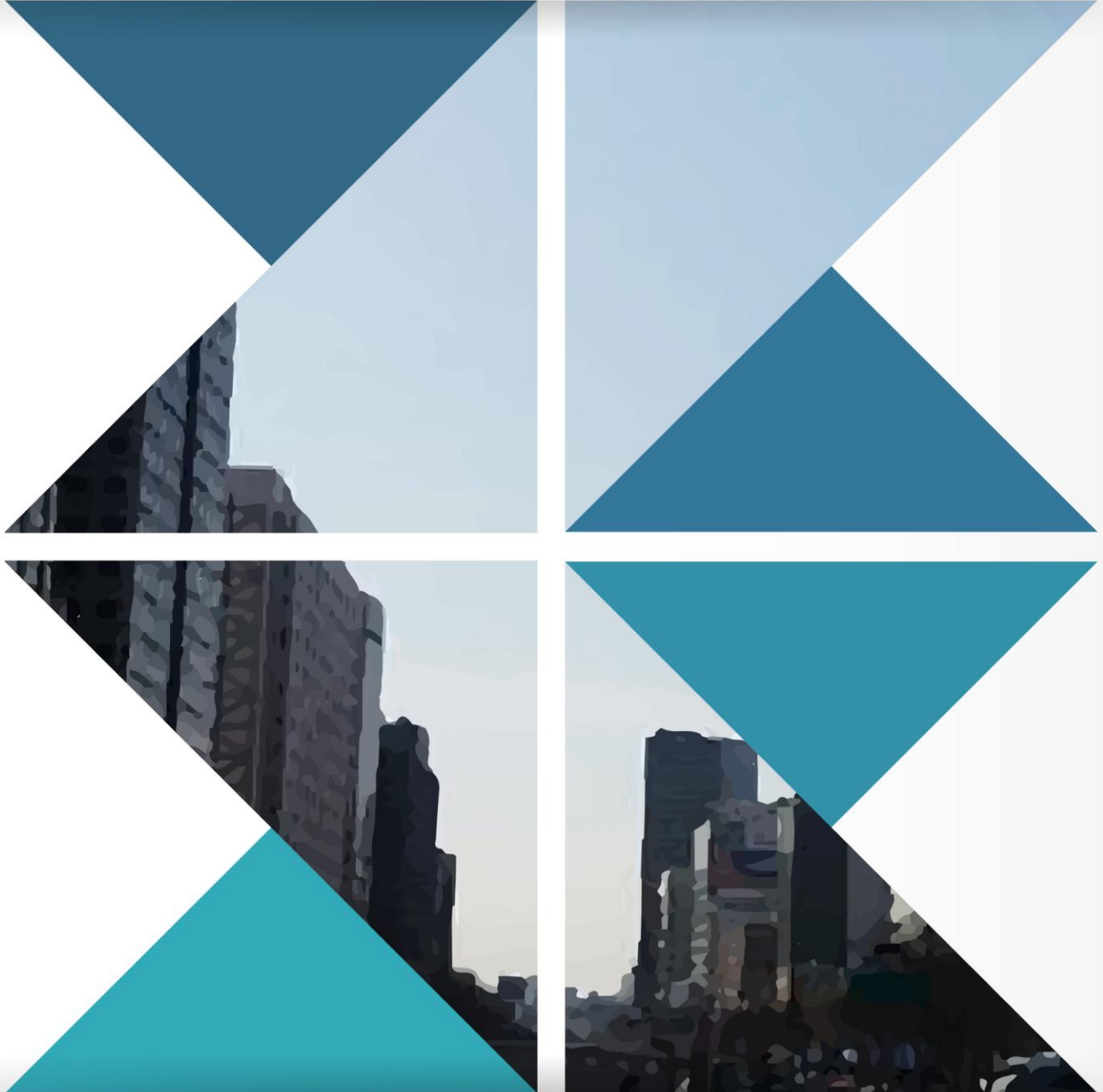


Contribution:



# Electromechanical Dynamics

James R. Melcher  
Herbert H. Woodson



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# **ELECTROMECHANICAL DYNAMICS**

## **Part I: Discrete Systems**



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To our parents





# PREFACE

## Part I: Discrete Systems

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To be effective a core curriculum subject must be broad enough to be germane to the many directions an electrical engineer may go professionally, yet it must have adequate depth to be of lasting value. At the same time, the subject must be related to the real world by examples of application. This is true because students learn by seeing material in a familiar context, and engineering students are motivated largely by the relevance of the material to the realities of the world around them.

In the organization of the core curriculum in electrical engineering at M.I.T. electromechanics is one major component. As our core curriculum has evolved, there have been changes in emphasis and a broadening of the topic. The basic text in electromechanics until 1954, when a new departure was made, was *Electric Machinery* by Fitzgerald and Kingsley. This change produced *Electromechanical Energy Conversion* by White and Woodson, which was used until 1961. At that time we started the revision that resulted in the present book. During this period we went through many versions of notes while teaching the material three semesters a year.

Our objective has always been to teach a subject that combines classical mechanics with the fundamentals of electricity and magnetism. Thus the subject offers the opportunity to teach both mechanics and electromagnetic theory in a context vital to much of the electrical engineering community.

Our choice of material was to some extent determined by a desire to give the student a breadth of background sufficient for further study of almost any type of electromechanical interaction, whether in rotating machinery,

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Before discussing how the material can be used to achieve these ends, a review of the contents is in order. The student who uses this book is assumed to have a background in electrostatics and magnetostatics. Consequently, Chapter 1 and Appendix B are essentially a review to define our starting point.

Chapter 2 is a generalization of the concepts of inductance and capacitance that are necessary to the treatment of electromechanical systems; it also provides a brief introduction to rigid-body mechanics. This treatment is included because many curricula no longer cover mechanics, other than particle mechanics in freshman physics. The basic ideas of Chapter 2 are repeated in Chapter 3 to establish some properties of electromechanical coupling in lumped-parameter systems and to obtain differential equations that describe the dynamics of lumped-parameter systems.

Next, the techniques of Chapters 2 and 3 are used to study rotating machines in Chapter 4. Physical models are defined, differential equations are written, machine types are classified, and steady-state characteristics are obtained and discussed. A separate chapter on rotating machines has been included not only because of the technological importance of machines but also because rotating machines are rich in examples of the kinds of phenomena that can be found in lumped-parameter electromechanical systems.

Chapter 5 is devoted to the study, with examples, of the dynamic behavior of lumped-parameter systems. Virtually all electromechanical systems are mathematically nonlinear; nonetheless, linear incremental models are useful for studying the stability of equilibria and the nature of the dynamical behavior in the vicinity of an equilibrium. The second half of this chapter develops the classic potential-well motions and loss-dominated dynamics in the context of electromechanics. These studies of nonlinear dynamics afford an opportunity to place linear models in perspective while forming further insights on the physical significance of, for example, flux conservation and state functions.

Chapter 6 represents our first departure from lumped-parameter systems into continuum systems with a discussion of how observers in relative motion will define and measure field quantities and the related effects of material motion on electromagnetic fields. It is our belief that dc rotating machines are most easily understood in this context. Certainly they are a good demonstration of field transformations at work.

As part of any continuum electromechanics problem, one must know how the electric and magnetic fields are influenced by excitations and motion. In quasi-static systems the distribution of charge and current are controlled by magnetic diffusion and charge relaxation, the subjects of Chapter 7. In Chapter 7 simple examples isolate significant cases of magnetic diffusion or charge relaxation, so that the physical processes involved can be better understood.

Chapters 6 and 7 describe the electrical side of a continuum electromechanical system with the material motion predetermined. The mechanical side of the subject is undertaken in Chapter 8 in a study of force densities of electric and magnetic origin. Because it is a useful concept in the analysis of many systems, we introduce the Maxwell stress tensor. The study of useful properties of tensors sets the stage for later use of mechanical stress tensors in elastic and fluid media.

At this point the additional ingredient necessary to the study of continuum electromechanics is the mechanical medium. In Chapter 9 we introduce simple elastic continua—longitudinal motion of a thin rod and transverse motion of wires and membranes. These models are used to study simple continuum mechanical motions (nondispersive waves) as excited electromechanically at boundaries.

Next, in Chapter 10 a string or membrane is coupled on a continuum basis to electric and magnetic fields and the variety of resulting dynamic behavior is studied. *The unifying thread of this treatment is the dispersion equation that relates complex frequency  $\omega$  with complex wavenumber  $k$ .* Without material convection there can be simple nondispersive waves, cut off or evanescent waves, absolute instabilities, and diffusion waves. The effect of material convection on evanescent waves and oscillations and on wave amplification are topics that make a strong connection with electron beam and plasma dynamics. The method of characteristics is introduced as a convenient tool in the study of wave propagation.

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In Chapter 12 we turn to a different mechanical medium, a fluid. We first study electromechanical interactions with inviscid, incompressible

fluids to establish essential phenomena in the simplest context. It is here that we introduce the basic notions of MHD energy conversion that can result when a conducting fluid flows through a transverse magnetic field. We also bring in electric-field interactions with fluids, in which ion drag phenomena are used as an example. In addition to these basically conducting processes, we treat the electromechanical consequences of polarization and magnetization in fluids. We demonstrate how highly conducting fluids immersed in magnetic fields can propagate Alfvén waves.

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Finally, in Chapter 14 we add viscosity to the fluid model and study the consequences in electromechanical interactions with steady flow. Hartmann flow demonstrates the effect of viscosity on the dc magnetohydrodynamic machine.

To be successful a text must have a theme; the material must be inter-related. Our philosophy has been to get into the subject where the student is most comfortable, with lumped-parameter (circuit) concepts. Thus many of the subtle approximations associated with quasi-statics are made naturally, and the student is faced with the implications of what he has assumed only after having become thoroughly familiar with the physical significance and usefulness of his approximations. By the time he reaches Chapter 4 he will have drawn a circle around at least a class of problems in which electromagnetic fields interact usefully with media in motion.

In dealing with physical and mathematical subjects, as we are here, in which the job is incomplete unless the student sees the physical laws put to work in some kind of physical embodiment, it is necessary for the thread of continuity to be woven into the material in diverse and subtle ways. A number of attempts have been made, to which we can add our early versions of notes, to write texts with one obvious, pedagogically logical basis for evolving the material; for example, it can be recognized that classes of physical phenomena could be grouped according to the differential equation that describes the pertinent dynamics. Thus we could treat magnetic diffusion, diffusion waves on elastic continua, and viscous diffusion waves in one chapter, even though the physical embodiments are entirely different. Alternatively, we could devise a subject limited to certain technological applications or cover superficially a wide range of basically unrelated topics such as “energy conversion” under one heading. This was the prevalent approach in engineering education a decade or so ago, even at the

undergraduate level. It seems clear to us that organizing material in a teachable and meaningful fashion is far more demanding than this. To confess our own mistakes, our material went originally from the general to the specific; it began with the relativistic form of Maxwell's equations, including the effects of motion, and ended with lumped-parameter devices as special cases. Even if this were a pedagogically tenable approach, which we found it was not, what a bad example to set for students who should be learning to distinguish between the essential and the superfluous! Ideas connected with the propagation of electromagnetic waves (relativistic ideas) must be included in the curriculum, but their connection with media in motion should be made after the student is aware of the first-order issues.

A meaningful presentation to *engineers* must interweave and interrelate mathematical concepts, physical characteristics, the modeling process, and the establishment of a physical "feel" for the world of reality. Our approach is to come to grips with each of these goals as quickly as possible (let the student "get wet" within the first two weeks) and then, while reinforcing what he has learned, continually add something new. Thus, if one looks, he will see the same ideas coming into the flow of material over and over again.

For the organization of this book one should look for many threads of different types. We can list here only a few, in the hope that the subtle reinforcing interplay of mathematical and physical threads will be made evident. Probably the essential theme is Maxwell's equations and the ideas of quasi-statics. The material introduced in Chapter 1 is completely abstract, but it is reinforced in the first few chapters with material that is close to home for the student. By the time he reaches Chapter 10 he will have learned that waves exist which intimately involve electric and magnetic fields that are altogether quasistatic. (This is something that comes as a surprise to many late in life.) Lumped-parameter ideas are based on the integral forms of Maxwell's equations, so that the dynamical effects found with lumped-parameter time constants  $L/R$  and  $RC$  in Chapter 5 are easily associated with the subjects of magnetic diffusion and charge relaxation. A close tie is made between the "speed voltage" of Chapter 5 and the effects of motion on magnetic fields, as described by field transformations in Chapters 6 to 14. Constant flux dynamics of a lumped coil in Chapter 5 are strongly associated with the dynamics of perfectly conducting continuous media; for example, Alfvén waves in Chapter 12.

Consider another thread of continuity. The book begins with the mathematics of circuit theory. The machines of Chapter 4 are essentially circuits in the sinusoidal steady state. In Chapter 5 we linearize to pursue lumped-parameter ideas of stability and other transient responses and then proceed to nonlinear dynamics, potential-well theory, and other approaches that should form a part of any engineer's mathematical background. By the time

the end of Chapter 10 is reached these ideas will have been carried into the continuum with the addition of tensor concepts, simple cases of the method of characteristics, and eigenvalue theory. The  $\omega$ - $k$  plot and its implication for all sorts of subjects in modern electrical engineering can be considered as a mathematical or a physical objective. The ideas of stability introduced with ordinary differential equations (*exp st*) in Chapter 5 evolve into the continuum stability studies of Chapter 10 [ $\exp j(\omega t - kx)$ ] and can be regarded as a mathematical or a physical thread in our treatment. We could list many other threads: witness the evolution of energy and thermodynamic notions from Chapters 3 to 5, 5 to 8, and 8 to 13.

We hope that this book is not just one more in the mathematics of electrical engineering or the technical aspects of rotating machines, transducers, delay lines, MHD converters, and so on, but rather that it is the mathematics, the physics, and, most of all, the engineering combined into one.

The material brought together here can be used in a variety of ways. It has been used by Professors C. N. Weygandt and F. D. Ketterer at the University of Pennsylvania for two subjects. The first restricts attention to Chapters 1 to 6 and Appendix B for a course in lumped-parameter electromechanics that both supplants the traditional one on rotating machines in the electrical engineering curriculum and gives the background required for further study in a second term (elective) covering Chapter 7 and beyond. Professors C. D. Hendricks and J. M. Crowley at the University of Illinois have used the material to follow a format that covers up through Chapter 10 in one term but omits much of the material in Chapter 7. Professor W. D. Getty at the University of Michigan has used the material to follow a one-term subject in lumped-parameter electromechanics taught from a different set of notes. Thus he has been able to use the early chapters as a review and to get well into the later chapters in a one-term subject.

At M.I.T. our curriculum seems always to be in a state of change. It is clear that much of the material, Chapters 1 to 10, will be part of our required (core) curriculum for the foreseeable future, but the manner in which it is packaged is continually changing. During the fall term, 1967, we covered Chapters 1 to 10 in a one-semester subject taught to juniors and seniors. The material from Chapters 4 and 6 on rotating machines was used selectively, so that students had "a foot solidly in the door" on this important subject but also that the coverage could retain an orientation toward the needs of all the diverse areas found in electrical engineering today. We have found the material useful as the basis for early graduate work and as a starting point in several courses related to electromechanics.

Finally, to those who open this book and then close it with the benediction, "good material but unteachable," we apologize because to them we have not made our point. Perhaps not as presented here, but certainly as it is

represented here, this material is rich in teaching possibilities. The demands on the teacher to see the subject in its total context, especially the related problems that lie between the lines, are significant. We have taught this subject many times to undergraduates, yet each term has been more enjoyable than the last. There are so many ways in which drama can be added to the material, and we do not need to ask the students (bless them) when we have been successful in doing so.

In developing this material we have found lecture demonstrations and demonstration films to be most helpful, both for motivation and for developing understanding. We have learned that when we want a student to see a particular phenomenon it is far better for us to do the experiment and let the student focus his attention on what he should see rather than on the wrong connections and blown fuses that result when he tries to do the experiment himself. The most successful experiments are often the simplest—those that give the student an opportunity to handle the apparatus himself. Every student should “chop up some magnetic field lines” with a copper “axe” or he will never really appreciate the subject. We have also found that some of the more complex demonstrations that are difficult and expensive to store and resurrect each semester come through very well in films. In addition to our own short films, three films have been produced professionally in connection with this material for the National Committee on Electrical Engineering Films, under a grant from the National Science Foundation, by the Education Development Center, Newton, Mass.

*Synchronous Machines: Electromechanical Dynamics* by H. H. Woodson  
*Complex Waves I: Propagation, Evanescence and Instability* by J. R. Melcher

*Complex Waves II: Instability, Convection and Amplification* by J. R. Melcher

An additional film is in the early stages of production. Other films that are useful have been produced by the Education Development Center for the National Committee on Fluid Mechanics Films and for the College Physics Film Program. Of particular interest, from the former series, is *Magnetohydrodynamics* by Arthur Shercliff.

A book like this can be produced only with plenty of assistance. We gratefully acknowledge the help we received from many directions and hope we have forgotten no one after seven years of work. First of all we want to acknowledge our students with whom we worked as the material developed. They are the one most essential ingredient in an effort of this sort. Next we want to thank Dr. S. I. Freedman, Professor H. H. Richardson, and Dr. C. V. Smith, Jr., for their assistance in framing worthwhile approaches to several of our key topics. In seven years we have had the help of many able

teachers in presenting this material to students. Their discussions and advice have been most useful. In this category we want particularly to mention Professors H. A. Haus, P. L. Penfield, D. C. White, G. L. Wilson, R. Gallager, and E. Pierson and Doctors J. Reynolds, W. H. Heiser, and A. Kusko. Professor Ketterer, who has taught this material at M.I.T. and the University of Pennsylvania, Professors C. D. Hendricks and J. M. Crowley, who have taught it at M.I.T. and the University of Illinois, and Professor W. D. Getty, who has taught it at M.I.T. and the University of Michigan, have been most generous with their comments. Messrs. Edmund Devitt, John Dressler, and Dr. Kent Edwards have checked the correctness of many of the mathematical treatments. Such a task as typing a manuscript repeatedly is enough to try the patience of anyone. Our young ladies of the keyboard, Miss M. A. Daly, Mrs. D. S. Figgins, Mrs. B. S. Morton, Mrs. E. M. Holmes, and Mrs. M. Mazroff, have been gentle and kind with us.

A lengthy undertaking of this sort can be successful only when it has the backing of a sympathetic administration. This work was started with the helpful support of Professor P. Elias, who was then head of the Department of Electrical Engineering at M.I.T. It was finished with the active encouragement of Professor L. D. Smullin, who is presently head of the Department.

Finally, and most sincerely, we want to acknowledge the perseverance of our families during this effort. Our wives, Blanche S. Woodson and Janet D. Melcher, have been particularly tolerant of the demands of this work.

This book appears in three separately bound, consecutively paged parts that can be used individually or in any combination. Flexibility is ensured by including with each part a complete Table of Contents and Index. In addition, for convenient reference, Parts II and III are supplemented by brief appendices which summarize the relevant material from the preceding chapters. Part I includes Chapters 1 to 6, hence emphasizes lumped-parameter models while developing background in field concepts for further studies.

*H. H. Woodson*  
*J. R. Melcher*

Cambridge, Massachusetts  
January 1968



# **ELECTROMECHANICAL DYNAMICS**

## **Part II: Fields, Forces, and Motion**



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*H. H. Woodson*  
*J. R. Melcher*

**ELECTROMECHANICAL DYNAMICS**

**Part III: Elastic and Fluid Media**

I  
II - B.L.T.





# **ELECTROMECHANICAL DYNAMICS**

**Part III: Elastic and Fluid Media**

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To our parents

D  
S - Blomk



# PREFACE

## Part III: Elastic and Fluid Media

In the early 1950's the option structure was abandoned and a common core curriculum was instituted for all electrical engineering students at M.I.T. The objective of the core curriculum was then, and is now, to provide a foundation in mathematics and science on which a student can build in his professional growth, regardless of the many opportunities in electrical engineering from which he may choose. In meeting this objective, core curriculum subjects cannot serve the needs of any professional area with respect to nomenclature, techniques, and problems unique to that area. Specialization comes in elective subjects, graduate study, and professional activities.

To be effective a core curriculum subject must be broad enough to be germane to the many directions an electrical engineer may go professionally, yet it must have adequate depth to be of lasting value. At the same time, the subject must be related to the real world by examples of application. This is true because students learn by seeing material in a familiar context, and engineering students are motivated largely by the relevance of the material to the realities of the world around them.

In the organization of the core curriculum in electrical engineering at M.I.T. electromechanics is one major component. As our core curriculum has evolved, there have been changes in emphasis and a broadening of the topic. The basic text in electromechanics until 1954, when a new departure was made, was *Electric Machinery* by Fitzgerald and Kingsley. This change produced *Electromechanical Energy Conversion* by White and Woodson, which was used until 1961. At that time we started the revision that resulted in the present book. During this period we went through many versions of notes while teaching the material three semesters a year.

Our objective has always been to teach a subject that combines classical mechanics with the fundamentals of electricity and magnetism. Thus the subject offers the opportunity to teach both mechanics and electromagnetic theory in a context vital to much of the electrical engineering community.

Our choice of material was to some extent determined by a desire to give the student a breadth of background sufficient for further study of almost any type of electromechanical interaction, whether in rotating machinery,

plasma dynamics, the electromechanics of biological systems, or magnetoelasticity. It was also chosen to achieve adequate depth while maintaining suitable unity, but, most important, examples were chosen that could be enlivened for the engineering student interested in the interplay of physical reality and the analytical model. There were many examples from which to choose, but only a few satisfied the requirement of being both mathematically lucid *and* physically demonstrable, so that the student could “push it or see it” and directly associate his observations with symbolic models. Among the areas of electrical engineering, electromechanics excels in offering the opportunity to establish that all-important “feel” for a physical phenomenon. Properly selected electromechanical examples can be the basis for discerning phenomena that are remote from human abilities to observe.

