	TT
Outcome Table		

Table 1.1: Tree diagram

Example 4

The life insurance policies of an insurance company are classified by:

- age of the insured: under 25 years, between 25 years and 50 years, over 50 years old;
- sex: M or F; and
- marital status: single or married.

What is the total number of classifications?

Solution

The events are:

E_1 = age of the insured:	3 age divisions, so $n(E_1) = 3$
E_2 = sex:	2 possibilities of sex, so $n(E_2) = 2$
E_3 = marital status:	2 possibilities of martial status, so $n(E_3) = 2$

Table 1.2:

All events E_1, E_2 and E_3 are independent, so the total number of different classifications is

$$\begin{aligned} E_1 \times E_2 \times E_3 &= 3 \times 2 \times 2 \\ &= 12 \end{aligned}$$

1.3.4 Permutations and combinations

What is the *difference between a combination and a permutation*?

The essential difference between a combination and a permutation has to do with the order in which objects are selected.

A combination focuses on the selection of objects without regard to the order in which they are selected. A permutation, in contrast, focuses on the arrangement of objects with regard to the order in which they are arranged.

1.3.4.1 Permutation

A permutation is an arrangement of all or part of a set of n objects, with regard to the order of the arrangement and without repetition. That is, the order in which we arrange objects is important.

Notations for permutations

1. The number of different arrangement of a set of n distinct objects is given by

$$(n)(n-1)(n-2)...(3)(2)(1) = n!\ ways,$$

where $n!$ is called the 'n factorial'.

2. The number of permutations of a set of n distinct objects taken r at a time, denoted the permutation by $P(n, r)$ or $P_r{}^n$, where repetitions are not allowed, is given by

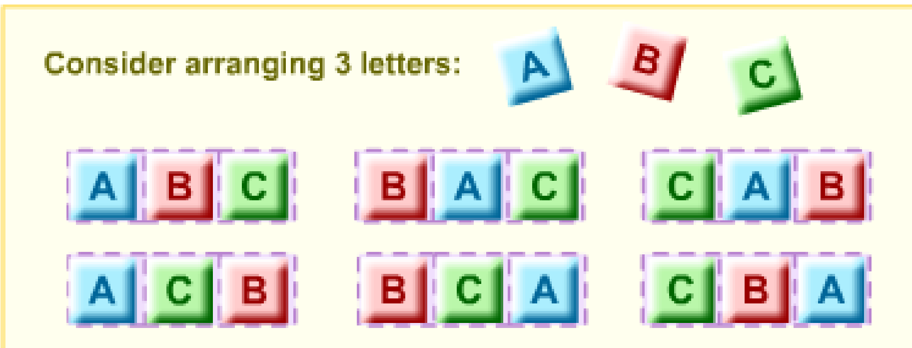
$$P(n, r) = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

Note: $P(n, r) = n!$

Example 5

Consider arranging 3 letters: A, B, C.

How many different ways can this be done, assuming that repetitions are not allowed?



As you can see, there are 6 possible permutations if we have three letters, and no repetitions.

We can get the answer without listing out all the elements by using the permutation formula

$$P(n, r) = \frac{n!}{(n-r)^{!}}$$

The number of different ways of a set of 3 distinct objects taken 3 at a time can be arranged without repetition is

$$\begin{aligned} P(3,\ 3) &= \frac{3!}{(3-3)^{!}} \\ &= \frac{3!}{0!} = 6 \end{aligned}$$

where 0! = 1.

1.3.4.2 Activity 1

In how many different ways can a supermarket manager display 5 brands of sweets in 3 spaces on a shelf? Repetitions are not allowed.

1.3.4.2.1 Activity 1 feedback

The number of different ways to display the 5 brands of sweets in 3 spaces is

$$\begin{aligned} P(5,\ 3) &= \frac{5!}{(5-3)!} \\ &= \frac{5!}{2!} = 60 \end{aligned}$$

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1.3.4.3 Combinations

What is a combination?

A combination is a selection of all or part of a set of n objects without repetition, and with regardless to the order in which objects are selected. That is, the order is not important.

For example, suppose you are asked to select a set of 4 different letters from the 26 English alphabet {A, B, C, D,. . . , Z} without regard to the order. Let's say you got

$$\{A, B, D, C\}, \{C, B, D, A\}, \{D, E, H, K\}$$

Actually, the set {A, B, D, C}, {C, B, D, A} have the same set of letters, even though the 4 letters are selected in different order. Hence, you have selected only two different sets of 4 letters (not 3 sets!).

Notations for combination

The number of combinations in which r objects can be selected from a set of n objects, where repetitions are not allowed and the ordering is not important, is denoted by

$$C(n, r) \ or \ C_r^n$$

$$C(n,\ r) = \frac{n!}{r!(n-r)!}$$

Notes:

1. $$C(n, r)P(n, r) \\ = r!$$
2. $$C(n, 0) = 1$$
3. $$C(n, n) = 1$$
4. $$C(n, r) = C(n, r-1)$$

Example 6

How many different ways of 4 letters can be selected from the 26 alphabet when repetitions are not allowed and the ordering of letters is not important?

Solution

There are *P*(26, 4) ways of arranging any 4 letters chosen from the 26 alphabet without repetitions. We can apply the permutation to get

$$P(26,\ 4) = \frac{26!}{(26-4)!} = \frac{26!}{22!} = 358\ 800\ ways$$

Since the order of any set of 4 letters chosen can be arranged in 4! different ways, the number of different sets of 4 letters is

$$\frac{P(26,\ 4)}{4!} = \frac{358\,800}{24} = 14\,950\ ways$$

The same result can be obtained if you use the combination formula

$$C(26,\ 4) = \frac{26!}{4!(26-4)!} = \frac{26!}{4!\ 22!} = 14\,950\ ways$$

1.3.4.4 Activity 2

In how many different ways can a group of 4 boys be selected from 10 if

1. the eldest boy is included in the group?
2. the eldest boy is excluded from the group?
3. What proportion of all possible groups contain the eldest boy?

1.3.4.4.1 Activity 2 feedback

1. Choose 3 from 9, since the eldest boy is fixed. The number of different ways is

$$C(9,\ 3) = \frac{9!}{3!(9-3)!} = \frac{9!}{3!6!} = 84$$

2. If the eldest boy is excluded, we actually choose 4 boys from 9. The number of different ways is

$$C(9,\ 4) = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = 126$$

3. The number of all possible groups is

$$C(10,\ 4) = \frac{10!}{4!6!} = 210$$

So the proportion of all possible groups containing the eldest boy is:

$$\frac{84}{210} = \frac{2}{5} = 40\%$$

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We have all faced the scenario where we've had to decide between two choices such as: do you go to the cinema with friends or do you read this module? Sometimes when you can't make the decision you opt to let 'lady luck' decide.

1.4 Probability

We have all faced the scenario where we've had to decide between two choices such as: do you go to the cinema with friends or do you read this module? Sometimes when you can't make the decision you opt to let 'lady luck' decide.

Animation/activity

Let's say you pull out a fair \$1-coin and assign one option to each side:

'flower' = go to cinema
'1' = read the module

You toss the fair coin and whichever side lands on top will be your choice. This method gives an even chance that either side will land on top, so either option is equally possible. Try 'tossing' the virtual coin here at least 10 times, or more.

However, just before you toss the coin, your mother calls asking you to come over for a meal. Now you have three choices, but your coin only has two sides. Assigning a meal with your mother to the coin landing on its edge and balancing there hardly seems fair! Therefore, you come up with the following plan: you take out a second fair \$1-coin and decide to toss both coins; you assign the following results to your choices

'flower' + 'flower' = go to cinema
'flower' + '1' = read the module
'1' + '1' = meal with mother

Now let's say you 'toss' the two coins together,

> You used two coins in the belief that each of the three options has the same chance of been chosen. But has it?

You should now have your own results from your 'virtual coin toss'. Let's assume I tried the same exercise, and that I also used two fair coins.

Here are my results from tossing my two coins 50 times:

	No. of times	%
'flower' + 'flower'	10	20
'flower' + '1'	27	54
'1' + '1'	13	26

This hardly looks like the three options are evenly distributed! We might expect some small variation, but not as much as 54% for one option!