Before discussing how the material can be used to achieve these ends, a review of the contents is in order. The student who uses this book is assumed to have a background in electrostatics and magnetostatics. Consequently, Chapter 1 and Appendix B are essentially a review to define our starting point.

Chapter 2 is a generalization of the concepts of inductance and capacitance that are necessary to the treatment of electromechanical systems; it also provides a brief introduction to rigid-body mechanics. This treatment is included because many curricula no longer cover mechanics, other than particle mechanics in freshman physics. The basic ideas of Chapter 2 are repeated in Chapter 3 to establish some properties of electromechanical coupling in lumped-parameter systems and to obtain differential equations that describe the dynamics of lumped-parameter systems.

Next, the techniques of Chapters 2 and 3 are used to study rotating machines in Chapter 4. Physical models are defined, differential equations are written, machine types are classified, and steady-state characteristics are obtained and discussed. A separate chapter on rotating machines has been included not only because of the technological importance of machines but also because rotating machines are rich in examples of the kinds of phenomena that can be found in lumped-parameter electromechanical systems.

Chapter 5 is devoted to the study, with examples, of the dynamic behavior of lumped-parameter systems. Virtually all electromechanical systems are mathematically nonlinear; nonetheless, linear incremental models are useful for studying the stability of equilibria and the nature of the dynamical behavior in the vicinity of an equilibrium. The second half of this chapter develops the classic potential-well motions and loss-dominated dynamics in the context of electromechanics. These studies of nonlinear dynamics afford an opportunity to place linear models in perspective while forming further insights on the physical significance of, for example, flux conservation and state functions.

Chapter 6 represents our first departure from lumped-parameter systems into continuum systems with a discussion of how observers in relative motion will define and measure field quantities and the related effects of material motion on electromagnetic fields. It is our belief that dc rotating machines are most easily understood in this context. Certainly they are a good demonstration of field transformations at work.

As part of any continuum electromechanics problem, one must know how the electric and magnetic fields are influenced by excitations and motion. In quasi-static systems the distribution of charge and current are controlled by magnetic diffusion and charge relaxation, the subjects of Chapter 7. In Chapter 7 simple examples isolate significant cases of magnetic diffusion or charge relaxation, so that the physical processes involved can be better understood.

Chapters 6 and 7 describe the electrical side of a continuum electromechanical system with the material motion predetermined. The mechanical side of the subject is undertaken in Chapter 8 in a study of force densities of electric and magnetic origin. Because it is a useful concept in the analysis of many systems, we introduce the Maxwell stress tensor. The study of useful properties of tensors sets the stage for later use of mechanical stress tensors in elastic and fluid media.

At this point the additional ingredient necessary to the study of continuum electromechanics is the mechanical medium. In Chapter 9 we introduce simple elastic continua—longitudinal motion of a thin rod and transverse motion of wires and membranes. These models are used to study simple continuum mechanical motions (nondispersive waves) as excited electromechanically at boundaries.

Next, in Chapter 10 a string or membrane is coupled on a continuum basis to electric and magnetic fields and the variety of resulting dynamic behavior is studied. The unifying thread of this treatment is the dispersion equation that relates complex frequency  $\omega$  with complex wavenumber  $k$ . Without material convection there can be simple nondispersive waves, cut off or evanescent waves, absolute instabilities, and diffusion waves. The effect of material convection on evanescent waves and oscillations and on wave amplification are topics that make a strong connection with electron beam and plasma dynamics. The method of characteristics is introduced as a convenient tool in the study of wave propagation.

In Chapter 11 the concepts and techniques of Chapters 9 and 10 are extended to three-dimensional systems. Strain displacement and stress-strain relations are introduced, with tensor concepts, and simple electromechanical examples of three-dimensional elasticity are given.

In Chapter 12 we turn to a different mechanical medium, a fluid. We first study electromechanical interactions with inviscid, incompressible

fluids to establish essential phenomena in the simplest context. It is here that we introduce the basic notions of MHD energy conversion that can result when a conducting fluid flows through a transverse magnetic field. We also bring in electric-field interactions with fluids, in which ion drag phenomena are used as an example. In addition to these basically conducting processes, we treat the electromechanical consequences of polarization and magnetization in fluids. We demonstrate how highly conducting fluids immersed in magnetic fields can propagate Alfvén waves.

In Chapter 13 we introduce compressibility to the fluid model. This can have a marked effect on electromechanical behavior, as demonstrated with the MHD conduction machine. With compressibility, a fluid will propagate longitudinal disturbances (acoustic waves). A transverse magnetic field and high electrical conductivity modify these disturbances to magnetoacoustic waves.

Finally, in Chapter 14 we add viscosity to the fluid model and study the consequences in electromechanical interactions with steady flow. Hartmann flow demonstrates the effect of viscosity on the dc magnetohydrodynamic machine.

To be successful a text must have a theme; the material must be inter-related. Our philosophy has been to get into the subject where the student is most comfortable, with lumped-parameter (circuit) concepts. Thus many of the subtle approximations associated with quasi-statics are made naturally, and the student is faced with the implications of what he has assumed only after having become thoroughly familiar with the physical significance and usefulness of his approximations. By the time he reaches Chapter 4 he will have drawn a circle around at least a class of problems in which electromagnetic fields interact usefully with media in motion.

In dealing with physical and mathematical subjects, as we are here, in which the job is incomplete unless the student sees the physical laws put to work in some kind of physical embodiment, it is necessary for the thread of continuity to be woven into the material in diverse and subtle ways. A number of attempts have been made, to which we can add our early versions of notes, to write texts with one obvious, pedagogically logical basis for evolving the material; for example, it can be recognized that classes of physical phenomena could be grouped according to the differential equation that describes the pertinent dynamics. Thus we could treat magnetic diffusion, diffusion waves on elastic continua, and viscous diffusion waves in one chapter, even though the physical embodiments are entirely different. Alternatively, we could devise a subject limited to certain technological applications or cover superficially a wide range of basically unrelated topics such as "energy conversion" under one heading. This was the prevalent approach in engineering education a decade or so ago, even at the



undergraduate level. It seems clear to us that organizing material in a teachable and meaningful fashion is far more demanding than this. To confess our own mistakes, our material went originally from the general to the specific; it began with the relativistic form of Maxwell's equations, including the effects of motion, and ended with lumped-parameter devices as special cases. Even if this were a pedagogically tenable approach, which we found it was not, what a bad example to set for students who should be learning to distinguish between the essential and the superfluous! Ideas connected with the propagation of electromagnetic waves (relativistic ideas) must be included in the curriculum, but their connection with media in motion should be made after the student is aware of the first-order issues.

A meaningful presentation to *engineers* must interweave and interrelate mathematical concepts, physical characteristics, the modeling process, and the establishment of a physical "feel" for the world of reality. Our approach is to come to grips with each of these goals as quickly as possible (let the student "get wet" within the first two weeks) and then, while reinforcing what he has learned, continually add something new. Thus, if one looks, he will see the same ideas coming into the flow of material over and over again.

For the organization of this book one should look for many threads of different types. We can list here only a few, in the hope that the subtle reinforcing interplay of mathematical and physical threads will be made evident. Probably the essential theme is Maxwell's equations and the ideas of quasi-statics. The material introduced in Chapter 1 is completely abstract, but it is reinforced in the first few chapters with material that is close to home for the student. By the time he reaches Chapter 10 he will have learned that waves exist which intimately involve electric and magnetic fields that are altogether quasistatic. (This is something that comes as a surprise to many late in life.) Lumped-parameter ideas are based on the integral forms of Maxwell's equations, so that the dynamical effects found with lumped-parameter time constants  $L/R$  and  $RC$  in Chapter 5 are easily associated with the subjects of magnetic diffusion and charge relaxation. A close tie is made between the "speed voltage" of Chapter 5 and the effects of motion on magnetic fields, as described by field transformations in Chapters 6 to 14. Constant flux dynamics of a lumped coil in Chapter 5 are strongly associated with the dynamics of perfectly conducting continuous media; for example, Alfvén waves in Chapter 12.

Consider another thread of continuity. The book begins with the mathematics of circuit theory. The machines of Chapter 4 are essentially circuits in the sinusoidal steady state. In Chapter 5 we linearize to pursue lumped-parameter ideas of stability and other transient responses and then proceed to nonlinear dynamics, potential-well theory, and other approaches that should form a part of any engineer's mathematical background. By the time

the end of Chapter 10 is reached these ideas will have been carried into the continuum with the addition of tensor concepts, simple cases of the method of characteristics, and eigenvalue theory. The  $\omega$ - $k$  plot and its implication for all sorts of subjects in modern electrical engineering can be considered as a mathematical or a physical objective. The ideas of stability introduced with ordinary differential equations ( $\exp st$ ) in Chapter 5 evolve into the continuum stability studies of Chapter 10 [ $\exp j(\omega t - kx)$ ] and can be regarded as a mathematical or a physical thread in our treatment. We could list many other threads: witness the evolution of energy and thermodynamic notions from Chapters 3 to 5, 5 to 8, and 8 to 13.

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*H. H. Woodson*  
*J. R. Melcher*

Cambridge, Massachusetts  
January 1968

# CONTENTS

## Part I: Discrete Systems

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.0	Introduction	1
1.0.1	Scope of Application	2
1.0.2	Objectives	4
1.1	Electromagnetic Theory	5
1.1.1	Differential Equations	6
1.1.2	Integral Equations	9
1.1.3	Electromagnetic Forces	12
1.2	Discussion	12
<b>2</b>	<b>Lumped Electromechanical Elements</b>	<b>15</b>
2.0	Introduction	15
2.1	Circuit Theory	16
2.1.1	Generalized Inductance	17
2.1.2	Generalized Capacitance	28
2.1.3	Discussion	34
2.2	Mechanics	35
2.2.1	Mechanical Elements	36
2.2.2	Mechanical Equations of Motion	49
2.3	Discussion	55
<b>3</b>	<b>Lumped-Parameter Electromechanics</b>	<b>60</b>
3.0	Introduction	60
3.1	Electromechanical Coupling	60
3.1.1	Energy Considerations	63
3.1.2	Mechanical Forces of Electric Origin	67
3.1.3	Energy Conversion	79
3.2	Equations of Motion	84
3.3	Discussion	88

<b>4 Rotating Machines</b>	<b>103</b>
4.0 Introduction	103
4.1 Smooth-Air-Gap Machines	104
4.1.1 Differential Equations	106
4.1.2 Conditions for Conversion of Average Power	110
4.1.3 Two-Phase Machine	111
4.1.4 Air-Gap Magnetic Fields	114
4.1.5 Discussion	117
4.1.6 Classification of Machine Types	119
4.1.7 Polyphase Machines	142
4.1.8 Number of Poles in a Machine	146
4.2 Salient-Pole Machines	150
4.2.1 Differential Equations	151
4.2.2 Conditions for Conversion of Average Power	154
4.2.3 Discussion of Saliency in Different Machine Types	156
4.2.4 Polyphase, Salient-Pole, Synchronous Machines	157
4.3 Discussion	165
<b>5 Lumped-Parameter Electromechanical Dynamics</b>	<b>179</b>
5.0 Introduction	179
5.1 Linear Systems	180
5.1.1 Linear Differential Equations	180
5.1.2 Equilibrium, Linearization, and Stability	182
5.1.3 Physical Approximations	206
5.2 Nonlinear Systems	213
5.2.1 Conservative Systems	213
5.2.2 Loss-Dominated Systems	227
5.3 Discussion	233
<b>6 Fields and Moving Media</b>	<b>251</b>
6.0 Introduction	251
6.1 Field Transformations	255
6.1.1 Transformations for Magnetic Field Systems	260
6.1.2 Transformations for Electric Field Systems	264
6.2 Boundary Conditions	267
6.2.1 Boundary Conditions for Magnetic Field Systems	270
6.2.2 Boundary Conditions for Electric Field Systems	277
6.3 Constituent Relations for Materials in Motion	283
6.3.1 Constituent Relations for Magnetic Field Systems	284
6.3.2 Constituent Relations for Electric Field Systems	289
6.4 DC Rotating Machines	291

6.4.1	Commutator Machines	292
6.4.2	Homopolar Machines	312
6.5	Discussion	317
<b>Appendix A Glossary of Commonly Used Symbols</b>		<b>A1</b>
<b>Appendix B Review of Electromagnetic Theory</b>		<b>B1</b>
B.1	Basic Laws and Definitions	B1
B.1.1	Coulomb's Law, Electric Fields and Forces	B1
B.1.2	Conservation of Charge	B5
B.1.3	Ampere's Law, Magnetic Fields and Forces	B6
B.1.4	Faraday's Law of Induction and the Potential Difference	B9
B.2	Maxwell's Equations	B12
B.2.1	Electromagnetic Waves	B13
B.2.2	Quasi-static Electromagnetic Field Equations	B19
B.3	Macroscopic Models and Constituent Relations	B25
B.3.1	Magnetization	B25
B.3.2	Polarization	B27
B.3.3	Electrical Conduction	B30
B.4	Integral Laws	B32
B.4.1	Magnetic Field System	B32
B.4.2	Electric Field System	B36
B.5	Recommended Reading	B37
<b>Appendix C Mathematical Identities and Theorems</b>		<b>C1</b>
<b>Index</b>		<b>1</b>
<b>Part II: Fields, Forces, and Motion</b>		
<b>7</b>	<b>Magnetic Diffusion and Charge Relaxation</b>	<b>330</b>
7.0	Introduction	330
7.1	Magnetic Field Diffusion	335
7.1.1	Diffusion as an Electrical Transient	338
7.1.2	Diffusion and Steady Motion	347
7.1.3	The Sinusoidal Steady-State in the Presence of Motion	355
7.1.4	Traveling Wave Diffusion in Moving Media	364
7.2	Charge Relaxation	370
7.2.1	Charge Relaxation as an Electrical Transient	372
7.2.2	Charge Relaxation in the Presence of Steady Motion	380
7.2.3	Sinusoidal Excitation and Charge Relaxation with Motion	392
7.2.4	Traveling-Wave Charge Relaxation in a Moving Conductor	397
7.3	Conclusion	401

<b>8</b>	<b>Field Description of Magnetic and Electric Forces</b>	<b>418</b>
8.0	Introduction	418
8.1	Forces in Magnetic Field Systems	419
8.2	The Stress Tensor	423
8.2.1	Stress and Traction	424
8.2.2	Vector and Tensor Transformations	434
8.3	Forces in Electric Field Systems	440
8.4	The Surface Force Density	445
8.4.1	Magnetic Surface Forces	447
8.4.2	Electric Surface Forces	447
8.5	The Magnetization and Polarization Force Densities	450
8.5.1	Examples with One Degree of Freedom	451
8.5.2	The Magnetization Force Density	456
8.5.3	The Stress Tensor	462
8.5.4	Polarization Force Density and Stress Tensor	463
8.6	Discussion	466
<b>9</b>	<b>Simple Elastic Continua</b>	<b>479</b>
9.0	Introduction	479
9.1	Longitudinal Motion of a Thin Rod	480
9.1.1	Wave Propagation Without Dispersion	487
9.1.2	Electromechanical Coupling at Terminal Pairs	498
9.1.3	Quasi-statics of Elastic Media	503
9.2	Transverse Motions of Wires and Membranes	509
9.2.1	Driven and Transient Response, Normal Modes	511
9.2.2	Boundary Conditions and Coupling at Terminal Pairs	522
9.3	Summary	535
<b>10</b>	<b>Dynamics of Electromechanical Continua</b>	<b>551</b>
10.0	Introduction	551
10.1	Waves and Instabilities in Stationary Media	554
10.1.1	Waves Without Dispersion	555
10.1.2	Cutoff or Evanescent Waves	556
10.1.3	Absolute or Nonconvective Instability	566
10.1.4	Waves with Damping, Diffusion Waves	576
10.2	Waves and Instabilities in the Presence of Material Motion	583
10.2.1	Fast and Slow Waves	586
10.2.2	Evanescence and Oscillation with Convection	596
10.2.3	Convective Instability or Wave Amplification	601
10.2.4	“Resistive Wall” Wave Amplification	608



10.3	Propagation	613
	10.3.1 Phase Velocity	613
	10.3.2 Group Velocity	614
	10.3.3 Characteristics and the Velocity of Wavefronts	618
10.4	Dynamics in Two Dimensions	621
	10.4.1 Membrane Dynamics: Two-Dimensional Modes	622
	10.4.2 Moving Membrane: Mach Lines	624
	10.4.3 A Kink Instability	627
10.5	Discussion	636
<b>Appendix D Glossary of Commonly Used Symbols</b>		<b>D1</b>
<b>Appendix E Summary of Part I and Useful Theorems</b>		<b>E1</b>
<b>Index</b>		<b>1</b>

### **Part III: Elastic and Fluid Media**

<b>11</b>	<b>Introduction to the Electromechanics of Elastic Media</b>	<b>651</b>
	11.0 Introduction	651
	11.1 Force Equilibrium	652
	11.2 Equations of Motion for Isotropic Media	653
	11.2.1 Strain-Displacement Relations	653
	11.2.2 Stress-Strain Relations	660
	11.2.3 Summary of Equations	666
	11.3 Electromechanical Boundary Conditions	668
	11.4 Waves in Isotropic Elastic Media	671
	11.4.1 Waves in Infinite Media	671
	11.4.2 Principal Modes of Simple Structures	679
	11.4.3 Elastic Vibrations of a Simple Guiding Structure	693
	11.5 Electromechanics and Elastic Media	696
	11.5.1 Electromagnetic Stresses and Mechanical Design	697
	11.5.2 Simple Continuum Transducers	704
	11.6 Discussion	717
<b>12</b>	<b>Electromechanics of Incompressible, Inviscid Fluids</b>	<b>724</b>
	12.0 Introduction	724
	12.1 Inviscid, Incompressible Fluids	726
	12.1.1 The Substantial Derivative	726
	12.1.2 Conservation of Mass	729
	12.1.3 Conservation of Momentum (Newton's Second Law)	731
	12.1.4 Constituent Relations	735

12.2	Magnetic Field Coupling with Incompressible Fluids	737
12.2.1	Coupling with Flow in a Constant-Area Channel	739
12.2.2	Coupling with Flow in a Variable-Area Channel	751
12.2.3	Alfvén Waves	759
12.2.4	Ferrohydrodynamics	772
12.3	Electric Field Coupling with Incompressible Fluids	776
12.3.1	Ion-Drag Phenomena	776
12.3.2	Polarization Interactions	783
12.4	Discussion	787
<b>13</b>	<b>Electromechanics of Compressible, Inviscid Fluids</b>	<b>813</b>
13.0	Introduction	813
13.1	Inviscid, Compressible Fluids	813
13.1.1	Conservation of Energy	814
13.1.2	Constituent Relations	815
13.2	Electromechanical Coupling with Compressible Fluids	820
13.2.1	Coupling with Steady Flow in a Constant-Area Channel	821
13.2.2	Coupling with Steady Flow in a Variable-Area Channel	828
13.2.3	Coupling with Propagating Disturbances	841
13.3	Discussion	854
<b>14</b>	<b>Electromechanical Coupling with Viscous Fluids</b>	<b>861</b>
14.0	Introduction	861
14.1	Viscous Fluids	862
14.1.1	Mathematical Description of Viscosity	862
14.1.2	Boundary Conditions	873
14.1.3	Fluid-Mechanical Examples	875
14.2	Electromechanical Coupling with Viscous Fluids	878
14.2.1	Electromechanical Coupling with Shear Flow	878
14.2.2	Electromechanical Coupling with Pressure-Driven Flow (Hartmann Flow)	884
14.3	Discussion	893
<b>Appendix F</b>	<b>Glossary of Commonly Used Symbols</b>	<b>F1</b>
<b>Appendix G</b>	<b>Summary of Parts I and II, and Useful Theorems</b>	<b>G1</b>
<b>Index</b>		<b>1</b>

# Appendix A

## GLOSSARY OF COMMONLY USED SYMBOLS

Section references indicate where symbols of a given significance are introduced; grouped symbols are accompanied by their respective references. The absence of a section reference indicates that a symbol has been applied for a variety of purposes. Nomenclature used in examples is not included.