The problem with tossing the two coins comes from poor counting of the outcomes. Let's consider a different situation for the moment.

Example 7

For the set

$A = \{a,b\}$

how many different combinations of the two elements in A are there if repetition is allowed, and order is important?

Solution

Step 1: there are two ways of choosing the first element.
Step 2: there are two ways of choosing the second element.

The Multiplication Principle gives a total of

$2 \times 2 = 4$

different ways, and they are:

aa
ab
ba
bb

So now let's reconsider your tossing two fair coins one by one. As in the example, we could get four possible outcomes:

First coin	Second coin
Flower	Flower
Flower	1
1	Flower
1	1

But - you may say - the second and third outcomes are the same!

In fact, they are *not* the same.

The first result in each outcome refers to the first coin, and the second result refers to the second coin. Therefore, each outcome represents a unique outcome.

The confusion arises because we originally assumed that the second and third outcomes represented the same choice. Thus, the second option, i.e. to 'read the module', is *twice* as likely to appear on a toss of the coins, which explains the results of our experiment.

To further demonstrate the fact that there are four outcomes, suppose we had to decide between the four choices: go to the cinema, read the module, meal with mother and sleep. We arrange to toss two fair coins. However, this time we toss the first coin to our left and the second coin to our right and record the result differently. This gives us four different outcomes:

Left coin	Right coin
'flower'	'flower'
'flower'	'1'
'1'	'flower'
'1'	'1'

We can assign these to our four different choices comfortable in the knowledge that each outcome is equally likely.

In mathematics we don't talk about chances, but about *probabilities*. As you saw in our last example, counting forms an essential part of determining the probability of each outcome. There are four outcomes, each of which is equally likely. We define the probability of each outcome as one chance in four outcomes, or:

$$\textit{probability of each outcome} = \frac{1}{4}$$

Example 8

A standard pack of cards contains four suits: clubs ♣, diamonds ◊, hearts ♡ and spades ♠.

Each suit contains 13 cards, and there are 52 cards in total. Assuming that the pack has been well shuffled, then the probability that the top card is of one particular suit is the same as the probability that it is of any of the other suits, or:

$$\begin{aligned} probability\ that\ the\ top\ card\ is\ a\ club &= \frac{1}{4} \\ probability\ that\ the\ top\ card\ is\ a\ diamond &= \frac{1}{4} \\ probability\ that\ the\ top\ card\ is\ a\ heart &= \frac{1}{4} \\ probability\ that\ the\ top\ card\ is\ a\ spade &= \frac{1}{4} \end{aligned}$$

Let's look at the last example more closely. If we are looking for a club, then when we turn over the top card success is measured by the card being any one of

{Ace♣,2♣,3♣,4♣,5♣,6♣,7♣,8♣,9♣,10♣,*Jack*♣,*Queen*♣,*King*♣}.

That is, when turning over the top card, there are 13 possible successful outcomes out of the total of 52 possible outcomes.

An important feature of the last example is that there is an equal chance that each card in the pack could be at the top. The position of the cards in a shuffled pack is considered to be random, i.e. there is no way of predicting which card will be in which position. Similarly, with our tossing the coin example the result of the toss is random as there is no way of knowing which side will appear on top after each toss.

Example 9

In a race at Happy Valley, there are ten horses running. Are the chances that any horse wins the same?

Solution

No. There are many different factors determining the winner of a horse race - the skill of the jockey, the state of the track, the training of the horse, etc.

Example 10

On a roulette wheel there are 38 equal divisions. Two divisions are numbered zero, and the rest are numbered from one to 36. Gamblers are allowed to bet on the ball landing in any of the divisions numbered one to 36. If the ball lands in one of the divisions marked '0', then the casino wins all bets. Are the chances that the ball will indicate any of the numbers zero to 36 the same?

Solution

No. Since there are two divisions marked '0' and only one for each of the other numbers, then the chances of landing on a '0' will be greater than for any of the other numbers.

1.4.1 The probability of one event

In order to investigate probabilities mathematically, we need to ensure that we have unambiguous definitions and terminology for describing problems. We start by identifying the scope of a problem.