Symbol	Meaning	Section
$A$	cross-sectional area	
$A_i$	coefficient in differential equation	5.1.1
$(A_n^+, A_n^-)$	complex amplitudes of components of $n$ th mode	9.2.1
$A_w$	cross-sectional area of armature conductor	6.4.1
$a$	spacing of pole faces in magnetic circuit	8.5.1
$a, (a_c, a_s)$	phase velocity of acoustic related waves	13.2.1, 11.4.1
$a_b$	Alfvén velocity	12.2.3
$(a, b, c)$	Lagrangian coordinates	11.1
$a_i$	constant coefficient in differential equation	5.1.1
$\mathbf{a}_p$	instantaneous acceleration of point $p$ fixed in material	2.2.1c
$B, B_r, B_s$	damping constant for linear, angular and square law dampers	2.2.1b, 4.1.1, 5.2.2
$\mathbf{B}, \mathbf{B}_i, B_0$	magnetic flux density	1.1.1a, 8.1, 6.4.2
$B_i$	induced flux density	7.0
$(B_r, B_{ra}, B_{rb}, B_{rm})$	radial components of air-gap flux densities	4.1.4
$[B_{rf}, (B_{rf})_{av}]$	radial flux density due to field current	6.4.1
$b$	width of pole faces in magnetic circuit	8.5
$b$	half thickness of thin beam	11.4.2b
$C$	contour of integration	1.1.2a
$C, (C_a, C_b), C_o$	capacitance	2.1.2, 7.2.1a, 5.2.1
$C$	coefficient in boundary condition	9.1.1
$\mathbf{C}$	the curl of the displacement	11.4
$(C^+, C^-)$	designation of characteristic lines	9.1.1

Symbol	Meaning	Section
$c_p$	specific heat capacity at constant pressure	13.1.2
$c_v$	specific heat capacity at constant volume	13.1.2
<b>D</b>	electric displacement	1.1.1a
$d$	length	
$da$	elemental area	1.1.2a
$d\mathbf{f}_n$	total elemental force on material in rigid body	2.2.1c
$d\mathbf{l}$	elemental line segment	1.1.2a
$d\mathbf{T}_n$	torque on elemental volume of material	2.2.1c
$dV$	elemental volume	1.1.2b
$E$	constant of motion	5.2.1
$E$	Young's modulus or the modulus of elasticity	9.1
$\mathbf{E}, E_0$	electric field intensity	1.1.1a, 5.1.2d
$E_f$	magnitude of armature voltage generated by field current in a synchronous machine	4.1.6a
$E_i$	induced electric field intensity	7.0
$e_{11}, e_{ij}$	strain tensor	9.1, 11.2
$\dot{e}_{ij}$	strain-rate tensor	14.1.1a
$F$	magnetomotive force (mmf)	13.2.2
<b>F</b>	force density	1.1.1a
$\hat{F}$	complex amplitude of $f(t)$	5.1.1
$F_0$	amplitude of sinusoidal driving force	9.1.3
$f$	equilibrium tension of string	9.2
$f$	driving function	5.1.1
$f, \mathbf{f}, f^e, f^s, f_j, f_i, f_1$	force	2.2.1, 2.2.1c, 3.1, 5.1.2a, 3.1.2b, 8.1, 9.1
$f$	arbitrary scalar function	6.1
$f'$	scalar function in moving coordinate system	6.1
$f$	three-dimensional surface	6.2
$f$	integration constant	11.4.2a
$G$	a constant	5.1.2c
$G$	shear modulus of elasticity	11.2.2
$G$	speed coefficient	6.4.1
$G$	conductance	3.1
$g$	air-gap length	5.2.1
$g, \mathbf{g}$	acceleration of gravity	5.1.2c, 12.1.3
$(\mathbf{H}, H_x, H_y, H_z)$	magnetic field intensity	1.1.1a
$h$	specific enthalpy	13.1.2
$\mathbf{I}, I, (I_r, I_s), I_f$	electrical current	10.4.3, 12.2.1a, 4.1.2, 6.4.1
$(i, i_1, i_2, \dots, i_k), (i_{ar}, i_{as}, i_{br}, i_{bs}), i_a, (i_a, i_b, i_c), (i_f, i_t), (i_r, i_s)$	electrical current	2.1, 4.1.3, 6.4.1, 4.1.7, 6.4.1, 4.1

Symbol	Meaning	Section
$\mathbf{i}_n$	unit vector perpendicular to area of integration	6.2.1
$\mathbf{i}_s$	unit vector normal to surface of integration	6.2.1
$(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z), (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$	unit vectors in coordinate directions	2.2.1c
$J, \mathbf{J}_f$	current density	7.0, 1.1.1a
$J, J_x, (J_x, J_y, J_z)$	moment of inertia	5.1.2b, 4.1.1, 2.2.1c
$J_{xx}, J_{yz}$	products of inertia	2.2.1c
$j$	$\sqrt{-1}$	4.1.6a
$K$	loading factor	13.2.2
$K, \mathbf{K}_f$	surface current density	7.0, 1.1.1a
$K$	linear or torsional spring constant	2.2.1a
$K_t$	induced surface current density	7.0
$k, k_o, (k_r, k_t)$	wavenumber	7.1.3, 10.1.3, 10.0
$k$	summation index	2.1.1
$k$	maximum coefficient of coupling	4.1.6b
$k_n$	$n$ th eigenvalue	9.2
$(L, L_1, L_2), (L_a, L_f), L_m, (L_0, L_2), (L_r, L_s, L_{sr}), L_{ss}$	inductance	2.1.1, 6.4.1, 2.1.1, 4.2.1, 4.1.1, 4.2.4
$L$	length of incremental line segment	6.2.1
$l$	value of relative displacement for which spring force is zero	2.2.1a
$l, l_w, l_y$	length	
$M$	Hartmann number	14.2.2
$M$	mass of one mole of gas in kilograms	13.1.2
$M$	Mach number	13.2.1
$M$	mass	2.2.1c
$M$	number of mechanical terminal pairs	2.1.1
$M, M_s$	mutual inductance	4.1.1, 4.2.4
$M$	magnetization density	1.1.1a
$m$	mass/unit length of string	9.2
$N$	number of electrical terminal pairs	2.1.1
$N$	number of turns	5.2.2
$n$	number density of ions	12.3.1
$n$	integer	7.1.1
$\mathbf{n}$	unit normal vector	1.1.2
$\mathbf{P}$	polarization density	1.1.1a
$P$	power	12.2.1a
$P$	number of pole pairs in a machine	4.1.8
$P$	power per unit area	14.2.1
$p$	pressure	5.1.2d and 12.1.4
$P_o, P_o, P_m, P_r$	power	4.1.6a, 4.1.6b, 4.1.2, 4.1.6b
$Q$	electric charge	7.2.1a
$q, q_i, q_k$	electric charge	1.1.3 and 2.1.2, 8.1, 2.1.2
$R, R_t, R_o$	radius	

Symbol	Meaning	Section
$R, R_a, R_b, R_f, R_r, R_s$	resistance	
$(R, R_g)$	gas constant	13.1.2
$R_e$	electric Reynolds number	7.0
$R_m$	magnetic Reynolds number	7.0
$r$	radial coordinate	
$\mathbf{r}$	position vector of material	2.2.1c
$\mathbf{r}'$	position vector in moving reference frame	6.1
$\mathbf{r}_m$	center of mass of rigid body	2.2.1c
$S$	reciprocal modulus of elasticity	11.5.2c
$S$	surface of integration	1.1.2a
$S$	normalized frequency	7.2.4
$S$	membrane tension	9.2
$S_z$	transverse force/unit length acting on string	9.2
$s$	complex frequency	5.1.1
$(s, s_{mT})$	slip	4.1.6b
$s_i$	$i$ th root of characteristic equation, a natural frequency	5.1.1
$T$	period of oscillation	5.2.1
$T$	temperature	13.1.2
$\mathbf{T}, T, T^e, T_{em}, T_m, T_0, T_1$	torque	2.2.1c, 5.1.2b, 3.1.1, 4.1.6b, 4.1.1, 6.4.1, 6.4.1
$\mathbf{T}$	surface force	8.4
$T_{ij}^m$	mechanical stress tensor	13.1.2
$T_{mn}$	the component of the stress-tensor in the $m$ th-direction on a cartesian surface with a normal vector in the $n$ th-direction	8.1
$T_{or}$	constant of coulomb damping	4.1.1
$T_0$	initial stress distribution on thin rod	9.1.1
$T$	longitudinal stress on a thin rod	9.1.1
$T_z$	transverse force per unit area on membrane	9.2
$T_2$	transverse force per unit area acting on thin beam	11.4.2b
$t$	time	1.1.1
$t'$	time measured in moving reference frame	6.1
$U$	gravitational potential	12.1.3
$U$	longitudinal steady velocity of string or membrane	10.2
$u$	internal energy per unit mass	13.1.1
$u$	surface coordinate	11.3
$u_0(x - x_0)$	unit impulse at $x = x_0$	9.2.1
$u$	transverse deflection of wire in $x$ -direction	10.4.3
$u_{-1}(t)$	unit step occurring at $t = 0$	5.1.2b
$V, V_m$	velocity	7.0, 13.2.3
$V$	volume	1.1.2
$V, V_a, V_f, V_o, V_s$	voltage	
$V$	potential energy	5.2.1

Symbol	Meaning	Section
$v, \mathbf{v}$	velocity	
$(v, v_1, \dots, v_k)$	voltage	2.1.1
$v', (v_a, v_b, v_c),$ $v_f, v_{oc}, v_l$	voltage	
$v_n$	velocity of surface in normal direction	6.2.1
$v_o$	initial velocity distribution on thin rod	9.1.1
$v_p$	phase velocity	9.1.1 and 10.2
$\mathbf{v}^r$	relative velocity of inertial reference frames	6.1
$v_s$	$\sqrt{f/m}$ for a string under tension $f$ and having mass/unit length $m$	10.1.1
$v$	longitudinal material velocity on thin rod	9.1.1
$v$	transverse deflection of wire in $y$ -direction	10.4.3
$(W_e, W_m)$	energy stored in electromechanical coupling	3.1.1
$(W'_e, W'_m, W')$	coenergy stored in electromechanical coupling	3.1.2b
$W''$	hybrid energy function	5.2.1
$w$	width	5.2.2
$w$	energy density	11.5.2c
$w'$	coenergy density	8.5
$X$	equilibrium position	5.1.2a
$(x, x_1, x_2, \dots, x_k)$	displacement of mechanical node	2.1.1
$x$	dependent variable	5.1.1
$x_p$	particular solution of differential equation	5.1.1
$(x_1, x_2, x_3), (x, y, z)$	cartesian coordinates	8.1, 6.1
$(x', y', z')$	cartesian coordinates of moving frame	6.1
$(\alpha, \beta)$	constants along $C^+$ and $C^-$ characteristics, respectively	9.1.1
$(\alpha, \beta)$	see (10.2.20) or (10.2.27)	
$\alpha$	transverse wavenumber	11.4.3
$(\alpha, \beta)$	angles used to define shear strain	11.2
$(\alpha, \beta)$	constant angles	4.1.6b
$\alpha$	space decay parameter	7.1.4
$\alpha$	damping constant	5.1.2b
$\alpha$	equilibrium angle of torsional spring	2.2.1a
$\gamma$	ratio of specific heats	13.1.2
$\gamma$	piezoelectric constant	11.5.2c
$\gamma, \gamma_0, \gamma'$	angular position	
$\Delta_d(t)$	slope excitation of string	10.2.1b
$\Delta_0$	amplitude of sinusoidal slope excitation	10.2.1b
$\Delta \mathbf{r}$	distance between unstressed material points	11.2.1a
$\Delta \mathbf{s}$	distance between stressed positions of material points	11.2.1a
$\delta( )$	incremental change in ( )	8.5
$\delta, \delta_1, \delta_0$	displacement of elastic material	11.1, 9.1, 11.4.2a
$\delta$	thickness of incremental volume element	6.2.1
$\delta$	torque angle	4.1.6a

Symbol	Meaning	Section
$\delta_{ij}$	Kronecker delta	8.1
$(\delta_+, \delta_-)$	wave components traveling in the $\pm x$ -directions	9.1.1
$\epsilon$	linear permittivity	1.1.1b
$\epsilon_0$	permittivity of free space	1.1.1a
$\eta$	efficiency of an induction motor	4.1.6b
$\eta$	second coefficient of viscosity	14.1.1c
$\theta, \theta_i, \theta_m$	angular displacement	2.1.1, 3.1.1, 5.2.1
$\theta$	power factor angle; phase angle between current and voltage	4.1.6a
$\theta$	equilibrium angle	5.2.1
$\dot{\theta}$	angular velocity of armature	6.4.1
$\theta_m$	maximum angular deflection	5.2.1
$(\lambda, \lambda_1, \lambda_2, \dots, \lambda_k)$	magnetic flux linkage	2.1.1, 6.4.1, 4.1.7,
$\lambda_a$		4.1.3, 4.1
$(\lambda_a, \lambda_b, \lambda_c)$		
$(\lambda_{ar}, \lambda_{as}, \lambda_{br}, \lambda_{bs})$		
$(\lambda_r, \lambda_s)$		
$\lambda$	Lamé constant for elastic material	11.2.3
$\lambda$	wavelength	7.1.4
$\mu$	linear permeability	1.1.1a
$\mu, (\mu_+, \mu_-)$	mobility	12.3.1, 1.1.1b
$\mu$	coefficient of viscosity	14.1.1
$\mu_d$	coefficient of dynamic friction	2.2.1b
$\mu_0$	permeability of free space	1.1.1a
$\mu_s$	coefficient of static friction	2.2.1b
$\nu$	Poisson's ratio for elastic material	11.2.2
$\nu$	damping frequency	10.1.4
$(\xi, \xi)$	continuum displacement	8.5
$\xi_0$	initial deflection of string	9.2
$\xi_d$	amplitude of sinusoidal driving deflection	9.2
$(\xi_n(x), \hat{\xi}_n(x))$	$n$ th eigenfunctions	9.2.1b
$(\xi_+, \xi_-)$	amplitudes of forward and backward traveling waves	9.2
$\dot{\xi}_0(x)$	initial velocity of string	9.2
$\rho$	mass density	2.2.1c
$\rho_f$	free charge density	1.1.1a
$\rho_s$	surface mass density	11.3
$\Sigma$	surface of discontinuity	6.2
$\sigma$	conductivity	1.1.1a
$\sigma_f$	free surface charge density	1.1.1a
$\sigma_m$	surface mass density of membrane	9.2
$\sigma_o$	surface charge density	7.2.3
$\sigma_s$	surface conductivity	1.1.1a
$\sigma_u$	surface charge density	7.2.3
$\tau$	surface traction	8.2.1
$\tau, \tau_d$	diffusion time constant	7.1.1, 7.1.2a
$\tau$	relaxation time	7.2.1a



Symbol	Meaning	Section
$\tau_e$	electrical time constant	5.2.2
$\tau_m$	time for air gap to close	5.2.2
$\tau_o$	time constant	5.1.3
$\tau_t$	traversal time	7.1.2a
$\phi$	electric potential	7.2
$\phi$	magnetic flux	2.1.1
$\phi$	cylindrical coordinate	2.1.1
$\phi$	potential for $\mathbf{H}$ when $\mathbf{J}_f = 0$	8.5.2
$\phi$	flow potential	12.2
$\chi_e$	electric susceptibility	1.1.1b
$\chi_m$	magnetic susceptibility	1.1.1a
$\psi$	the divergence of the material displacement	11.4
$\psi$	angle defined in Fig. 6.4.2	6.4.1
$\psi$	angular position in the air gap measured from stator winding ( <i>a</i> ) magnetic axis	4.1.4
$\psi$	electromagnetic force potential	12.2
$\psi$	angular deflection of wire	10.4.3
$\Omega$	equilibrium rotational speed	5.1.2b
$\mathbf{\Omega}$	rotation vector in elastic material	11.2.1a
$\Omega_n$	real part of eigenfrequency (10.1.47)	10.1.4
$\omega, (\omega_r, \omega_s)$	radian frequency of electrical excitation	4.1.6a, 4.1.2
$\omega$	natural angular frequency (Im $s$ )	5.1.2b
$\boldsymbol{\omega}, \omega_m$	angular velocity	2.2.1c, 4.1.2
$\omega_c$	cutoff frequency for evanescent waves	10.1.2
$\omega_d$	driving frequency	9.2
$\omega_n$	$n$ th eigenfrequency	9.2
$\omega_o$	natural angular frequency	5.1.3
$(\omega_r, \omega_i)$	real and imaginary parts of $\omega$	10.0
$\nabla$	nabla	6.1
$\nabla_\Sigma$	surface divergence	6.2.1



# Appendix B

## REVIEW OF ELECTROMAGNETIC THEORY

### B.1 BASIC LAWS AND DEFINITIONS

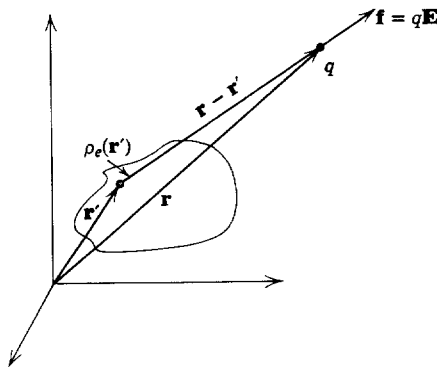
The laws of electricity and magnetism are empirical. Fortunately they can be traced to a few fundamental experiments and definitions, which are reviewed in the following sections. The rationalized MKS system of units is used.

#### B.1.1 Coulomb's Law, Electric Fields and Forces

Coulomb found that when a charge  $q$  (coulombs) is brought into the vicinity of a distribution of *charge density*  $\rho_e(\mathbf{r}')$  (coulombs per cubic meter), as shown in Fig. B.1.1, a force of repulsion  $\mathbf{f}$  (newtons) is given by

$$\mathbf{f} = q\mathbf{E}, \quad (\text{B.1.1})$$

where the *electric field intensity*  $\mathbf{E}$  (volts per meter) is evaluated at the position



**Fig. B.1.1** The force  $\mathbf{f}$  on the point charge  $q$  in the vicinity of charges with density  $\rho_e(\mathbf{r}')$  is represented by the electric field intensity  $\mathbf{E}$  times  $q$ , where  $\mathbf{E}$  is found from (B.1.2).

$\mathbf{r}$  of the charge  $q$  and determined from the distribution of charge density by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho_e(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \tag{B.1.2}$$

In the rationalized MKS system of units the permittivity  $\epsilon_0$  of free space is

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9} \text{ F/m}. \tag{B.1.3}$$

Note that the integration of (B.1.2) is carried out over all the charge distribution (excluding  $q$ ), hence represents a superposition (at the location  $\mathbf{r}$  of  $q$ ) of the electric field intensities due to elements of charge density at the positions  $\mathbf{r}'$ .

As an example, suppose that the charge distribution  $\rho_e(\mathbf{r}')$  is simply a point charge  $Q$  (coulombs) at the origin (Fig. B.1.2); that is,

$$\rho_e = Q \delta(\mathbf{r}'), \tag{B.1.4}$$

where  $\delta(\mathbf{r}')$  is the *delta function* defined by

$$\begin{aligned} \delta(\mathbf{r}') &= 0, & \mathbf{r}' &\neq 0, \\ \int_{V'} \delta(\mathbf{r}') dV' &= 1. \end{aligned} \tag{B.1.5}$$

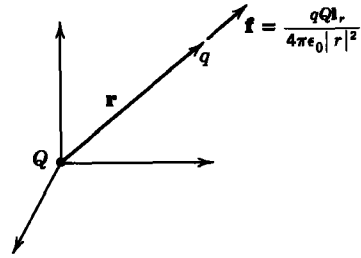


Fig. B.1.2 Coulomb's law for point charges  $Q$  (at the origin) and  $q$  (at the position  $\mathbf{r}$ ).

For the charge distribution of (B.1.4) integration of (B.1.2) gives

$$\mathbf{E}(\mathbf{r}) = \frac{Q\mathbf{r}}{4\pi\epsilon_0|r|^3}. \tag{B.1.6}$$

Hence the force on the point charge  $q$ , due to the point charge  $Q$ , is from (B.1.1)

$$\mathbf{f} = \frac{qQ\mathbf{r}}{4\pi\epsilon_0|r|^3}. \tag{B.1.7}$$

This expression takes the familiar form of *Coulomb's law* for the force of repulsion between point charges of like sign.

We know that electric charge occurs in integral multiples of the electronic charge ( $1.60 \times 10^{-19}$  C). The charge density  $\rho_e$ , introduced with (B.1.2), is defined as

$$\rho_e(\mathbf{r}) = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \sum_i q_i, \tag{B.1.8}$$

where  $\delta V$  is a small volume enclosing the point  $\mathbf{r}$  and  $\sum_i q_i$  is the algebraic sum of charges within  $\delta V$ . The charge density is an example of a continuum model. To be valid the limit  $\delta V \rightarrow 0$  must represent a volume large enough to contain a large number of charges  $q_i$ , yet small enough to appear infinitesimal when compared with the significant dimensions of the system being analyzed. This condition is met in most electromechanical systems.

For example, in copper at a temperature of 20°C the number density of free electrons available for carrying current is approximately  $10^{23}$  electrons/cm<sup>3</sup>. If we consider a typical device dimension to be on the order of 1 cm, a reasonable size for  $\delta V$  would be a cube with 1-mm sides. The number of electrons in  $\delta V$  would be  $10^{20}$ , which certainly justifies the continuum model.