Definition 1*: An experiment is a process that yields an outcome.*

Definition 2*: An event, E, is an outcome or combination of outcomes from an experiment.*

Definition 3*: A sample space, S, is the event consisting of all possible outcomes.*

Example 11

In the single toss of a fair coin, we assume the 'flower' represents 'the side on top after the toss is the flower', etc.

The sample space is: S = {'flower', '1'}

And one event is: E_1 = {'flower'}

Another event is: E_2 = {'1'}

Example 12

In the single toss of two fair coins:

$$S = \{'flower' + 'flower', 'flower' + '1', '1' + 'flower', '1' + '1'\}$$

And one event is:

$$\begin{aligned} E_1 &= \{one\ coin\ 'flower', the\ other\ coin\ '1'\} \\ &= \{'flower' + '1', '1' + 'flower'\} \end{aligned}$$

Another event is:

$$\begin{aligned} E_2 &= \{left\ coin\ 'flower', right\ coin\ '1'\} \\ &= \{'flower' + '1'\} \end{aligned}$$

E_1 and E_2 are therefore very different events, as you saw earlier.

Example 13

For the pack of cards example:

$$S = \left\{ \begin{array}{l} Ace\clubsuit, 2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, Jack\clubsuit, Queen\clubsuit, King\clubsuit, \\ Ace\diamondsuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, Jack\diamondsuit, Queen\diamondsuit, King\diamondsuit, \\ Ace\heartsuit, 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, Jack\heartsuit, Queen\heartsuit, King\heartsuit, \\ Ace\spadesuit, 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, Jack\spadesuit, Queen\spadesuit, King\spadesuit \end{array} \right\}$$

If E_1 is the event that the top card is a club, then this is the subset:

E_1 = {*Ace*♣,2♣,3♣,4♣,5♣,6♣,7♣,8♣,9♣,10♣,*Jack*♣,*Queen*♣,*King*♣}

For the event that the top card is a face card, then:

$$E_2 = \left\{\begin{matrix} 10\clubsuit, Ace\clubsuit, Jack\clubsuit, \ Queen\clubsuit, King\clubsuit, 10\Diamond, Ace\Diamond, Jack\Diamond, Queen\Diamond, King\Diamond, \\ 10\heartsuit, Ace\heartsuit, Jack\heartsuit, Queen\heartsuit, King\heartsuit, 10\spadesuit, Ace\spadesuit, Jack\spadesuit, Queen\spadesuit, King\spadesuit \end{matrix}\right\}$$

Definition 4: *If S is a finite sample space in which all outcomes are equally likely, and E is an event in S, then the probability of E is:*

$$p(E) = \frac{number\ of\ outcomes\ in\ E}{total\ number\ of\ outcomes\ in\ S}$$

Or, if we define a 'successful outcome' as one in E, then:

$$p(E) = \frac{number\ of\ auccessful\ outcomes}{total\ number\ of\ outcomes\ in\ S}$$

Example 14

In a single toss of a fair coin there is one 'successful outcome' in both E_1 and E_2 and two outcomes in S. Therefore:

$$p('flower') = \frac{1}{2}$$
$$p('1') = \frac{1}{2}$$

Example 15

In the single toss of two fair coins independently, there are four outcomes in *S*. Therefore:

$$\begin{aligned} p(E_1) &= p(\{one\ coin\ 'flower',\ the\ other\ coin\ '1'\}) \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

And for:

$$\begin{aligned} p(E_2) &= p(\{left\ coin\ 'flower',\ right\ coin\ '1'\}) \\ &= \frac{1}{4} \end{aligned}$$

Example 16

In investigating the value of the top card of a shuffled pack, *S* contains 52 possible outcomes. For the top card to be a club, then there are 13 'successful outcomes', so:

$$p(E_1) = \frac{13}{52} = \frac{1}{4}$$

For the top card to be a face card, there are 20 'successful outcomes', so:

$$p(E_2) = \frac{20}{52} = \frac{5}{13}$$

As you can now see, for many problems, such as tossing two coins, the difficulties arise in *accurately identifying the outcomes*.