The force, as expressed by (B.1.1), gives the total force on a single test charge in vacuum and, as such, is not appropriate for use in a continuum model of electromechanical systems. It is necessary to use an *electric force density*  $\mathbf{F}$  (newtons per cubic meter) that can be found by averaging (B.1.1) over a small volume.

$$\mathbf{F} = \lim_{\delta V \rightarrow 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \rightarrow 0} \frac{\sum q_i \mathbf{E}_i}{\delta V}. \quad (\text{B.1.9})$$

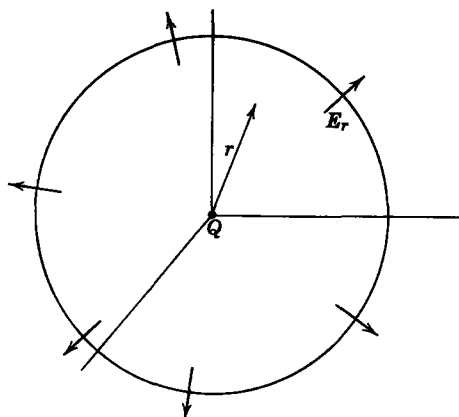
Here  $q_i$  represents all of the charges in  $\delta V$ ,  $\mathbf{E}_i$  is the electric field intensity acting on the  $i$ th charge, and  $\mathbf{f}_i$  is the force on the  $i$ th charge. As in the charge density defined by (B.1.8), the limit of (B.1.9) leads to a continuum model if the volume  $\delta V$  can be defined so that it is small compared with macroscopic dimensions of significance, yet large enough to contain many electronic charges. Further, there must be a sufficient amount of charge external to the volume  $\delta V$  that the electric field experienced by each of the test charges is essentially determined by the sources of field outside the volume. Fortunately these requirements are met in almost all physical situations that lead to useful electromechanical interactions. Because all charges in the volume  $\delta V$  experience essentially the same electric field  $\mathbf{E}$ , we use the definition of free charge density given by (B.1.8) to write (B.1.9) as

$$\mathbf{F} = \rho_e \mathbf{E}. \quad (\text{B.1.10})$$

Although the static electric field intensity  $\mathbf{E}$  can be computed from (B.1.2), it is often more convenient to state the relation between charge density and field intensity in the form of *Gauss's law*:

$$\oint_S \epsilon_0 \mathbf{E} \cdot \mathbf{n} \, da = \int_V \rho_e \, dV. \quad (\text{B.1.11})$$

In this integral law  $\mathbf{n}$  is the outward-directed unit vector normal to the surface  $S$ , which encloses the volume  $V$ . It is not our purpose in this brief review to show that (B.1.11) is implied by (B.1.2). It is helpful, however, to note that



**Fig. B.1.3** A hypothetical sphere of radius  $r$  encloses a charge  $Q$  at the origin. The integral of  $\epsilon_0 E_r$  over the surface of the sphere is equal to the charge  $Q$  enclosed.

in the case of a point charge  $Q$  at the origin it predicts the same electric field intensity (B.1.6) as found by using (B.1.2). For this purpose the surface  $S$  is taken as the sphere of radius  $r$  centered at the origin, as shown in Fig. B.1.3. By symmetry the only component of  $\mathbf{E}$  is radial ( $E_r$ ), and this is constant at a given radius  $r$ . Hence (B.1.11) becomes

$$4\pi r^2 E_r \epsilon_0 = Q. \quad (\text{B.1.12})$$

Here the integration of the charge density over the volume  $V$  enclosed by  $S$  is the total charge enclosed  $Q$  but can be formally taken by using (B.1.4) with the definition provided by (B.1.5). It follows from (B.1.12) that

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}, \quad (\text{B.1.13})$$

a result that is in agreement with (B.1.6).

Because the volume and surface of integration in (B.1.11) are arbitrary, the integral equation implies a differential law. This is found by making use of the *divergence theorem*\*

$$\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV \quad (\text{B.1.14})$$

to write (B.1.11) as

$$\int_V (\nabla \cdot \epsilon_0 \mathbf{E} - \rho_e) \, dV = 0. \quad (\text{B.1.15})$$

\* For a discussion of the divergence theorem see F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 312.

Since the volume of integration is arbitrary, it follows that

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e. \quad (\text{B.1.16})$$

From this discussion it should be apparent that this *differential* form of *Gauss's law* is implied by Coulomb's law, with the electric field intensity defined as a force per unit charge.

### B.1.2 Conservation of Charge

Experimental evidence supports the postulate that electric charge is conserved. When a negative charge appears (e.g., when an electron is removed from a previously neutral atom), an equal positive charge also appears (e.g., the positive ion remaining when the electron is removed from the atom).

We can make a mathematical statement of this postulate in the following way. Consider a volume  $V$  enclosed by a surface  $S$ . If charge is conserved, the net rate of flow of electric charge out through the surface  $S$  must equal the rate at which the total charge in the volume  $V$  decreases. The current density  $\mathbf{J}$  (coulombs per square meter-second) is defined as having the direction of flow of positive charge and a magnitude proportional to the net rate of flow of charge per unit area. Then the statement of conservation of charge is

$$\oint_S \mathbf{J} \cdot \mathbf{n} \, da = - \frac{d}{dt} \int_V \rho_e \, dV. \quad (\text{B.1.17})$$

Once again it follows from the arbitrary nature of  $S$  (which is fixed in space) and the divergence theorem (B.1.14) that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \quad (\text{B.1.18})$$

It is this equation that is used as a *differential* statement of *conservation of charge*.

To express conservation of charge it has been necessary to introduce a new continuum variable, the current density  $\mathbf{J}$ . Further insight into the relation between this quantity and the charge density  $\rho_e$  is obtained by considering a situation in which two types of charge contribute to the current, charges  $q_+$  with velocity  $\mathbf{v}_+$  and charges  $q_-$  with velocity  $\mathbf{v}_-$ . The current density  $\mathbf{J}_+$  that results from the flow of positive charge is

$$\mathbf{J}_+ = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \sum_i q_{+i} \mathbf{v}_{+i}. \quad (\text{B.1.19})$$

If we define a *charge-average velocity*  $\mathbf{v}_+$  for the positive charges as

$$\mathbf{v}_+ = \frac{\sum_i q_{+i} \mathbf{v}_{+i}}{\sum_i q_{+i}} \quad (\text{B.1.20})$$

and the density  $\rho_+$  of positive charges from (B.1.8) as

$$\rho_+ = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \sum_i q_{+i}, \quad (\text{B.1.21})$$

we can write the current density of (B.1.19) as

$$\mathbf{J}_+ = \rho_+ \mathbf{v}_+. \quad (\text{B.1.22})$$

Similar definitions for the charge-average velocity  $\mathbf{v}_-$  and charge density  $\rho_-$  of negative charges yields the component of current density

$$\mathbf{J}_- = \rho_- \mathbf{v}_-. \quad (\text{B.1.23})$$

The total current density  $\mathbf{J}$  is the vector sum of the two components

$$\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-. \quad (\text{B.1.24})$$

Now consider the situation of a material that contains charge densities  $\rho_+$  and  $\rho_-$  which have charge-average velocities  $\mathbf{v}_+$  and  $\mathbf{v}_-$  with respect to the material. Assume further that the material is moving with a velocity  $\mathbf{v}$  with respect to an observer who is to measure the current. The net average velocities of positive and negative charges as seen by the observer are  $\mathbf{v}_+ + \mathbf{v}$  and  $\mathbf{v}_- + \mathbf{v}$ , respectively. The current density measured by the observer is then from (B.1.24)

$$\mathbf{J} = (\rho_+ \mathbf{v}_+ + \rho_- \mathbf{v}_-) + \rho_e \mathbf{v}, \quad (\text{B.1.25})$$

where the net charge density  $\rho_e$  is given by

$$\rho_e = \rho_+ + \rho_-. \quad (\text{B.1.26})$$

The first term of (B.1.25) is a net flow of charge with respect to the material and is normally called a *conduction current*. (It is often described by Ohm's law.) The last term represents the transport of net charge and is conventionally called a *convection current*. It is crucial that *net flow of charge* be distinguished from *flow of net charge*. The net charge may be zero but a current can still be accounted for by the conduction term. This is the case in metallic conductors.

### B.1.3 Ampère's Law, Magnetic Fields and Forces

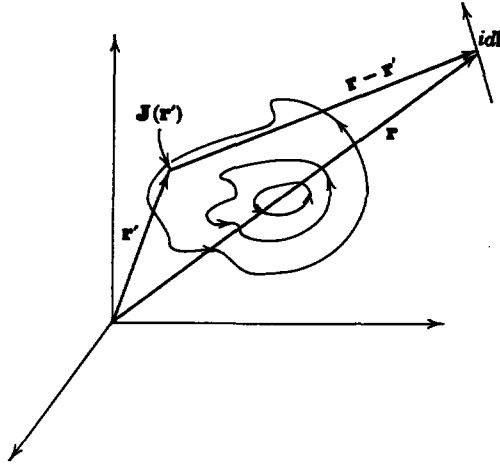
The *magnetic flux density*  $\mathbf{B}$  is defined to express the force on a current element  $i d\mathbf{l}$  placed in the vicinity of other currents. This element is shown in Fig. B.1.4 at the position  $\mathbf{r}$ . Then, according to Ampère's experiments, the force is given by

$$\mathbf{f} = i d\mathbf{l} \times \mathbf{B}, \quad (\text{B.1.27})$$

where

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (\text{B.1.28})$$



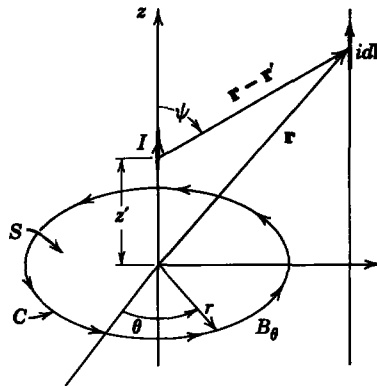


**Fig. B.1.4** A distribution of current density  $J(r')$  produces a force on the current element  $idl$  which is represented in terms of the magnetic flux density  $B$  by (B.1.27) and (B.1.28).

Hence the flux density at the position  $r$  of the current element  $idl$  is the superposition of fields produced by currents at the positions  $r'$ . In this expression the permeability of free space  $\mu_0$  is

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.} \quad (\text{B.1.29})$$

As an example, suppose that the distribution of current density  $J$  is composed of a current  $I$  (amperes) in the  $z$  direction and along the  $z$ -axis, as shown in Fig. B.1.5. The magnetic flux density at the position  $r$  can be computed



**Fig. B.1.5** A current  $I$  (amperes) along the  $z$ -axis produces a magnetic field at the position  $r$  of the current element  $idl$ .

from (B.1.28), which for this case reduces to\*

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{i}_z \times (\mathbf{r} - z'\mathbf{i}_z)}{|\mathbf{r} - z'\mathbf{i}_z|^3} dz' \quad (\text{B.1.30})$$

Here the coordinate of the source current  $I$  is  $z'$ , as shown in Fig. B.1.5, whereas the coordinate  $\mathbf{r}$  that designates the position at which  $\mathbf{B}$  is evaluated can be written in terms of the cylindrical coordinates  $(r, \theta, z)$ . Hence (B.1.30) becomes

$$\mathbf{B} = \frac{\mu_0 I \mathbf{i}_\theta}{4\pi} \int_{-\infty}^{+\infty} \frac{\sin \psi \sqrt{(z - z')^2 + r^2}}{[(z - z')^2 + r^2]^{3/2}} dz', \quad (\text{B.1.31})$$

where, from Fig. B.1.5,  $\sin \psi = r/\sqrt{(z - z')^2 + r^2}$ . Integration on  $z'$  gives the magnetic flux density

$$\mathbf{B} = \frac{\mu_0 I \mathbf{i}_\theta}{2\pi r}. \quad (\text{B.1.32})$$

It is often more convenient to relate the magnetic flux density to the current density  $\mathbf{J}$  by the integral of *Ampère's law* for static fields, which takes the form

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} \, da. \quad (\text{B.1.33})$$

Here  $C$  is a closed contour of line integration and  $S$  is a surface enclosed by  $C$ . We wish to present a review of electromagnetic theory and therefore we shall not embark on a proof that (B.1.33) is implied by (B.1.28). Our purpose is served by recognizing that (B.1.33) can also be used to predict the flux density in the situation in Fig. B.1.5. By symmetry we recognize that  $\mathbf{B}$  is azimuthally directed and independent of  $\theta$  and  $z$ . Then, if we select the contour  $C$  in a plane  $z$  equals constant and at a radius  $r$ , as shown in Fig. B.1.5, (B.1.33) becomes

$$2\pi r B_\theta = \mu_0 I. \quad (\text{B.1.34})$$

Solution of this expression for  $B_\theta$  gives the same result as predicted by (B.1.28). [See (B.1.32).]

The contour  $C$  and surface  $S$  in (B.1.33) are arbitrary and therefore the equation can be cast in a differential form. This is done by using Stokes' theorem†,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{A}) \, da, \quad (\text{B.1.35})$$

\* Unit vectors in the coordinate directions are designated by  $\mathbf{i}$ . Thus  $\mathbf{i}_z$  is a unit vector in the  $z$ -direction.

† See F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 318.

to write (B.1.33) as

$$\int_S (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) \cdot \mathbf{n} \, da = 0, \quad (\text{B.1.36})$$

from which the differential form of Ampère's law follows as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \quad (\text{B.1.37})$$

So far the assumption has been made that the current  $\mathbf{J}$  is constant in time. Maxwell's contribution consisted in recognizing that if the sources  $\rho_e$  and  $\mathbf{J}$  (hence the fields  $\mathbf{E}$  and  $\mathbf{B}$ ) are time varying the displacement current  $\epsilon_0 \partial \mathbf{E} / \partial t$  must be included on the right-hand side of (B.1.37). Thus for dynamic fields Ampère's law takes the form

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}. \quad (\text{B.1.38})$$

This alteration of (B.1.37) is necessary if conservation of charge expressed by (B.1.18) is to be satisfied. Because the divergence of any vector having the form  $\nabla \times \mathbf{A}$  is zero, the divergence of (B.1.38) becomes

$$\nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \epsilon_0 \mathbf{E})}{\partial t} = 0. \quad (\text{B.1.39})$$

Then, if we recall that  $\rho_e$  is related to  $\mathbf{E}$  by Gauss's law (B.1.16), the conservation of charge equation (B.1.18) follows. The displacement current in (B.1.38) accounts for the rate of change of  $\rho_e$  in (B.1.18).

We shall make considerable use of Ampère's law, as expressed by (B.1.38), with Maxwell's displacement current included. From our discussion it is clear that the static form of this law results from the force law of interaction between currents. The magnetic flux density is defined in terms of the force produced on a current element. Here we are interested primarily in a continuum description of the force, hence require (B.1.27) expressed as a force density. With the same continuum restrictions implied in writing (B.1.10), we write the magnetic force density (newtons per cubic meter) as

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \quad (\text{B.1.40})$$

In view of our remarks it should be clear that this force density is not something that we have derived but rather arises from the definition of the flux density  $\mathbf{B}$ . Further remarks on this subject are found in Section 8.1.

#### B.1.4 Faraday's Law of Induction and the Potential Difference

Two extensions of static field theory are required to describe dynamic fields. One of these, the introduction of the displacement current in Ampère's law, was discussed in the preceding section. Much of the significance of this

generalization stems from the apparent fact that an electric field can lead to the generation of a magnetic field. As a second extension of static field theory, Faraday discovered that, conversely, time-varying magnetic fields can lead to the generation of electric fields.

*Faraday's law of induction* can be written in the integral form

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da, \quad (\text{B.1.41})$$

where again  $C$  is a contour that encloses the surface  $S$ . The contour and surface are arbitrary; hence it follows from Stokes' theorem (B.1.35) that Faraday's law has the differential form

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.1.42})$$

Note that in the static case this expression reduces to  $\nabla \times \mathbf{E} = 0$ , which is, in addition to Gauss's law, a condition on the static electric field. That this further equation is consistent with the electric field, as given by (B.1.2), is not shown in this review. Clearly the one differential equation represented by Gauss's law could not alone determine the three components of  $\mathbf{E}$ .

In regions in which the magnetic field is either static or negligible the electric field intensity can be derived as the gradient of a scalar potential  $\phi$ :

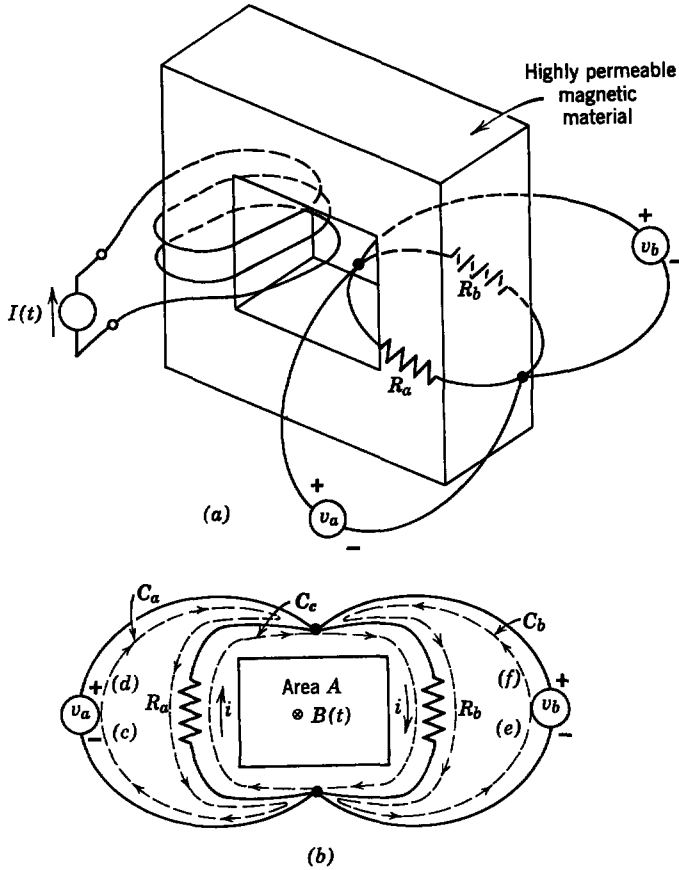
$$\mathbf{E} = -\nabla\phi. \quad (\text{B.1.43})$$

This is true because the curl of the gradient is zero and (B.1.42) is satisfied. The difference in potential between two points, say  $a$  and  $b$ , is a measure of the line integral of  $\mathbf{E}$ , for

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = - \int_a^b \nabla\phi \cdot d\mathbf{l} = \phi_a - \phi_b. \quad (\text{B.1.44})$$

The potential difference  $\phi_a - \phi_b$  is referred to as the voltage of point  $a$  with respect to  $b$ . If there is no magnetic field  $\mathbf{B}$  in the region of interest, the integral of (B.1.44) is independent of path. In the presence of a time-varying magnetic field the integral of  $\mathbf{E}$  around a closed path is not in general zero, and if a potential is defined in some region by (B.1.43) the path of integration will in part determine the measured potential difference.

The physical situation shown in Fig. B.1.6 serves as an illustration of the implications of Faraday's law. A magnetic circuit is excited by a current source  $I(t)$  as shown. Because the magnetic material is highly permeable, the induced flux density  $B(t)$  is confined to the cross section  $A$  which links a circuit formed by resistances  $R_a$  and  $R_b$  in series. A cross-sectional view of the



**Fig. B.1.6** (a) A magnetic circuit excited by  $I(t)$  so that flux  $AB(t)$  links the resistive loop (b) a cross-sectional view of the loop showing connection of the voltmeters.

circuit is shown in Fig. B.1.6b, in which high impedance voltmeters  $v_a$  and  $v_b$  are shown connected to the same nodes. Under the assumption that no current is drawn by the voltmeters, and given the flux density  $B(t)$ , we wish to compute the voltages that would be indicated by  $v_a$  and  $v_b$ .

Three contours of integration  $C$  are defined in Fig. B.1.6b and are used with Faraday's integral law (B.1.41). The integral of  $\mathbf{E}$  around the contour  $C_e$  is equal to the drop in potential across both of the resistances, which carry the same current  $i$ . Hence, since this path encloses a total flux  $AB(t)$ , we have

$$i(R_a + R_b) = - \frac{d}{dt} [AB(t)]. \tag{B.1.45}$$

The paths of integration  $C_a$  and  $C_b$  do not enclose a magnetic flux; hence for

these paths (B.1.41) gives

$$v_a = -iR_a = \frac{R_a}{R_a + R_b} \frac{d}{dt} [AB(t)] \quad \text{for } C_a, \quad (\text{B.1.46})$$

$$v_b = iR_b = \frac{-R_b}{R_a + R_b} \frac{d}{dt} [AB(t)] \quad \text{for } C_b, \quad (\text{B.1.47})$$

where the current  $i$  is evaluated by using (B.1.45). The most obvious attribute of this result is that although the voltmeters are connected to the same nodes they do not indicate the same values. In the presence of the magnetic induction the contour of the voltmeter leads plays a role in determining the voltage indicated.

The situation shown in Fig. B.1.6 can be thought of as a transformer with a single turn secondary. With this in mind, it is clear that Faraday's law plays an essential role in electrical technology.

The divergence of an arbitrary vector  $\nabla \times \mathbf{A}$  is zero. Hence the divergence of (B.1.42) shows that the divergence of  $\mathbf{B}$  is constant. This fact also follows from (B.1.28), from which it can be shown that this constant is zero. Hence an additional differential equation for  $\mathbf{B}$  is

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{B.1.48})$$

Integration of this expression over an arbitrary volume  $V$  and use of the divergence theorem (B.1.14) gives

$$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0. \quad (\text{B.1.49})$$

This integral law makes more apparent the fact that there can be no net magnetic flux emanating from a given region of space.

## B.2 MAXWELL'S EQUATIONS

The generality and far-reaching applications of the laws of electricity and magnetism are not immediately obvious; for example, the law of induction given by (B.1.42) was recognized by Faraday as true when applied to a conducting circuit. The fact that (B.1.42) has significance even in regions of space unoccupied by matter is a generalization that is crucial to the theory of electricity and magnetism. We can summarize the differential laws introduced in Section B.1 as

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e, \quad (\text{B.2.1})$$

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0, \quad (\text{B.2.2})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}, \quad (\text{B.2.3})$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{B.2.4})$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{B.2.5})$$

Taken together, these laws are called *Maxwell's equations* in honor of the man who was instrumental in recognizing that they have a more general significance than any one of the experiments from which they originate. For example, we can think of a time-varying magnetic flux that induces an electric field according to (B.2.4) even in the absence of a material circuit. Similarly, (B.2.3) is taken to mean that even in regions of space in which there is no circuit, hence  $\mathbf{J} = 0$ , a time-varying electric field leads to an induced magnetic flux density  $\mathbf{B}$ .

The coupling between time-varying electric and magnetic fields, as predicted by (B.2.1 to B.2.5), accounts for the existence of electromagnetic waves, whether they be radio or light waves or even gamma rays. As we might guess from the electromechanical origins of electromagnetic theory, the propagation of electromagnetic waves is of secondary importance in the study of most electromechanical phenomena. This does not mean that electromechanical interactions are confined to frequencies that are low compared with radio frequencies. Indeed, electromechanical interactions of practical significance extend into the gigahertz range of frequencies.

To take a mature approach to the study of electromechanics it is necessary that we discriminate at the outset between essential and nonessential aspects of interactions between fields and media. This makes it possible to embark immediately on a study of nontrivial interactions. An essential purpose of this section is the motivation of approximations used in this book.

Although electromagnetic waves usually represent an unimportant consideration in electromechanics and are not discussed here in depth, they are important to an understanding of the quasi-static approximations that are introduced in Section B.2.2. Hence we begin with a brief simplified discussion of electromagnetic waves.

### B.2.1 Electromagnetic Waves

Consider fields predicted by (B.2.3) and (B.2.4) in a region of free space in which  $\mathbf{J} = 0$ . In particular, we confine our interest to situations in which the fields depend only on  $(x, t)$  (the fields are one-dimensional) and write the  $y$ -component of (B.2.3) and the  $z$ -component of (B.2.4)

$$-\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}, \quad (\text{B.2.6})$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \quad (\text{B.2.7})$$

This pair of equations, which make evident the coupling between the dynamic electric and magnetic fields, is sufficient to determine the field components  $B_z$  and  $E_y$ . In fact, if we take the time derivative of (B.2.6) and use the resulting

expression to eliminate  $B_z$  from the derivative with respect to  $x$  of (B.2.7), we obtain

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}, \quad (\text{B.2.8})$$

where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ (m/sec).}$$

This equation for  $E_y$  is called the *wave equation* because it has solutions in the form of

$$E_y(x, t) = E_+(x - ct) + E_-(x + ct). \quad (\text{B.2.9})$$

That this is true may be verified by substituting (B.2.9) into (B.2.8). Hence solutions for  $E_y$  can be analyzed into components  $E_+$  and  $E_-$  that represent waves traveling, respectively, in the  $+x$ - and  $-x$ -directions with the *velocity of light*  $c$ , given by (B.2.8). The prediction of electromagnetic wave propagation is a salient feature of Maxwell's equations. It results, as is evident from the derivation, because time-varying magnetic fields can induce electric fields [Faraday's law, (B.2.7)] while at the same time dynamic electric fields induce magnetic fields [Ampère's law with the displacement current included (B.2.6)]. It is also evident from the derivation that if we break this two-way coupling by leaving out the displacement current *or* omitting the magnetic induction term electromagnetic waves are not predicted.

Electromechanical interactions are usually not appreciably affected by the propagational character of electromagnetic fields because the velocity of propagation  $c$  is very large. Suppose that we are concerned with a system whose largest dimension is  $l$ . The time  $l/c$  required for the propagation of a wave between extremes of the system is usually short compared with characteristic dynamical times of interest; for example, in a device in which  $l = 0.3$  m the time  $l/c$  equals  $10^{-9}$  sec. If we were concerned with electromechanical motions with a time constant of a microsecond (which is extremely short for a device characterized by 30 cm), it would be reasonable to ignore the wave propagation. In the absence of other dynamic effects this could be done by assuming that the fields were established everywhere within the device instantaneously.

Even though it is clear that the propagation of electromagnetic waves has nothing to do with the dynamics of interest, it is not obvious how to go about simplifying Maxwell's equations to remove this feature of the dynamics. A pair of particular examples will help to clarify approximations made in the next section. These examples, which are considered simultaneously so that they can be placed in contrast, are shown in Fig. B.2.1.



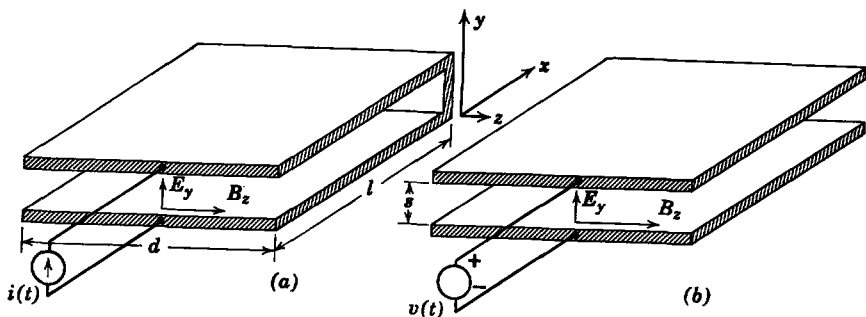


Fig. B.2.1 Perfectly conducting plane-parallel electrodes driven at  $x = -l$ : (a)  $i(t) = i_0 \cos \omega t$ ; (b)  $v(t) = v_0 \cos \omega t$ .

A pair of perfectly conducting parallel plates has the spacing  $s$  which is much smaller than the  $x$ - $z$  dimensions  $l$  and  $d$ . The plates are excited at  $x = -l$  by

a current source  
 $i(t) = i_0 \cos \omega t$  (amperes). (B.2.10a)

a voltage source  
 $v(t) = v_0 \cos \omega t$  (volts). (B.2.10b)

At  $x = 0$ , the plates are terminated in  
 a perfectly conducting short circuit plate.

an open circuit.

If we assume that the spacing  $s$  is small enough to warrant ignoring the effects of fringing and that the driving sources at  $x = -l$  are distributed along the  $z$ -axis, the one-dimensional fields  $B_z$  and  $E_y$  predicted by (B.2.6) and (B.2.7) represent the fields between the plates. Hence we can think of the current and voltage sources as exciting electromagnetic waves that propagate along the  $x$ -axis between the plates. The driving sources impose conditions on the fields at  $x = -l$ . They are obtained by

integrating (B.1.33) around the contour  $C$  (Fig. B.2.2a) which encloses the upper plate adjacent to the current source. (The surface  $S$  enclosed by  $C$  is very thin so that negligible displacement current links the loop).

integrating the electric field between (a) and (b) in Fig. B.2.2b to relate the potential difference of the voltage source to the electric field intensity  $E_y(-l, t)$ .

$$B_z(-l, t) = -\mu_0 K = -\frac{\mu_0 i(t)}{d} \quad \text{(B.2.11a)}$$

$$\int_s^0 E_y dy = -s E_y(-l, t) = v(t). \quad \text{(B.2.11b)}$$

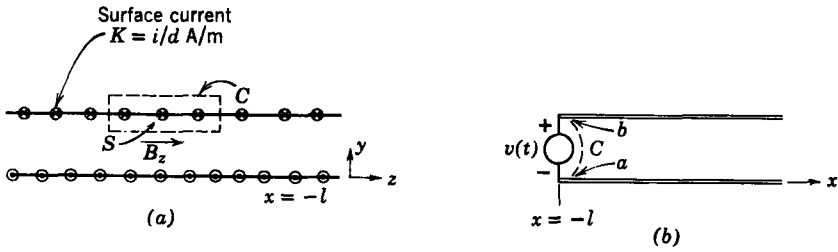


Fig. B.2.2 Boundary conditions for the systems in Fig. B.2.1

Similar conditions used at  $x = 0$  give the boundary conditions

$$E_y(0, t) = 0 \qquad (B.2.12a) \quad | \quad B_z(0, t) = 0 \qquad (B.2.12b)$$

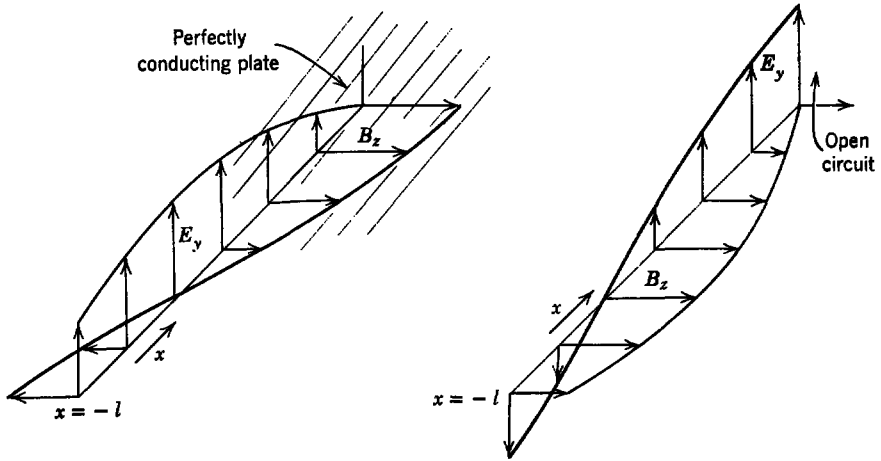
It is not our purpose in this chapter to become involved with the formalism of solving the wave equation [or (B.2.6) and (B.2.7)] subject to the boundary conditions given by (B.2.11) and (B.2.12). There is ample opportunity to solve boundary value problems for electromechanical systems in the text, and the particular problem at hand forms a topic within the context of transmission lines and waveguides. For our present purposes, it suffices to guess solutions to these equations that will satisfy the appropriate boundary conditions. Then direct substitution into the differential equations will show that we have made the right choice.

$$E_y = -i_o \frac{\sin \omega t \sin (\omega x/c)}{d \epsilon_0 c \cos (\omega l/c)}, \qquad E_y = - \frac{v_o \cos \omega t \cos (\omega x/c)}{s \cos (\omega l/c)}, \qquad (B.2.13a) \qquad (B.2.13b)$$

$$B_z = - \frac{\mu_0 i_o \cos \omega t \cos (\omega x/c)}{d \cos (\omega l/c)}, \qquad B_z = - \frac{v_o \sin \omega t \sin (\omega x/c)}{c s \cos (\omega l/c)} \qquad (B.2.14a) \qquad (B.2.14b)$$

Note that at  $x = -l$  the boundary conditions B.2.11 are satisfied, whereas at  $x = 0$  the conditions of (B.2.12) are met. One way to show that Maxwell's equations are satisfied also (aside from direct substitution) is to use trigonometric identities\* to rewrite these standing wave solutions as the superposition of two traveling waves in the form of (B.2.9). Our solutions are sinusoidal, steady-state solutions, so that with the understanding that the amplitude of the field at any point along the  $x$ -axis is varying sinusoidally with time we can obtain an impression of the dynamics by plotting the instantaneous amplitudes, as shown in Fig. B.2.3. In general, the fields have the sinusoidal distribution along the  $x$ -axis of a standing wave. From (B.2.13 to B.2.14) it

\* For example in (B.2.13a)  $\sin \omega t \sin (\omega x/c) \equiv \frac{1}{2} \{ \cos [\omega(t - x/c)] - \cos [\omega(t + x/c)] \}$ .



**Fig. B.2.3** Amplitude of the electric field intensity and magnetic flux density along the  $x$ -axis of the parallel-plate structures shown in Fig. B.2.1 For these plots  $\omega l/c = 3\pi/4$ .

is clear that as a function of time the electric field reaches its maximum amplitude when  $B_z = 0$  and vice versa. Hence the amplitudes of  $E_y$  and  $B_z$  shown in Fig. B.2.3 are for different instants of time. The fields near  $x = 0$  do not in general have the same phase as those excited at  $x = -l$ . If, however, we can make the approximation that times of interest (which in this case are  $1/\omega$ ) are much longer than the propagation time  $l/c$ ,

$$\frac{l/c}{1/\omega} = \frac{\omega l}{c} \ll 1. \tag{B.2.15}$$

The sine functions can then be approximated by their arguments (which are small compared with unity) and the cosine functions are essentially equal to unity. Hence, when (B.2.15) is satisfied, the field distributions (B.2.13) and (B.2.14) become

$$E_y \simeq -\frac{i_0 \sin \omega t}{d \epsilon_0 c} \left( \frac{\omega x}{c} \right), \tag{B.2.16a}$$

$$E_y \simeq -\frac{v_0}{s} \cos \omega t, \tag{B.2.16b}$$

$$B_z \simeq -\frac{\mu_0 i_0 \cos \omega t}{d}, \tag{B.2.17a}$$

$$B_z \simeq -\frac{v_0}{cs} \sin \omega t \left( \frac{\omega x}{c} \right). \tag{B.2.17b}$$

The distribution of field amplitudes in this limit is shown in Fig. B.2.4. The most significant feature of the limiting solutions is that

the magnetic field between the short-circuited plates has the same distribution as if the excitation current were static.

the electric field between the open-circuited plates has the same distribution as if the excitation voltage were constant.

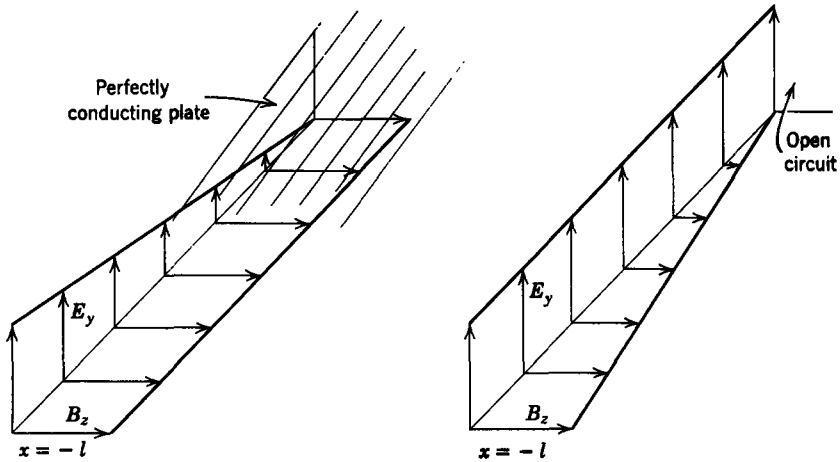


Fig. B.2.4 The distribution of field amplitudes between the parallel plates of Fig. B.2.1 in the limit in which  $(\omega l/c) \ll 1$ .

Note that the fields as they are excited at  $x = -l$  retain the same phase everywhere between the plates. This simply reflects the fact that according to the approximate equations there is no time lag between an excitation at  $x = -l$  and the field response elsewhere along the  $x$ -axis. It is in this limit that the ideas of circuit theory are applicable, for if we now compute

the voltage  $v(t)$  at  $x = -l$

$$v(t) = -sE_y(-l, t) \quad (B.2.18a)$$

we obtain the terminal equation for an inductance

$$v = L \frac{d}{dt} (i_o \cos \omega t), \quad (B.2.19a)$$

where the inductance  $L$  is

$$L = \frac{s\mu_0}{d}.$$

the current  $i(t)$  at  $x = -l$

$$i(t) = -B_z(-l, t) \frac{d}{\mu_0} \quad (B.2.18b)$$

we obtain the terminal equation for a capacitance

$$i(t) = C \frac{d}{dt} (v_o \cos \omega t), \quad (B.2.19b)$$

where the capacitance  $C$  is

$$C = \frac{\epsilon_0 d l}{s}.$$

A comparison of the examples will be useful for motivating many of the somewhat subtle ideas introduced in the main body of the book. One of the most important points that we can make here is that even though we have solved the same pair of Maxwell's equations (B.2.6) and (B.2.7) for both examples, subject to the same approximation that  $\omega l/c \ll 1$  (B.2.15), we have been led to very different physical results. The difference between these

two examples arises from the boundary condition at  $x = 0$ . In the case of

a short circuit a static excitation leads to a uniform magnetic field but no electric field. The electric field is generated by Faraday's law because the magnetic field is in fact *only quasi-static* and varies slowly with time.

an open circuit a static excitation results in a uniform electric field but no magnetic field. The magnetic field is induced by the displacement current in Ampère's law because the electric field is, in fact, *only quasi-static* and varies slowly with time.

### B.2.2 Quasi-Static Electromagnetic Field Equations

As long as we are not interested in phenomena related to the propagation of electromagnetic waves, it is helpful to recognize that most electromechanical situations are in one of two classes, exemplified by the two cases shown in Fig. B.2.1. In the situation in which the plates are short-circuited together (Fig. B.2.1a) the limit  $\omega/c \ll 1$  means that the *displacement current* is of *negligible* importance. A characteristic of this system is that with a static excitation a large current results; hence there is a large static magnetic field. For this reason it exemplifies a *magnetic field system*. By contrast, in the case in which the plates are open-circuited, as shown in Fig. B.2.1b, a static excitation gives rise to a static electric field but no magnetic field. This example exemplifies an *electric field system*, in which the *magnetic induction* of Faraday's law is of *negligible* importance. To emphasize these points consider how we can use these approximations at the outset to obtain the approximate solutions of (B.2.19). Suppose that the excitations in Fig. B.2.1 were static. The fields between the plates are then independent of  $x$  and given by

$$E_y = 0, \quad (\text{B.2.20a}) \quad \left| \quad E_y = -\frac{v}{s}, \quad (\text{B.2.20b})\right.$$

$$B_z = -\frac{\mu_0 i}{d}, \quad (\text{B.2.21a}) \quad \left| \quad B_z = 0. \quad (\text{B.2.21b})\right.$$

Now suppose that the fields vary slowly with time [the systems are quasi-static in the sense of a condition like (B.2.15)]. Then  $i$  and  $v$  in these equations are time-varying, hence

<p><math>B_z</math> is a function of time. From Faraday's law of induction as expressed by (B.2.7)</p> $\frac{\partial E_y}{\partial x} = \frac{\mu_0}{d} \frac{di}{dt}. \quad (\text{B.2.22a})$		<p><math>E_y</math> is a function of time. From Ampère's law, as expressed by (B.2.6)</p> $\frac{\partial B_z}{\partial x} = \frac{\mu_0 \epsilon_0}{s} \frac{dv}{dt}. \quad (\text{B.2.22b})$
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Now the right-hand side of each of these equations is independent of  $x$ ; hence they can be integrated on  $x$ . At the same time, we recognize that

$$E_y(0, t) = 0, \quad (\text{B.2.23a}) \quad \left| \quad B_z(0, t) = 0, \quad (\text{B.2.23b})\right.$$

so that integration gives

$$E_y = \frac{\mu_0 x}{d} \frac{di}{dt}, \quad (\text{B.2.24a}) \quad \left| \quad B_z = \frac{\mu_0 \epsilon_0 x}{s} \frac{dv}{dt}. \quad (\text{B.2.24b})\right.$$

Recall how the terminal voltage and current are related to these field quantities (B.2.18) and these equations become

$$v(t) = L \frac{di}{dt}, \quad (\text{B.2.25a}) \quad \left| \quad i(t) = C \frac{dv}{dt}, \quad (\text{B.2.25b})\right.$$

where again the inductance  $L$  and capacitance  $C$  are defined as following (B.2.19). Hence making these approximations at the outset has led to the same approximate results as those found in the preceding section by computing the exact solution and taking the limits appropriate to  $\omega l/c \ll 1$ .

The simple example in Fig. B.2.1 makes it plausible that Maxwell's equations can be written in two quasi-static limits appropriate to the analysis of two major classes of electromechanical interaction:

Magnetic Field Systems	Electric Field Systems
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (\text{B.2.26a})$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (\text{B.2.26b})$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (\text{B.2.27a})$	$\nabla \times \mathbf{E} = 0, \quad (\text{B.2.27b})$
$\nabla \cdot \mathbf{B} = 0, \quad (\text{B.2.28a})$	$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e, \quad (\text{B.2.28b})$
$\nabla \cdot \mathbf{J} = 0, \quad (\text{B.2.29a})$	$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \quad (\text{B.2.29b})$

Here the displacement current has been omitted from Ampère's law in the magnetic field system, whereas the magnetic induction has been dropped from Faraday's law in the electric field system. Note that if the displacement current is dropped from (B.2.26a) the charge density must be omitted from the conservation of charge equation (B.2.29a) because the latter expression is the divergence of (B.2.26a).

We have not included Gauss's law for the charge density in the magnetic field system or the divergence equation for  $\mathbf{B}$  in the electric field system because in the respective situations these expressions are of no interest. In fact, only the divergence of (B.2.26b) is of interest in determining the dynamics of most electric field systems and that is (B.2.29b).

It must be emphasized that the examples of Fig. B.2.1 serve only to motivate the approximations introduced by (B.2.26 to B.2.29). The two systems of equations have a wide range of application. The recognition that a given physical situation can be described as a magnetic field system, as opposed to an electric field system, requires judgment based on experience. A major intent of this book is to establish that kind of experience.

In the cases of Fig. B.2.1 we could establish the accuracy of the approximate equations by calculating effects induced by the omitted terms; for example, in the magnetic field system of Fig. B.2.1*a* we ignored the displacement current to obtain the quasi-static solution of (B.2.21*a*) and (B.2.24*a*). We could now compute the correction  $B_z^c$  to the quasi-static magnetic field induced by the displacement current by using (B.2.6), with  $E$  given by (B.2.24*a*). This produces

$$\frac{\partial B_z^c}{\partial x} = -\frac{\mu_0^2 \epsilon_0 x}{d} \frac{d^2 i}{dt^2}. \quad (\text{B.2.30})$$

Because the right-hand side of this expression is a known function of  $x$ , it can be integrated. The constant of integration is evaluated by recognizing that the quasi-static solution satisfies the driving condition at  $x = -l$ ; hence the correction field  $B_z^c$  must be zero there and

$$B_z^c = -\frac{\mu_0^2 \epsilon_0 (x^2 - l^2)}{2d} \frac{d^2 i}{dt^2}. \quad (\text{B.2.31})$$

Now, to determine the error incurred in ignoring this field we take the ratio of its largest value (at  $x = 0$ ) to the quasi-static field of (B.2.21*a*):

$$\frac{|B_z^c|}{|B_z|} = \frac{l^2}{2c^2} \frac{|d^2 i/dt^2|}{|i|}. \quad (\text{B.2.32})$$

If this ratio is small compared with 1, the quasi-static solution is adequate. It is evident that in this case the ratio depends on the time rate of change of the excitation. In Section B.2.1, in which  $i = i_0 \cos \omega t$ , (B.2.32) becomes

$$\frac{|B_z^c|}{|B_z|} = \frac{1}{2} \left( \frac{\omega l}{c} \right)^2 \ll 1, \quad (\text{B.2.33})$$

which is essentially the same condition given by (B.2.15).