A roulette wheel is a common tool for gambling in casinos and knowledge of how probabilities apply is essential for the educated gambler. The wheel contains pockets at its edge large enough to hold the ball that forms part of the equipment. Each hole is labelled with a number, and is coloured either black or red. At the start of each play gamblers must place a bet on a number, group of numbers or colour. The wheel is then spun in one direction and the ball is made to travel in the opposite direction in a slot above the wheel. Once the speed of the ball drops to a point where it can no longer stay in the slot it falls onto the wheel where it bounces around until it settles into one of the pockets to determine the winning number and colour. Usually the casino will reserve the slot numbered '0' for itself - when the ball lands in that slot the casino wins all bets. When the ball lands on any other number the payout is normally related to the associated probability.

Example 17

For a roulette wheel with two zeros, if we denote the two zeros as separate outcomes '01' and '02', then *S* has 38 different outcomes, which are all equally likely:

$$S = \left\{ \begin{array}{l} 0_1, 0_2, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, \\ 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36 \end{array} \right\}$$

and if we interested in the probability of the ball landing on the '8', then:

$$p('8') = \frac{1}{38}$$

Note, however:

$$p('0') = \frac{2}{38} = \frac{1}{19}$$

If you think that the 'house advantage' built into roulette makes little difference, try the Internet activity (Page 19).

In addition to coins, cards and roulette wheels, another common device that appears in games of chance is the die (the more familiar plural is 'dice'). By a 'fair dice' we mean that there is an equal probability of any of the six sides appearing at the top after a throw.

Example 18

Let E_i be the event when the die ends with the side showing the value *i* at the top (i=1,2,3,4,5,6), then:

$$p(E_i) = \frac{1}{6}$$

Games with dice become more interesting when more than one die is involved. Such games are normally based on the combined values of the sides appearing on top after a throw.

Example 19

What is the probability of the total of the top sides of two dice being 10?

Solution

First of all, let's calculate the size of our sample space. As each die has 6 sides and the result on one die is not dependent on the other we can use the Multiplication Principle to determine that there are 6 × 6 = 36 possible combinations, presented here as ordered pairs to indicate the values of the first and second die:

(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)

These possibilities generate the following 36 totals:

2,3,4,5,6,7,3,4,5,6,7,8,
4,5,6,7,8,9,5,6,7,8,9,10,
6,7,8,9,10,11,7,8,9,10,11,12

We can see that there are three possible outcomes that generate a total of 10: (4,6),(5,5),(6,4) giving:

$$\begin{aligned} p(total\ of\ 10) &= \frac{3}{36} \\ &= \frac{1}{12} \end{aligned}$$

In the previous example, it was crucial that we handled the possible outcomes as ordered pairs to ensure that we capture the two successful outcomes (4,6) and (6,4).

Example 20

One game with dice is based on the player throwing two dice and obtaining a total of 8 or greater. What is the probability of the player winning?

Solution

There are 15 successful outcomes:

$$p(win) = \frac{15}{36}$$

As you can see, the possibility of winning is less than 50%.

Probabilities are also used extensively in quality control especially in production lines.

Example 21

A company manufacturers computer circuit boards. The process produces 20 defective boards in every 1000. In taking a sample of 5 from the 1000, what is the probability that the sample will contain no defectives?

Solution

Since order is not involved here, the selection of 5 boards relates to combinations. There are C(1000,5) ways of selecting the 5 boards - this is our sample space.

There are 980 non-defective boards, and there are C(980,5) ways of selecting 5 boards from these - these are our successful outcomes.

Therefore, the probability, to five decimal places, is:

$$\begin{aligned} p(no\,defectives\,in\,the\,sample) &= \frac{C(980,5)}{C(1000,5)} \\ &= 0.90374 \end{aligned}$$

Most people in Hong Kong have participated at least once in the Mark Six lottery.

Example 22

What are the actual chances of winning the Mark Six?

Solution

Each play involves the drawing of 6 numbers from 49, and the order that the numbers are drawn is irrelevant. This means that there are C(49,6) possible outcomes - this is our sample space. There is only one winning combination, so:

$$\begin{aligned} p(win) &= \frac{1}{C(49,6)} \\ &= \frac{6!}{49 \times 48 \times 47 \times 46 \times 45 \times 44} \\ &= 0.000000071511 \end{aligned}$$

1.4.1.1 Internet activity

Internet activity

Try out a roulette simulator (https://roulette-simulator.info/en).

By default, the simulator will run 100,000 spins of the roulette wheel in just a few seconds. See how well you do against the house, given the odds we've just explored.

This simulator also allows you to choose the type of bet you wish to place, and the system of roulette rules you'd like to play under. Explanations for all of these options are provided.

Try out some different combinations of bets and rules: which ones give you the best chance of winning in the long run?

Please contribute an account of your experience with the roulette simulator on our discussion board.

Go to Discussion Board. (http://freecourseware.ouhk.edu.hk/fc/php/forum_index.php?course=&lang=c)

1.4.1.2 Exercise 1

1. When throwing two dice, what are the successful outcomes for a total of 4?
2. What is the probability of throwing an even number when using only one die?
3. What is the probability of finding a Jack when randomly selecting one card from a standard pack of 52 cards?
4. In producing circuit boards, there are 10 defectives in a box of 100. If 4 are randomly selected from the box, what is the probability that there is exactly one defective board in the sample of 4?

1.4.1.2.1 Feedback - Exercise 1

1. When throwing 2 dice together, all possible outcomes for a total of 4 are (1,3), (2,2), (3,1).
2. E = {♣J,àJ,J,♠J} There are four successful outcomes out of the possible 52. Therefore
$$p(E) = \frac{4}{52} = \frac{1}{13}$$
3. E = {♣J,àJ,J,♠J} There are four successful outcomes out of the possible 52. Therefore
$$p(E) = \frac{4}{52} = \frac{1}{13}$$
4. One defective microprocessor and three good microprocessors are required to choose. We have 10 defective microprocessors and 90 (=100 - 10) good microprocessors, so the successful outcomes. Since any:
$$E = C(10,1), C(90,3)$$

   Since any four microprocessors can be chosen in *C*(100,4) ways, so:
$$p(E) = \frac{C(10,1) \times C(90,3)}{C(100,4)} \simeq 0.2996$$

1.4.1.3 Exercise 2

Suppose that the chances of any baby being male or female are even. A newly married couple plan to have three children.

1. Denoting a male baby by *M* and a female baby by *F*, write down the sample space for the possible sex of all three children collectively. For example, *MFM* would denote the first and third babies were males and the second baby was female.
2. What is the probability that only one child will be male?
3. What is the probability that at least one child will be male?

1.4.1.3.1 Feedback - Exercise 2

1. *S* = {MMM,MMF,MFM,FMM,MFF,FMF,FFM,FFF}
2. *E* = {MFF,FMF,FFM}

   There are three successful outcomes out of the possible eight. Therefore:

   $$p(E) = \frac{3}{8}$$
3. *E* = {MMM,MMF,MFM,FMM,MFF,FMF,FFM} There are seven successful outcomes out of the possible eight. Therefore:

   $$p(E) = \frac{7}{8}$$

1.4.1.4 Exercise 3

A woman is pregnant and expects to give birth in a leap year (366 days). Given that January 1 is a Saturday, what is the probability that the baby will be born on a Saturday?

1.4.1.4.1 Feedback - Exercise 3

If January 1 is a Saturday, then there are 53 Saturdays in the year. Therefore, there are 313 other days. The days of the year are the sample space and the 53 Saturdays are the event. Therefore:

$$p(E) = \frac{53}{366}$$

1.5 Conclusion

This module has taken you through some of the basics of counting and probability. You should realize by now how useful an understanding of probability is in real life, from a simple coin flip to the Mark Six.

If you would like to learn more on this subject, you are welcome to enrol in MATH S121 A Foundation in Pure Mathematics (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&lang=eng&ncode=MATH%20S121&shopping=Y&TYPE=CI&CODE=M121) offered by the School of Science and Technology (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcSubWeb&l=C_ST&lid=191133000200&lang=eng) of the OUHK.