Once the fields have been determined by using either the magnetic field or the electric field representation it is possible to calculate the effects of the omitted terms. This procedure results in a condition characterized by (B.2.33). For this example, if the device were 30 cm long and driven at 1 MHz (this

is an extremely high frequency for anything 30 cm long to respond to electro-mechanically) (B.2.33) becomes

$$\frac{1}{2} \left( \frac{\omega l}{c} \right)^2 = \frac{1}{2} \left( \frac{2 \cdot \pi \cdot 10^6 \cdot 0.3}{3 \times 10^8} \right)^2 = 2\pi^2 \times 10^{-6} \ll 1, \quad (\text{B.2.34})$$

and the quasi-static approximation is extremely good.

It is significant that the magnetic and electric field systems can be thought of in terms of their respective modes of electromagnetic energy storage. In the quasi-static systems the energy that can be attributed to the electromagnetic fields is stored either in the magnetic or electric field. This can be seen by using (B.2.26 to B.2.27) to derive Poynting's theorem for the conservation of electromagnetic energy. If the equations in (B.2.27) are multiplied by  $\mathbf{B}/\mu_0$  and subtracted from the equations in (B.2.26) multiplied by  $\mathbf{E}/\mu_0$ , it follows that

$$\frac{\mathbf{E}}{\mu_0} \cdot \nabla \times \mathbf{B} - \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} = \mathbf{E} \cdot \mathbf{J} + \frac{\mathbf{B}}{\mu_0} \cdot \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.2.35a})$$

$$\frac{\mathbf{E}}{\mu_0} \cdot \nabla \times \mathbf{B} - \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} = \mathbf{E} \cdot \mathbf{J} + \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \quad (\text{B.2.35b})$$

Then, because of a vector identity,\* these equations take the form

$$-\nabla \cdot \left( \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) = \mathbf{E} \cdot \mathbf{J} + \frac{\partial}{\partial t} \left( \frac{1}{2} \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0} \right). \quad (\text{B.2.36a})$$

$$-\nabla \cdot \left( \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) = \mathbf{E} \cdot \mathbf{J} + \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E} \right). \quad (\text{B.2.36b})$$

Now, if we integrate these equations over a volume  $V$  enclosed by a surface  $S$ , the divergence theorem (B.1.14) gives

$$-\oint_S \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \cdot \mathbf{n} \, da = \int_V \mathbf{E} \cdot \mathbf{J} \, dV + \frac{\partial}{\partial t} \int_V w \, dV, \quad (\text{B.2.37})$$

where

$$w = \frac{1}{2} \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0}. \quad (\text{B.2.38a})$$

$$w = \frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E}. \quad (\text{B.2.38b})$$

The term on the left in (B.2.37) (including the minus sign) can be interpreted as the flux of energy into the volume  $V$  through the surface  $S$ . This energy is either dissipated within the volume  $V$ , as expressed by the first term on the right, or stored in the volume  $V$ , as expressed by the second term. Hence

\*  $\nabla \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{C}$ .



( $w$ ) can be interpreted as an electromagnetic energy density. The electromagnetic energy of the magnetic field system is stored in the magnetic field alone. Similarly, an electric field system is one in which the electromagnetic energy is stored in the electric field.

The familiar elements of electrical circuit theory illustrate the division of interactions into those defined as magnetic field systems and those defined as electric field systems. From the discussion in this and the preceding section it is evident that the short-circuited plates in Fig. B.2.1 constitute an inductor, whereas the open-circuited plates can be represented as a capacitor. This fact is the basis for the development of electromechanical interactions undertaken in Chapter 2. From this specific example it is evident that the magnetic field system includes interactions in which we can define lumped-parameter variables like the inductance, but it is not so evident that this model also describes the magnetohydrodynamic interactions of a fluid and some plasmas with a magnetic field and the magnetoelastic interactions of solids in a magnetic field, even including electromechanical aspects of microwave magnetics.

Similarly, the electric field system includes not only the electromechanics of systems that can be modeled in terms of circuit concepts like the capacitance but ferroelectric interactions between solids and electric fields, the electrohydrodynamics of a variety of liquids and slightly ionized gases in an electric field, and even the most important oscillations of an electron beam. Of course, if we are interested in the propagation of an electromagnetic wave through an ionospheric plasma or through the slightly ionized wake of a space vehicle, the full set of Maxwell's equations must be used.

There are situations in which the propagational aspects of the electromagnetic fields are not of interest, yet neither of the quasi-static systems is appropriate. This is illustrated by short-circuiting the parallel plates of Fig. B.2.1 at  $x = 0$  by a resistive sheet. A static current or voltage applied to the plates at  $x = -l$  then leads to both electric and magnetic fields between the plates. If the resistance of the sheet is small, the electric field between the plates is also small, and use of the exact field equations would show that we are still justified in ignoring the displacement current. In this case the inductance of Fig. B.2.1a is in series with a resistance. In the opposite extreme, if the resistance of the resistive sheet were very high, we would still be justified in ignoring the magnetic induction of Faraday's law. The situation shown in Fig. B.2.1b would then be modeled by a capacitance shunted by a resistance. The obvious questions are, when do we make a transition from the first case to the second and why is not this intermediate case of more interest in electromechanics?

The purpose of practical electromechanical systems is either the conversion of an electromagnetic excitation into a force that can perform work on a

mechanical system or the reciprocal generation of electromagnetic energy from a force of mechanical origin. From (B.1.10) and (B.1.40) there are two fundamental types of electromagnetic force. Suppose that we are interested in producing a force of electrical origin on the upper of the two plates in Fig. B.2.1. We have the option of imposing a large current to interact with its induced magnetic field or of using a large potential to create an electric field that would interact with induced charges on the upper plate. Clearly, we are not going to impose a large potential on the plates if they are terminated in a small resistance or attempt to drive a large current through the plates with an essentially open circuit at  $x = 0$ . The electrical dissipation in both cases would be prohibitively large. More likely, if we intended to use the force  $\mathbf{J} \times \mathbf{B}$ , we would make the resistance as small as possible to minimize the dissipation of electric power and approach the case of Fig. B.2.1a. The essentially open circuit shown in Fig. B.2.1b would make it possible to use a large potential to create a significant force of the type  $\rho_e \mathbf{E}$  without undue power dissipation. In the intermediate case the terminating resistance could be adjusted to make the electric and magnetic forces about equal. As a practical matter, however, the resulting device would probably melt before it served any useful electromechanical function. The power dissipated in the termination resistance would be a significant fraction of any electric power converted to mechanical form.\*

The energy densities of (B.2.38) provide one means of determining when the problem shown in Fig. B.2.1 (but with a resistive sheet terminating the plates at  $x = 0$ ) is intermediate between a magnetic and an electric field system. In the intermediate case the energy densities are equal

$$\frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E} = \frac{1}{2} \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0}. \quad (\text{B.2.39})$$

Now, if the resistive sheet has a total resistance of  $R$ , then from (B.2.18a) applied at  $x = 0$

$$E_y s = -iR. \quad (\text{B.2.40})$$

The current can be evaluated in terms of the magnetic field at  $x = 0$  by using (B.2.18b):

$$E_y s = B_z \frac{dR}{\mu_0}. \quad (\text{B.2.41})$$

Substitution of the electric field, as found from this expression into (B.2.39), gives

$$\frac{\epsilon_0}{2} B_z^2 \left( \frac{Rd}{s\mu_0} \right)^2 = \frac{1}{2} \frac{B_z^2}{\mu_0}. \quad (\text{B.2.42})$$

\* It is interesting that for this particular intermediate case the electric force tends to pull the plates together, whereas the magnetic force tends to push them apart. Hence, because the two forces are equal in magnitude, they just cancel.

Hence, if the energy densities are equal, we obtain the following relation among the physical parameters of the system:

$$\frac{dR}{s} = \left( \frac{\mu_0}{\epsilon_0} \right)^{1/2}. \quad (\text{B.2.43})$$

It would be a digression to pursue this point here, but (B.2.43) is the condition that must be satisfied if an electromagnetic wave launched between the plates at  $x = -l$  is to be absorbed, without reflection, by the resistive sheet\*; that is, the intermediate case is one in which all the power fed into the system, regardless of the frequency or time constant, is dissipated by the resistive sheet.

### B.3 MACROSCOPIC MODELS AND CONSTITUENT RELATIONS

When solids, liquids, and gases are placed in electromagnetic fields, they influence the field distribution. This is another way of saying that the force of interaction between charges or between currents is influenced by the presence of media. The effect is not surprising because the materials are comprised of charged particles.

Problems of physical significance can usually be decomposed into parts with widely differing scales. At the molecular or submolecular level we may be concerned with the dynamics of individual charges or of the atoms or molecules to which they are attached. These systems tend to have extremely small dimensions when compared with the size of a physical device. On the macroscopic scale we are not interested in the detailed behavior of the microscopic constituents of a material but rather only a knowledge of the average behavior of variables, since only these averages are observable on a macroscopic scale. The charge and current densities introduced in Section B.1 are examples of such variables, hence it is a macroscopic picture of fields and media that we require here.

There are three major ways in which media influence macroscopic electromagnetic fields. Hence the following sections undertake a review of magnetization, polarization, and conduction in common materials.

#### B.3.1 Magnetization

The macroscopic motions of electrons, even though associated with individual atoms or molecules, account for aggregates of charge and current

\* The propagation of an electromagnetic wave on structures of this type is discussed in texts concerned with transmission lines or TEM wave guide modes. For a discussion of this matching problem see R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, p. 111, or S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, p. 27.

(when viewed at the macroscopic level) that induce electric and magnetic fields. These field sources are not directly accessible; for example, the equivalent currents within the material cannot be circulated through an external circuit. The most obvious sources of magnetic field that are inaccessible in this sense are those responsible for the field of a permanent magnet. The earliest observations on magnetic fields involved the lodestone, a primitive form of the permanent magnet. Early investigators such as Oersted found that magnetic fields produced by a permanent magnet are equivalent to those induced by a circulating current. In the formulation of electromagnetic theory we must distinguish between fields due to sources within the material and those from applied currents simply because it is only the latter sources that can be controlled directly. Hence we divide the source currents into *free currents* (with the density  $\mathbf{J}_f$ ) and *magnetization currents* (with the density  $\mathbf{J}_m$ ). Ampère's law then takes the form

$$\nabla \times \left( \frac{\mathbf{B}}{\mu_0} \right) = \mathbf{J}_m + \mathbf{J}_f. \quad (\text{B.3.1})$$

By convention it is also helpful to attribute a fraction of the field induced by these currents to the magnetization currents in the material. Hence (B.3.1) is written as

$$\nabla \times \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J}_f, \quad (\text{B.3.2})$$

where the *magnetization density*  $\mathbf{M}$  is defined by

$$\nabla \times \mathbf{M} = \mathbf{J}_m. \quad (\text{B.3.3})$$

Up to this point in this chapter it has been necessary to introduce only two field quantities to account for interactions between charges and between currents. To account for the macroscopic properties of media we have now introduced a new field quantity, the magnetization density  $\mathbf{M}$ , and in the next section similar considerations concerning electric polarization of media lead to the introduction of the polarization density  $\mathbf{P}$ . It is therefore apparent that macroscopic field theory is formulated in terms of four field variables. In our discussion these variables have been  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{M}$ , and  $\mathbf{P}$ . An alternative representation of the fields introduces the *magnetic field intensity*  $\mathbf{H}$ , in our development *defined as*

$$\mathbf{H} = \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right). \quad (\text{B.3.4})$$

From our definition it is clear that we could just as well deal with  $\mathbf{B}$  and  $\mathbf{H}$  as the macroscopic magnetic field vectors rather than with  $\mathbf{B}$  and  $\mathbf{M}$ . This is

particularly appealing, for then (B.3.2) takes the simple form

$$\nabla \times \mathbf{H} = \mathbf{J}_f. \quad (\text{B.3.5})$$

When the source quantities  $\mathbf{J}_f$  and  $\mathbf{M}$  are specified independently, the magnetic field intensity  $\mathbf{H}$  (or magnetic flux density  $\mathbf{B}$ ) can be found from the quasi-static magnetic field equations. A given constant magnetization density corresponds to the case of the permanent magnet. In most cases, however, the source quantities are functions of the field vectors, and these functional relations, called *constituent relations*, must be known before the problems can be solved. The constituent relations represent the constraints placed on the fields by the internal physics of the media being considered. Hence it is these relations that make it possible to separate the microscopic problem from the macroscopic one of interest here.

The simplest form of constituent relation for a magnetic material arises when it can be considered *electrically linear* and *isotropic*. Then the *permeability*  $\mu$  is constant in the relation

$$\mathbf{B} = \mu \mathbf{H}. \quad (\text{B.3.6})$$

The material is isotropic because  $\mathbf{B}$  is collinear with  $\mathbf{H}$  and a particular constant ( $\mu$ ) times  $\mathbf{H}$ , regardless of the direction of  $\mathbf{H}$ . A material that is *homogeneous* and isotropic will in addition have a permeability  $\mu$  that does not vary with position in the material. Another way of expressing (B.3.6) is to define a magnetic susceptibility  $\chi_m$  (dimensionless) such that

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (\text{B.3.7})$$

where

$$\mu = \mu_0(1 + \chi_m). \quad (\text{B.3.8})$$

Magnetic materials are commonly found with  $\mathbf{B}$  not a linear function of  $\mathbf{H}$  and the constitutive law takes the general form

$$\mathbf{B} = \mathbf{B}(\mathbf{H}). \quad (\text{B.3.9})$$

We deal with some problems involving materials of this type, but with few exceptions confine our examples to situations in which  $\mathbf{B}$  is a single-valued function of  $\mathbf{H}$ . In certain magnetic materials in some applications the  $\mathbf{B}$ - $\mathbf{H}$  curve must include hysteresis and (B.3.9) is not single-valued.\*

The differential equations for a magnetic field system in the presence of moving magnetized media are summarized in Table 1.2.

### B.3.2 Polarization

The force between a charge distribution and a test charge is observed to change if a dielectric material is brought near the region occupied by the test

\* G. R. Slemon, *Magnetolectric Devices*, Wiley, New York, 1966, p. 115.

charge. Like the test charge, the charged particles which compose the dielectric material experience forces due to the applied field. Although these charges remain identified with the molecules of the material, their positions can be distorted incrementally by the electric force and thus lead to a polarization of the molecules.

The basic sources of the electric field are charges. Hence it is natural to define a *polarization charge density*  $\rho_p$  as a source of a fraction of the electric field which can be attributed to the inaccessible sources within the media. Thus Gauss's law (B.1.16) is written

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_f + \rho_p, \quad (\text{B.3.10})$$

where the *free charge density*  $\rho_f$  resides on conducting electrodes and other parts of the system capable of supporting conduction currents. The free charges do not remain attached to individual molecules but rather can be conducted from one point to another in the system.

In view of the form taken by Gauss's law, it is convenient to identify a field induced by the polarization charges by writing (B.3.10) as

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f, \quad (\text{B.3.11})$$

where the *polarization density*  $\mathbf{P}$  is related to the polarization charge density by

$$\rho_p = -\nabla \cdot \mathbf{P}. \quad (\text{B.3.12})$$

As in Section B.3.1, it is convenient to define a new vector field that serves as an alternative to  $\mathbf{P}$  in formulating the electrodynamics of polarized media. This is the *electric displacement*  $\mathbf{D}$ , defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (\text{B.3.13})$$

In terms of this field, Gauss's law for electric fields (B.3.11) becomes

$$\nabla \cdot \mathbf{D} = \rho_f. \quad (\text{B.3.14})$$

The simple form of this expression makes it desirable to use  $\mathbf{D}$  rather than  $\mathbf{P}$  in the formulation of problems.

If a polarization charge model is to be used to account for the effects of polarizable media on electric fields, we must recognize that the motion of these charges can lead to a current. In fact, now that two classes of charge density have been identified we must distinguish between two classes of current density. The free current density  $\mathbf{J}_f$  accounts for the conservation of free charge so that (B.1.18) can be written as

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0. \quad (\text{B.3.15})$$

In view of (B.3.11), this expression becomes

$$\nabla \cdot \mathbf{J}_f + \frac{\partial}{\partial t} \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = 0. \quad (\text{B.3.16})$$

Now, if we write Ampère's law (B.2.26*b*) as

$$\nabla \times \left( \frac{\mathbf{B}}{\mu_0} \right) = \mathbf{J}_f + \mathbf{J}_p + \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}, \quad (\text{B.3.17})$$

where  $\mathbf{J}_p$  is a current density due to the motion of polarization charges, the divergence of (B.3.17) must give (B.3.16). Therefore

$$\nabla \cdot \mathbf{J}_p + \frac{\partial}{\partial t} (-\nabla \cdot \mathbf{P}) = 0. \quad (\text{B.3.18})$$

which from (B.3.12) is an expression for the conservation of polarization charge. This expression does not fully determine the polarization current density  $\mathbf{J}_p$ , because in general we could write

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{A}, \quad (\text{B.3.19})$$

where  $\mathbf{A}$  is an arbitrary vector, and still satisfy (B.3.18). At this point we could derive the quantity  $\mathbf{A}$  (which would turn out to be  $\mathbf{P} \times \mathbf{v}$ , where  $\mathbf{v}$  is the velocity of the polarized medium). It is important, however, to recognize that this represents an unnecessary digression. In the electric field system the magnetic field appears in only one of the equations of motion—Ampère's law. It does not appear in (B.2.27*b*) to (B.2.29*b*), nor will it appear in any constitutive law used in this book. For this reason the magnetic field serves simply as a quantity to be calculated once the electromechanical problem has been solved. We might just as well lump the quantity  $\mathbf{A}$  with the magnetic field in writing Ampère's law. In fact, if we are consistent, the magnetic field intensity  $\mathbf{H}$  can be defined as given by

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (\text{B.3.20})$$

with no loss of physical significance. In an electric field system the magnetic field is an alternative representation of the current density  $\mathbf{J}_f$ . A review of the quasi-static solutions for the system in Fig. B.2.1*b* illustrates this point.

In some materials (ferroelectrics) the polarization density  $\mathbf{P}$  is constant. In most common dielectrics, however, the polarization density is a function of  $\mathbf{E}$ . The simplest constituent relation for a dielectric is that of linear and isotropic material,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad (\text{B.3.21})$$

where  $\chi_e$  is the *dielectric susceptibility* (dimensionless) that may be a function of space but not of  $\mathbf{E}$ . For such a material we define the *permittivity*  $\epsilon$  as

$$\epsilon = \epsilon_0(1 + \chi_e). \quad (\text{B.3.22})$$

and then write the relation between  $\mathbf{D}$  and  $\mathbf{E}$  as [see (B.3.13)]

$$\mathbf{D} = \epsilon\mathbf{E}. \quad (\text{B.3.23})$$

This mathematical model of polarizable material is used extensively in this book.

The differential equations for the electric field system, in the presence of moving polarized media, are summarized in Table 1.2.

### B.3.3 Electrical Conduction

In both magnetic and electric field systems the conduction process accounts for the free current density  $\mathbf{J}_f$  in a fixed conductor. The most common model for this process is appropriate in the case of an isotropic, linear, conducting medium which, when stationary, has the constituent relation (often called *Ohm's law*)

$$\mathbf{J}_f = \sigma\mathbf{E}. \quad (\text{B.3.24})$$

Although (B.3.24) is the most widely used mathematical model of the conduction process, there are important electromechanical systems for which it is not adequate. This becomes apparent if we attempt to derive (B.3.24), an exercise that will contribute to our physical understanding of Ohm's law.

In many materials the conduction process involves two types of charge carrier (say, ions and electrons). As discussed in Section B.1.2, a macroscopic model for this case would recognize the existence of free charge densities  $\rho_+$  and  $\rho_-$  with charge average velocities  $\mathbf{v}_+$  and  $\mathbf{v}_-$ , respectively. Then

$$\mathbf{J}_f = \rho_+\mathbf{v}_+ + \rho_-\mathbf{v}_-. \quad (\text{B.3.25})$$

The problem of relating the free current density to the electric field intensity is thus a problem in electromechanics in which the velocities of the particles carrying the free charge must be related to the electric fields that apply forces to the charges.

The charge carriers have finite mass and thus accelerate when subjected to a force. In this case there are forces on the positive and negative carriers, respectively, given by (B.1.10) (here we assume that effects from a magnetic field are ignorable):

$$\mathbf{F}_+ = \rho_+\mathbf{E}, \quad (\text{B.3.26})$$

$$\mathbf{F}_- = \rho_-\mathbf{E}. \quad (\text{B.3.27})$$



As the charge carriers move, their motion is retarded by collisions with other particles. On a macroscopic basis the retarding force of collisions can be thought of as a viscous damping force that is proportional to velocity. Hence we can picture the conduction process in two extremes. With no collisions between particles the electric force densities of (B.3.26 and B.3.27) continually accelerate the charges, for the only retarding forces are due to acceleration expressed by Newton's law. In the opposite extreme a charge carrier suffers collisions with other particles so frequently that its average velocity quickly reaches a limiting value, which in view of (B.3.26 and B.3.27) is proportional to the applied electric field. It is in this second limiting case that Ohm's law assumes physical significance. By convention *mobilities*  $\mu_+$  and  $\mu_-$  which relate these limiting velocities to the field  $\mathbf{E}$  are defined

$$\mathbf{v}_+ = \mu_+ \mathbf{E}, \quad (\text{B.3.28})$$

$$\mathbf{v}_- = \mu_- \mathbf{E}. \quad (\text{B.3.29})$$

In terms of these quantities, (B.3.25) becomes

$$\mathbf{J}_f = (\rho_+ \mu_+ + \rho_- \mu_-) \mathbf{E}. \quad (\text{B.3.30})$$

It is important to recognize that it is only when the collisions between carriers and other particles dominate the accelerating effect of the electric field that the conduction current takes on a form in which it is dependent on the instantaneous value of  $\mathbf{E}$ . Fortunately, (B.3.30) is valid in a wide range of physical situations. In fact, in a metallic conductor the number of charge carriers is extremely high and very nearly independent of the applied electric field. The current carriers in most metals are the electrons, which are detached from atoms held in the lattice structure of the solid. Therefore the negatively charged electrons move in a background field of positive charge and, to a good approximation,  $\rho_+ = -\rho_-$ . Then (B.3.30) becomes

$$\mathbf{J} = \sigma \mathbf{E}, \quad (\text{B.3.31})$$

where the conductivity is defined as

$$\rho_+ (\mu_+ - \mu_-). \quad (\text{B.3.32})$$

The usefulness of the conductivity as a parameter stems from the fact that both the number of charges available for conduction and the net mobility (essentially that of the electrons) are constant. This makes the conductivity essentially independent of the electric field, as assumed in (B.3.24).\*

\* We assume here that the temperature remains constant. A worthwhile qualitative description of conduction processes in solids is given in J. M. Ham and G. R. Slemon, *Scientific Basis of Electrical Engineering*, Wiley, New York, 1961, p. 453.

In some types of material (notably slightly ionized gases) which behave like insulators, the conduction process cannot be described simply by Ohm's law. In such materials the densities of charge carriers and even the mobilities may vary strongly with electric field intensity.

## B.4 INTEGRAL LAWS

The extensive use of circuit theory bears testimony to the usefulness of the laws of electricity and magnetism in integral form. Physical situations that would be far too involved to describe profitably in terms of field theory have a lucid and convenient representation in terms of circuits. Conventional circuit elements are deduced from the integral forms of the field equations. The description of lumped-parameter electromechanical systems, as undertaken in Chapter 2, requires that we generalize the integral laws to include time-varying surfaces and contours of integration. Hence it is natural that we conclude this appendix with a discussion of the integral laws.

### B.4.1 Magnetic Field Systems

Faraday's law of induction, as given by (B.1.42), has the differential form

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.4.1})$$

This expression can be integrated over a surface  $S$  enclosed by the contour  $C$ . Then, according to Stokes's theorem,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da. \quad (\text{B.4.2})$$

Now, if  $S$  and  $C$  are fixed in space, the time derivative on the right can be taken either before or after the surface integral of  $\mathbf{B} \cdot \mathbf{n}$  is evaluated. Note that  $\int_S \mathbf{B} \cdot \mathbf{n} \, da$  is only a function of time. For this reason (B.1.41) could be written with the total derivative outside the surface integral. It is implied in the integral equation (B.1.41) that  $S$  is fixed in space.

Figure B.4.1 shows an example in which it is desirable to be able to use (B.4.2), with  $S$  and  $C$  varying in position as a function of time. The contour  $C$  is a rectangular loop that encloses a surface  $S$  which makes an angle  $\theta(t)$  with the horizontal. Although the induction law is not limited to this case, the loop could comprise a one-turn coil, in which case it is desirable to be able to use (B.4.2) with  $C$  fixed to the coil. The integral law of induction would be much more useful if it could be written in the form

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (\text{B.4.3})$$

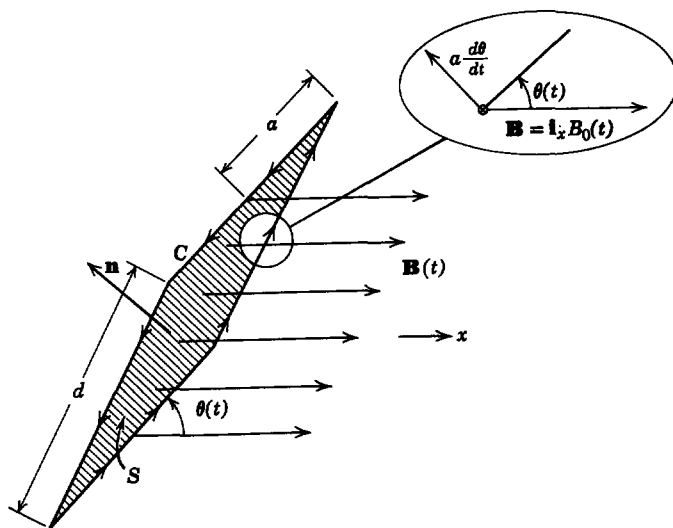


Fig. B.4.1 Contour  $C$  enclosing a surface  $S$  which varies as a function of time. The rectangular loop links no magnetic flux when  $\theta = 0, \pi, \dots$

In this form the quantity on the right is the negative time rate of change of the flux linked by the contour  $C$ , whereas  $E'$  is interpreted as the electric field measured in the moving frame of the loop. An integral law of induction in the form of (B.4.3) is essential to the lumped-parameter description of magnetic field systems. At this point we have two choices. We can accept (B.4.3) as an empirical result of Faraday's original investigations or we can show mathematically that (B.4.2) takes the form of (B.4.3) if

$$E' = E + v \times B, \tag{B.4.4}$$

where  $v$  is the velocity of  $d\mathbf{l}$  at the point of integration. In any case this topic is pursued in Chapter 6 to clarify the significance of electric and magnetic fields measured in different frames of reference.

The mathematical connection between (B.4.2) and (B.4.3) is made by using the integral theorem

$$\frac{d}{dt} \int_S \mathbf{A} \cdot \mathbf{n} \, da = \int_S \left[ \frac{\partial \mathbf{A}}{\partial t} + (\nabla \cdot \mathbf{A})\mathbf{v} \right] \cdot \mathbf{n} \, da + \oint_C (\mathbf{A} \times \mathbf{v}) \cdot d\mathbf{l}, \tag{B.4.5}$$

where  $v$  is the velocity of  $S$  and  $C$  and in the case of (B.4.3),  $\mathbf{A} \rightarrow \mathbf{B}$ . Before we embark on a proof of this theorem, an example will clarify its significance.

**Example B.4.1.** The coil shown in Fig. B.4.1 rotates with the angular deflection  $\theta(t)$  in a uniform magnetic flux density  $\mathbf{B}(t)$ , directed as shown. We wish to compute the rate of change of the flux linked by the coil in two ways: first by computing  $\int_S \mathbf{B} \cdot \mathbf{n} \, da$  and taking

its derivative [the left-hand side of (B.4.5)], then by using the surface and contour integrations indicated on the right-hand side of (B.4.5). This illustrates how the identity allows us to carry out the surface integration before rather than after the time derivative is taken. From Fig. B.4.1 we observe that

$$\int_S \mathbf{B} \cdot \mathbf{n} \, da = -B_0(t)2ad \sin \theta, \tag{a}$$

so that the first calculation gives

$$\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da = -2ad \sin \theta \frac{dB_0}{dt} - B_0 2ad \cos \theta \frac{d\theta}{dt}. \tag{b}$$

To evaluate the right-hand side of (B.4.5) observe that  $\nabla \cdot \mathbf{B} = 0$  and [from (a)]

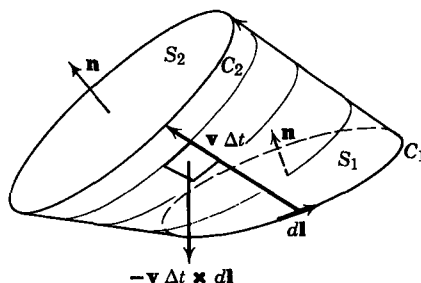
$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da = -2ad \sin \theta \frac{dB_0}{dt}. \tag{c}$$

The quantity  $\mathbf{B} \times \mathbf{v}$  is collinear with the axis of rotation in Fig. B.4.1; hence there is no contribution to the line integral along the pivoted ends of the loop. Because both the velocity  $\mathbf{v} = \mathbf{i}_\theta a (d\theta/dt)$  and line elements  $d\mathbf{l}$  are reversed in going from the upper to the lower horizontal contours, the line integral reduces to twice the value from the upper contour.

$$\oint_C \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l} = -2B_0 a d \cos \theta \frac{d\theta}{dt} \tag{d}$$

From (c) and (d) it follows that the right-hand side of (B.4.5) also gives (b). Thus, at least for this example, (B.4.5) provides alternative ways of evaluating the time rate of change of the flux linked by the contour  $C$ .

The integral theorem of (B.4.5) can be derived by considering the deforming surface  $S$  shown at two instants of time in Fig. B.4.2. In the incremental time interval  $\Delta t$  the surface  $S$  moves from  $S_1$  to  $S_2$ , and therefore by



**Fig. B.4.2** When  $t = t$ , the surface  $S$  enclosed by the contour  $C$  is as indicated by  $S_1$  and  $C_1$ . By the time  $t = t + \Delta t$  this surface has moved to  $S_2$ , where it is enclosed by the contour  $C_2$ .

definition

$$\frac{d}{dt} \int_S \mathbf{A} \cdot \mathbf{n} \, da = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left( \int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da - \int_{S_1} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da \right). \quad (\text{B.4.6})$$

Here we have been careful to show that when the integral on  $S_2$  is evaluated  $t = t + \Delta t$ , in contrast to the integration on  $S_1$ , which is carried out when  $t = t$ .

The expression on the right in (B.4.6) can be evaluated at a given instant in time by using the divergence theorem (B.1.14) to write

$$\int_V \nabla \cdot \mathbf{A} \, dV \cong \int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da - \int_{S_1} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da - \Delta t \oint_{C_1} \mathbf{A} \cdot \mathbf{v} \times d\mathbf{l} \quad (\text{B.4.7})$$

for the volume  $V$  traced out by the surface  $S$  in the time  $\Delta t$ . Here we have used the fact that  $-\mathbf{v} \, \Delta t \times d\mathbf{l}$  is equivalent to a surface element  $\mathbf{n} \, da$  on the surface traced out by the contour  $C$  in going from  $C_1$  to  $C_2$  in Fig. B.4.2. To use (B.4.7) we make three observations. First, as  $\Delta t \rightarrow 0$ ,

$$\int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da \cong \int_{S_2} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da + \int_{S_1} \frac{\partial \mathbf{A}}{\partial t} \Big|_t \Delta t \cdot \mathbf{n} \, da + \dots \quad (\text{B.4.8})$$

Second, it is a vector identity that

$$\mathbf{A} \cdot \mathbf{v} \times d\mathbf{l} = \mathbf{A} \times \mathbf{v} \cdot d\mathbf{l}. \quad (\text{B.4.9})$$

Third, an incremental volume  $dV$  swept out by the surface  $da$  is essentially the base times the perpendicular height or

$$dV = \Delta t \mathbf{v} \cdot \mathbf{n} \, da. \quad (\text{B.4.10})$$

From these observations (B.4.7) becomes

$$\begin{aligned} \Delta t \int_{S_1} (\nabla \cdot \mathbf{A}) \mathbf{v} \cdot \mathbf{n} \, da &\cong \int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da - \int_{S_1} \Delta t \frac{\partial \mathbf{A}}{\partial t} \Big|_t \cdot \mathbf{n} \, da \\ &\quad - \int_{S_1} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da - \Delta t \oint_{C_1} \mathbf{A} \times \mathbf{v} \cdot d\mathbf{l}. \end{aligned} \quad (\text{B.4.11})$$

This expression can be solved for the quantity on the right in (B.4.6) to give

$$\frac{d}{dt} \int_S \mathbf{A} \cdot \mathbf{n} \, da = \lim_{\Delta t \rightarrow 0} \left\{ \int_{S_1} \left[ (\nabla \cdot \mathbf{A}) \mathbf{v} + \frac{\partial \mathbf{A}}{\partial t} \right] \cdot \mathbf{n} \, da + \oint_{C_1} \mathbf{A} \times \mathbf{v} \cdot d\mathbf{l} \right\}. \quad (\text{B.4.12})$$

The limit of this expression produces the required relation (B.4.5).

Use of (B.4.5) to express the right-hand side of (B.4.2) results in

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da - \int_S (\nabla \cdot \mathbf{B}) \mathbf{v} \cdot \mathbf{n} \, da - \oint_C (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l}. \quad (\text{B.4.13})$$

Because  $\nabla \cdot \mathbf{B} = 0$ , (B.4.2) then reduces to (B.4.3), with  $\mathbf{E}'$  given by (B.4.4).

The integral laws for the magnetic field system are summarized in Table 1.2 at the end of Chapter 1. In these equations surfaces and contours of integration can, in general, be time-varying.

### B.4.2 Electric Field System

Although the integral form of Faraday's law can be taken as an empirical fact, we require (B.4.5) to write Ampère's law in integral form for an electric field system. If we integrate (B.3.20) over a surface  $S$  enclosed by a contour  $C$ , by Stokes's theorem it becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n} \, da. \quad (\text{B.4.14})$$

As with the induction law for the magnetic field system, this expression can be generalized to include the possibility of a deforming surface  $S$  by using (B.4.13) with  $\mathbf{B} \rightarrow \mathbf{D}$  to rewrite the last term. If, in addition, we use (B.3.14) to replace  $\nabla \cdot \mathbf{D}$  with  $\rho_f$ , (B.4.14) becomes

$$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da, \quad (\text{B.4.15})$$

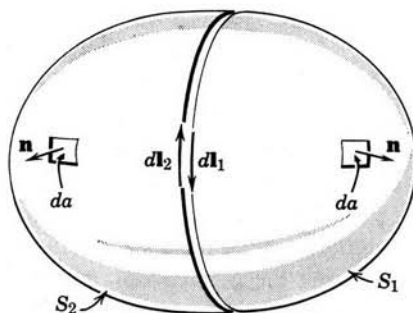
where

$$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}, \quad (\text{B.4.16})$$

$$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}. \quad (\text{B.4.17})$$

The fields  $\mathbf{H}'$  and  $\mathbf{J}'_f$  can be interpreted as the magnetic field intensity and free current density measured in the moving frame of the deforming contour. The significance of these field transformations is discussed in Chapter 6. Certainly the relationship between  $\mathbf{J}'_f$  (the current density in a frame moving with a velocity  $\mathbf{v}$ ) and the current density  $\mathbf{J}_f$  (measured in a fixed frame), as given by (B.4.17), is physically reasonable. The free charge density appears as a current in the negative  $\mathbf{v}$ -direction when viewed from a frame moving at the velocity  $\mathbf{v}$ . It was reasoning of this kind that led to (B.1.25).

As we have emphasized, it is the divergence of Ampère's differential law that assumes the greatest importance in electric field systems, for it accounts for conservation of charge. The integral form of the conservation of charge



**Fig. B.4.3** The sum of two surfaces  $S_1$  and  $S_2$  “spliced” together at the contour to enclose the volume  $V$ .

equation, including the possibility of a deforming surface of integration, is obtained by using (B.4.15). For this purpose integrations are considered over two deforming surfaces,  $S_1$  and  $S_2$ , as shown in Fig. B.4.3. These surfaces are chosen so that they are enclosed by the same contour  $C$ . Hence, taken together,  $S_1$  and  $S_2$  enclose a volume  $V$ .

Integration of (B.4.15) over each surface gives

$$\oint_C \mathbf{H}' \cdot d\mathbf{l}_1 = \int_{S_1} \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_{S_1} \mathbf{D} \cdot \mathbf{n} \, da. \quad (\text{B.4.18})$$

$$\oint_C \mathbf{H}' \cdot d\mathbf{l}_2 = \int_{S_2} \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_{S_2} \mathbf{D} \cdot \mathbf{n} \, da. \quad (\text{B.4.19})$$

Now, if  $\mathbf{n}$  is defined so that it is directed out of the volume  $V$  on each surface, the line integral enclosing  $S_1$  will be the negative of that enclosing  $S_2$ . Then the sum of (B.4.18 and B.4.19) gives the desired integral form of the conservation of charge equation:

$$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_V \rho_f \, dV = 0. \quad (\text{B.4.20})$$

In writing this expression we have used Gauss’s theorem and (B.3.14) to show the explicit dependence of the current density through the deforming surface on the enclosed charge density.

The integral laws for electric field systems are summarized in Table 1.2 at the end of Chapter 1.

## B.5 RECOMMENDED READING

The following texts treat the subject of electrodynamics and provide a comprehensive development of the fundamental laws of electricity and magnetism.

R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960; J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1962; S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, 1965; W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley, Reading, Mass., 1956; J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941.

Many questions arise in the study of the effects of moving media on electric and magnetic fields concerning the macroscopic representation of polarized and magnetized media; for example, in this appendix we introduced the fields  $\mathbf{E}$  and  $\mathbf{B}$  as the quantities defined by the force law. Then  $\mathbf{P}$  and  $\mathbf{M}$  (or  $\mathbf{D}$  and  $\mathbf{H}$ ) were introduced to account for the effects of polarization and magnetization. Hence the effect of the medium was accounted for by equivalent polarization charges  $\rho_p$  and magnetization currents  $\mathbf{J}_m$ . Other representations can be used in which a different pair of fundamental vectors is taken, as defined by the force law (say,  $\mathbf{E}$  and  $\mathbf{H}$ ), and in which the effects of media are accounted for by an equivalent magnetic charge instead of an equivalent current. If we are consistent in using the alternative formulations of the field equations, they predict the same physical results, including the force on magnetized and polarized media. For a complete discussion of these matters see P. Penfield, and H. Haus, *Electrodynamics of Moving Media*, M.I.T. Press, Cambridge, Mass., 1967.



## Appendix C

# SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

### IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi \nabla\psi + \psi \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A},$$

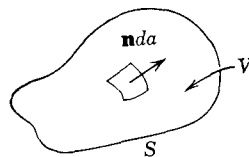
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

## THEOREMS

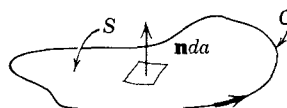
$$\int_a^b \nabla \phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



Divergence theorem  $\oint_S \mathbf{A} \cdot \mathbf{n} da = \int_V \nabla \cdot \mathbf{A} dV$



Stokes's theorem  $\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} da$



## Appendix D

# GLOSSARY OF COMMONLY USED SYMBOLS

Section references indicate where symbols of a given significance are introduced; grouped symbols are accompanied by their respective references. The absence of a section reference indicates that a symbol has been applied for a variety of purposes. Nomenclature used in examples is not included.

Symbol	Meaning	Section
$A$	cross-sectional area	
$A_i$	coefficient in differential equation	5.1.1
$(A_n^+, A_n^-)$	complex amplitudes of components of $n$ th mode	9.2.1
$A_w$	cross-sectional area of armature conductor	6.4.1
$a$	spacing of pole faces in magnetic circuit	8.5.1
$a, (a_c, a_s)$	phase velocity of acoustic related waves	13.2.1, 11.4.1
$a_b$	Alfvén velocity	12.2.3
$(a, b, c)$	Lagrangian coordinates	11.1
$a_i$	constant coefficient in differential equation	5.1.1
$\mathbf{a}_p$	instantaneous acceleration of point $p$ fixed in material	2.2.1c
$B, B_r, B_s$	damping constant for linear, angular and square law dampers	2.2.1b, 4.1.1, 5.2.2
$\mathbf{B}, \mathbf{B}_i, B_0$	magnetic flux density	1.1.1a, 8.1, 6.4.2
$B_i$	induced flux density	7.0
$(B_r, B_{ra}, B_{rb}, B_{rm})$	radial components of air-gap flux densities	4.1.4
$[B_{rf}, (B_{rf})_{av}]$	radial flux density due to field current	6.4.1
$b$	width of pole faces in magnetic circuit	8.5
$b$	half thickness of thin beam	11.4.2b
$C$	contour of integration	1.1.2a
$C, (C_a, C_b), C_o$	capacitance	2.1.2, 7.2.1a, 5.2.1
$C$	coefficient in boundary condition	9.1.1
$\mathbf{C}$	the curl of the displacement	11.4
$(C^+, C^-)$	designation of characteristic lines	9.1.1

Symbol	Meaning	Section
$c_p$	specific heat capacity at constant pressure	13.1.2
$c_v$	specific heat capacity at constant volume	13.1.2
<b>D</b>	electric displacement	1.1.1a
$d$	length	
$da$	elemental area	1.1.2a
$d\mathbf{f}_n$	total elemental force on material in rigid body	2.2.1c
$d\mathbf{l}$	elemental line segment	1.1.2a
$d\mathbf{T}_n$	torque on elemental volume of material	2.2.1c
$dV$	elemental volume	1.1.2b
$E$	constant of motion	5.2.1
$E$	Young's modulus or the modulus of elasticity	9.1
$\mathbf{E}, E_0$	electric field intensity	1.1.1a, 5.1.2d
$E_f$	magnitude of armature voltage generated by field current in a synchronous machine	4.1.6a
$E_i$	induced electric field intensity	7.0
$e_{11}, e_{ij}$	strain tensor	9.1, 11.2
$\dot{e}_{ij}$	strain-rate tensor	14.1.1a
$F$	magnetomotive force (mmf)	13.2.2
<b>F</b>	force density	1.1.1a
$\hat{F}$	complex amplitude of $f(t)$	5.1.1
$F_0$	amplitude of sinusoidal driving force	9.1.3
$f$	equilibrium tension of string	9.2
$f$	driving function	5.1.1
$f, \mathbf{f}, f^e, f^s, f_j, f_i, f_1$	force	2.2.1, 2.2.1c, 3.1, 5.1.2a, 3.1.2b, 8.1, 9.1
$f$	arbitrary scalar function	6.1
$f'$	scalar function in moving coordinate system	6.1
$f$	three-dimensional surface	6.2
$f$	integration constant	11.4.2a
$G$	a constant	5.1.2c
$G$	shear modulus of elasticity	11.2.2
$G$	speed coefficient	6.4.1
$G$	conductance	3.1
$g$	air-gap length	5.2.1
$g, \mathbf{g}$	acceleration of gravity	5.1.2c, 12.1.3
$(\mathbf{H}, H_x, H_y, H_z)$	magnetic field intensity	1.1.1a
$h$	specific enthalpy	13.1.2
<b>I, I, (I<sub>r</sub>, I<sub>s</sub>), I<sub>f</sub></b>	electrical current	10.4.3, 12.2.1a, 4.1.2, 6.4.1
$(i, i_1, i_2, \dots, i_k), (i_{ar}, i_{as}, i_{br}, i_{bs}), i_a, (i_a, i_b, i_c), (i_f, i_t), (i_r, i_s)$	electrical current	2.1, 4.1.3, 6.4.1, 4.1.7, 6.4.1, 4.1

Symbol	Meaning	Section
$i_n$	unit vector perpendicular to area of integration	6.2.1
$i_s$	unit vector normal to surface of integration	6.2.1
$(i_x, i_y, i_z), (i_1, i_2, i_3)$	unit vectors in coordinate directions	2.2.1c
$J, J_f$	current density	7.0, 1.1.1a
$J, J_r, (J_x, J_y, J_z)$	moment of inertia	5.1.2b, 4.1.1, 2.2.1c
$J_{xx}, J_{yz}$	products of inertia	2.2.1c
$j$	$\sqrt{-1}$	4.1.6a
$K$	loading factor	13.2.2
$K, K_f$	surface current density	7.0, 1.1.1a
$K$	linear or torsional spring constant	2.2.1a
$K_t$	induced surface current density	7.0
$k, k_c, (k_r, k_t)$	wavenumber	7.1.3, 10.1.3, 10.0
$k$	summation index	2.1.1
$k$	maximum coefficient of coupling	4.1.6b
$k_n$	$n$ th eigenvalue	9.2
$(L, L_1, L_2), (L_a, L_f),$ $L_m, (L_0, L_2),$ $(L_r, L_s, L_{sr}), L_{ss}$	inductance	2.1.1, 6.4.1, 2.1.1, 4.2.1, 4.1.1, 4.2.4
$L$	length of incremental line segment	6.2.1
$l$	value of relative displacement for which spring force is zero	2.2.1a
$l, l_w, l_y$	length	
$M$	Hartmann number	14.2.2
$M$	mass of one mole of gas in kilograms	13.1.2
$M$	Mach number	13.2.1
$M$	mass	2.2.1c
$M$	number of mechanical terminal pairs	2.1.1
$M, M_s$	mutual inductance	4.1.1, 4.2.4
$M$	magnetization density	1.1.1a
$m$	mass/unit length of string	9.2
$N$	number of electrical terminal pairs	2.1.1
$N$	number of turns	5.2.2
$n$	number density of ions	12.3.1
$n$	integer	7.1.1
$n$	unit normal vector	1.1.2
$P$	polarization density	1.1.1a
$P$	power	12.2.1a
$p$	number of pole pairs in a machine	4.1.8
$p$	power per unit area	14.2.1
$p$	pressure	5.1.2d and 12.1.4
$P_e, P_g, P_m, P_r$	power	4.1.6a, 4.1.6b, 4.1.2, 4.1.6b
$Q$	electric charge	7.2.1a
$q, q_i, q_k$	electric charge	1.1.3 and 2.1.2, 8.1, 2.1.2
$R, R_t, R_o$	radius	

Symbol	Meaning	Section
$R, R_a, R_b, R_f, R_r, R_s$	resistance	
$(R, R_\rho)$	gas constant	13.1.2
$R_e$	electric Reynolds number	7.0
$R_m$	magnetic Reynolds number	7.0
$r$	radial coordinate	
$\mathbf{r}$	position vector of material	2.2.1c
$\mathbf{r}'$	position vector in moving reference frame	6.1
$\mathbf{r}_m$	center of mass of rigid body	2.2.1c
$S$	reciprocal modulus of elasticity	11.5.2c
$S$	surface of integration	1.1.2a
$S$	normalized frequency	7.2.4
$S$	membrane tension	9.2
$S_z$	transverse force/unit length acting on string	9.2
$s$	complex frequency	5.1.1
$(s, s_{mT})$	slip	4.1.6b
$s_i$	$i$ th root of characteristic equation, a natural frequency	5.1.1
$T$	period of oscillation	5.2.1
$T$	temperature	13.1.2
$\mathbf{T}, T, T^e, T_{em}, T_m, T_0, T_1$	torque	2.2.1c, 5.1.2b, 3.1.1, 4.1.6b, 4.1.1, 6.4.1, 6.4.1
$\mathbf{T}$	surface force	8.4
$T_{ij}^m$	mechanical stress tensor	13.1.2
$T_{mn}$	the component of the stress-tensor with the $m$ th-direction on a cartesian surface with a normal vector in the $n$ th-direction	8.1
$T_{or}$	constant of coulomb damping	4.1.1
$T_0$	initial stress distribution on thin rod	9.1.1
$T$	longitudinal stress on a thin rod	9.1.1
$T_z$	transverse force per unit area on membrane	9.2
$T_2$	transverse force per unit area acting on thin beam	11.4.2b
$t$	time	1.1.1
$t'$	time measured in moving reference frame	6.1
$U$	gravitational potential	12.1.3
$U$	longitudinal steady velocity of string or membrane	10.2
$u$	internal energy per unit mass	13.1.1
$u$	surface coordinate	11.3
$u_0(x - x_0)$	unit impulse at $x = x_0$	9.2.1
$u$	transverse deflection of wire in $x$ -direction	10.4.3
$u_{-1}(t)$	unit step occurring at $t = 0$	5.1.2b
$V, V_m$	velocity	7.0, 13.2.3
$V$	volume	1.1.2
$V, V_a, V_f, V_o, V_s$	voltage	
$V$	potential energy	5.2.1

Symbol	Meaning	Section
$v, \mathbf{v}$	velocity	
$(v, v_1, \dots, v_k)$	voltage	2.1.1
$v', (v_a, v_b, v_c),$ $v_f, v_{oc}, v_t$	voltage	
$v_n$	velocity of surface in normal direction	6.2.1
$v_o$	initial velocity distribution on thin rod	9.1.1
$v_p$	phase velocity	9.1.1 and 10.2
$\mathbf{v}^r$	relative velocity of inertial reference frames	6.1
$v_s$	$\sqrt{f/m}$ for a string under tension $f$ and having mass/unit length $m$	10.1.1
$v$	longitudinal material velocity on thin rod	9.1.1
$v$	transverse deflection of wire in $y$ -direction	10.4.3
$(W_e, W_m)$	energy stored in electromechanical coupling	3.1.1
$(W'_e, W'_m, W')$	coenergy stored in electromechanical coupling	3.1.2b
$W''$	hybrid energy function	5.2.1
$w$	width	5.2.2
$w$	energy density	11.5.2c
$w'$	coenergy density	8.5
$X$	equilibrium position	5.1.2a
$(x, x_1, x_2, \dots, x_k)$	displacement of mechanical node	2.1.1
$x$	dependent variable	5.1.1
$x_p$	particular solution of differential equation	5.1.1
$(x_1, x_2, x_3), (x, y, z)$	cartesian coordinates	8.1, 6.1
$(x', y', z')$	cartesian coordinates of moving frame	6.1
$(\alpha, \beta)$	constants along $C^+$ and $C^-$ characteristics, respectively	9.1.1
$(\alpha, \beta)$	see (10.2.20) or (10.2.27)	
$\alpha$	transverse wavenumber	11.4.3
$(\alpha, \beta)$	angles used to define shear strain	11.2
$(\alpha, \beta)$	constant angles	4.1.6b
$\alpha$	space decay parameter	7.1.4
$\alpha$	damping constant	5.1.2b
$\alpha$	equilibrium angle of torsional spring	2.2.1a
$\gamma$	ratio of specific heats	13.1.2
$\gamma$	piezoelectric constant	11.5.2c
$\gamma, \gamma_0, \gamma'$	angular position	
$\Delta_a(t)$	slope excitation of string	10.2.1b
$\Delta_0$	amplitude of sinusoidal slope excitation	10.2.1b
$\Delta r$	distance between unstressed material points	11.2.1a
$\Delta s$	distance between stressed positions of material points	11.2.1a
$\delta( )$	incremental change in ( )	8.5
$\delta, \delta_1, \delta_0$	displacement of elastic material	11.1, 9.1, 11.4.2a
$\delta$	thickness of incremental volume element	6.2.1
$\delta$	torque angle	4.1.6a

Symbol	Meaning	Section
$\delta_{ij}$	Kronecker delta	8.1
$(\delta_+, \delta_-)$	wave components traveling in the $\pm x$ -directions	9.1.1
$\epsilon$	linear permittivity	1.1.1b
$\epsilon_0$	permittivity of free space	1.1.1a
$\eta$	efficiency of an induction motor	4.1.6b
$\eta$	second coefficient of viscosity	14.1.1c
$\theta, \theta_i, \theta_m$	angular displacement	2.1.1, 3.1.1, 5.2.1
$\theta$	power factor angle; phase angle between current and voltage	4.1.6a
$\theta$	equilibrium angle	5.2.1
$\dot{\theta}$	angular velocity of armature	6.4.1
$\theta_m$	maximum angular deflection	5.2.1
$(\lambda, \lambda_1, \lambda_2, \dots, \lambda_k)$	magnetic flux linkage	2.1.1, 6.4.1, 4.1.7,
$\lambda_a$		4.1.3, 4.1
$(\lambda_a, \lambda_b, \lambda_c)$		
$(\lambda_{ar}, \lambda_{as}, \lambda_{br}, \lambda_{bs})$		
$(\lambda_r, \lambda_s)$		
$\lambda$	Lamé constant for elastic material	11.2.3
$\lambda$	wavelength	7.1.4
$\mu$	linear permeability	1.1.1a
$\mu, (\mu_+, \mu_-)$	mobility	12.3.1, 1.1.1b
$\mu$	coefficient of viscosity	14.1.1
$\mu_d$	coefficient of dynamic friction	2.2.1b
$\mu_0$	permeability of free space	1.1.1a
$\mu_s$	coefficient of static friction	2.2.1b
$\nu$	Poisson's ratio for elastic material	11.2.2
$\nu$	damping frequency	10.1.4
$(\xi, \xi)$	continuum displacement	8.5
$\xi_0$	initial deflection of string	9.2
$\xi_d$	amplitude of sinusoidal driving deflection	9.2
$(\xi_n(x), \xi_n(x))$	$n$ th eigenfunctions	9.2.1b
$(\xi_+, \xi_-)$	amplitudes of forward and backward traveling waves	9.2
$\dot{\xi}_0(x)$	initial velocity of string	9.2
$\rho$	mass density	2.2.1c
$\rho_f$	free charge density	1.1.1a
$\rho_s$	surface mass density	11.3
$\Sigma$	surface of discontinuity	6.2
$\sigma$	conductivity	1.1.1a
$\sigma_f$	free surface charge density	1.1.1a
$\sigma_m$	surface mass density of membrane	9.2
$\sigma_o$	surface charge density	7.2.3
$\sigma_s$	surface conductivity	1.1.1a
$\sigma_u$	surface charge density	7.2.3
$\tau$	surface traction	8.2.1
$\tau, \tau_d$	diffusion time constant	7.1.1, 7.1.2a
$\tau$	relaxation time	7.2.1a



Symbol	Meaning	Section
$\tau_e$	electrical time constant	5.2.2
$\tau_m$	time for air gap to close	5.2.2
$\tau_o$	time constant	5.1.3
$\tau_t$	traversal time	7.1.2a
$\phi$	electric potential	7.2
$\phi$	magnetic flux	2.1.1
$\phi$	cylindrical coordinate	2.1.1
$\phi$	potential for H when $J_f = 0$	8.5.2
$\phi$	flow potential	12.2
$\chi_e$	electric susceptibility	1.1.1b
$\chi_m$	magnetic susceptibility	1.1.1a
$\psi$	the divergence of the material displacement	11.4
$\psi$	angle defined in Fig. 6.4.2	6.4.1
$\psi$	angular position in the air gap measured from stator winding ( $a$ ) magnetic axis	4.1.4
$\psi$	electromagnetic force potential	12.2
$\psi$	angular deflection of wire	10.4.3
$\Omega$	equilibrium rotational speed	5.1.2b
$\Omega$	rotation vector in elastic material	11.2.1a
$\Omega_n$	real part of eigenfrequency (10.1.47)	10.1.4
$\omega, (\omega_r, \omega_s)$	radian frequency of electrical excitation	4.1.6a, 4.1.2
$\omega$	natural angular frequency (Im $s$ )	5.1.2b
$\omega, \omega_m$	angular velocity	2.2.1c, 4.1.2
$\omega_c$	cutoff frequency for evanescent waves	10.1.2
$\omega_d$	driving frequency	9.2
$\omega_n$	$n$ th eigenfrequency	9.2
$\omega_o$	natural angular frequency	5.1.3
$(\omega_r, \omega_t)$	real and imaginary parts of $\omega$	10.0
$\nabla$	nabla	6.1
$\nabla_\Sigma$	surface divergence	6.2.1

## Appendix E

# SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

### IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi \nabla\psi + \psi \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

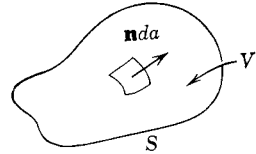
**THEOREMS**

$$\int_a^b \nabla \phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



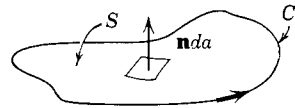
Divergence theorem

$$\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV$$



Stokes's theorem

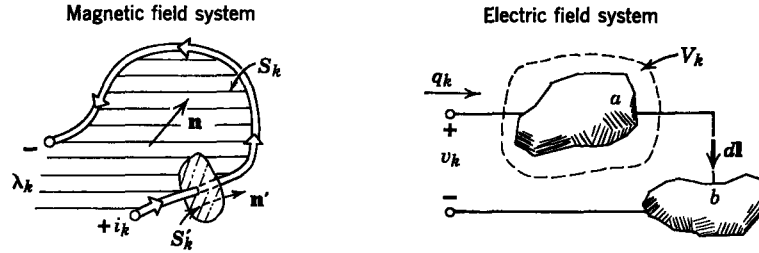
$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da$$



**Table 1.2 Summary of Quasi-Static Electromagnetic Equations**

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da$	(1.1.23)
			where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V \rho_f \, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da$	(1.1.27)
		where $\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}$		
		$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$		

**Table 2.1 Summary of Terminal Variables and Terminal Relations**



**Definition of Terminal Variables**

Flux

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da$$

Current

$$i_k = \int_{S'_k} \mathbf{J}_f \cdot \mathbf{n}' \, da$$

Charge

$$q_k = \int_{V_k} \rho_f \, dV$$

Voltage

$$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

**Terminal Conditions**

$$v_k = \frac{d\lambda_k}{dt}$$

$$i_k = \frac{dq_k}{dt}$$

$$\lambda_k = \lambda_k(i_1 \cdots i_N; \text{geometry})$$

$$q_k = q_k(v_1 \cdots v_N; \text{geometry})$$

$$i_k = i_k(\lambda_1 \cdots \lambda_N; \text{geometry})$$

$$v_k = v_k(q_1 \cdots q_N; \text{geometry})$$

**and M Mechanical Terminal Pairs\***

Magnetic Field Systems

Electric Field Systems

Conservation of Energy

$$dW_m = \sum_{j=1}^N i_j d\lambda_j - \sum_{j=1}^M f_j^e dx_j$$

(a)

$$dW_e = \sum_{j=1}^N v_j dq_j - \sum_{j=1}^M f_j^e dx_j$$

(b)

$$dW'_m = \sum_{j=1}^N \lambda_j di_j + \sum_{j=1}^M f_j^e dx_j$$

(c)

$$dW'_e = \sum_{j=1}^N q_j dv_j + \sum_{j=1}^M f_j^e dx_j$$

(d)

Forces of Electric Origin,  $j = 1, \dots, M$

$$f_j^e = - \frac{\partial W_m(\lambda_1, \dots, \lambda_N; x_1, \dots, x_M)}{\partial x_j}$$

(e)

$$f_j^e = - \frac{\partial W_e(q_1, \dots, q_N; x_1, \dots, x_M)}{\partial x_j}$$

(f)

$$f_j^e = \frac{\partial W'_m(i_1, \dots, i_N; x_1, \dots, x_M)}{\partial x_j}$$

(g)

$$f_j^e = \frac{\partial W'_e(v_1, \dots, v_N; x_1, \dots, x_M)}{\partial x_j}$$

(h)

Relation of Energy to Coenergy

$$W_m + W'_m = \sum_{j=1}^N \lambda_j i_j$$

(i)

$$W_e + W'_e = \sum_{j=1}^N v_j q_j$$

(j)

Energy and Coenergy from Electrical Terminal Relations

$$W_m = \sum_{j=1}^N \int_0^{\lambda_j} i_j(\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, 0, \dots, 0; x_1, \dots, x_M) d\lambda'_j \quad (k)$$

$$W_e = \sum_{j=1}^N \int_0^{q_j} v_j(q_1, \dots, q_{j-1}, q'_j, 0, \dots, 0; x_1, \dots, x_M) dq'_j \quad (l)$$

$$W'_m = \sum_{j=1}^N \int_0^{i_j} \lambda_j(i_1, \dots, i_{j-1}, i'_j, 0, \dots, 0; x_1, \dots, x_M) di'_j \quad (m)$$

$$W'_e = \sum_{j=1}^N \int_0^{v_j} q_j(v_1, \dots, v_{j-1}, v'_j, 0, \dots, 0; x_1, \dots, x_M) dv'_j \quad (n)$$

\* The mechanical variables  $f_j$  and  $x_j$  can be regarded as the  $j$ th force and displacement or the  $j$ th torque  $T_j$  and angular displacement  $\theta_j$ .

**Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media**

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}'_f = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_{\Sigma} \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho'_f = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_{\Sigma} \cdot \mathbf{K}_f = v_n(\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)			

# Appendix F

## GLOSSARY OF COMMONLY USED SYMBOLS

Section references indicate where symbols of a given significance are introduced; grouped symbols are accompanied by their respective references. The absence of a section reference indicates that a symbol has been applied for a variety of purposes. Nomenclature used in examples is not included.

Symbol	Meaning	Section
$A$	cross-sectional area	
$A_i$	coefficient in differential equation	5.1.1
$(A_n^+, A_n^-)$	complex amplitudes of components of $n$ th mode	9.2.1
$A_w$	cross-sectional area of armature conductor	6.4.1
$a$	spacing of pole faces in magnetic circuit	8.5.1
$a, (a_c, a_s)$	phase velocity of acoustic related waves	13.2.1, 11.4.1
$a_b$	Alfvén velocity	12.2.3
$(a, b, c)$	Lagrangian coordinates	11.1
$a_i$	constant coefficient in differential equation	5.1.1
$a_p$	instantaneous acceleration of point $p$ fixed in material	2.2.1c
$B, B_r, B_s$	damping constant for linear, angular and square law dampers	2.2.1b, 4.1.1, 5.2.2
$B, B_i, B_0$	magnetic flux density	1.1.1a, 8.1, 6.4.2
$B_i$	induced flux density	7.0
$(B_r, B_{ra}, B_{rb}, B_{rm})$	radial components of air-gap flux densities	4.1.4
$[B_{rf}, (B_{rf})_{av}]$	radial flux density due to field current	6.4.1
$b$	width of pole faces in magnetic circuit	8.5
$b$	half thickness of thin beam	11.4.2b
$C$	contour of integration	1.1.2a
$C, (C_a, C_b), C_0$	capacitance	2.1.2, 7.2.1a, 5.2.1
$C$	coefficient in boundary condition	9.1.1
$C$	the curl of the displacement	11.4
$(C^+, C^-)$	designation of characteristic lines	9.1.1



Symbol	Meaning	Section
$c_p$	specific heat capacity at constant pressure	13.1.2
$c_v$	specific heat capacity at constant volume	13.1.2
<b>D</b>	electric displacement	1.1.1a
$d$	length	
$da$	elemental area	1.1.2a
$d\mathbf{f}_n$	total elemental force on material in rigid body	2.2.1c
$d\mathbf{l}$	elemental line segment	1.1.2a
$d\mathbf{T}_n$	torque on elemental volume of material	2.2.1c
$dV$	elemental volume	1.1.2b
$E$	constant of motion	5.2.1
$E$	Young's modulus or the modulus of elasticity	9.1
$\mathbf{E}, E_0$	electric field intensity	1.1.1a, 5.1.2d
$E_f$	magnitude of armature voltage generated by field current in a synchronous machine	4.1.6a
$E_i$	induced electric field intensity	7.0
$e_{11}, e_{ij}$	strain tensor	9.1, 11.2
$\dot{e}_{ij}$	strain-rate tensor	14.1.1a
$F$	magnetomotive force (mmf)	13.2.2
<b>F</b>	force density	1.1.1a
$\hat{F}$	complex amplitude of $f(t)$	5.1.1
$F_0$	amplitude of sinusoidal driving force	9.1.3
$f$	equilibrium tension of string	9.2
$f$	driving function	5.1.1
$f, \mathbf{f}, f^e, f^s, f_j, f_i, f_1$	force	2.2.1, 2.2.1c, 3.1, 5.1.2a, 3.1.2b, 8.1, 9.1
$f$	arbitrary scalar function	6.1
$f'$	scalar function in moving coordinate system	6.1
$f$	three-dimensional surface	6.2
$f$	integration constant	11.4.2a
$G$	a constant	5.1.2c
$G$	shear modulus of elasticity	11.2.2
$G$	speed coefficient	6.4.1
$G$	conductance	3.1
$g$	air-gap length	5.2.1
$g, \mathbf{g}$	acceleration of gravity	5.1.2c, 12.1.3
$(\mathbf{H}, H_x, H_y, H_z)$	magnetic field intensity	1.1.1a
$h$	specific enthalpy	13.1.2
$\mathbf{I}, I, (I_r, I_\theta), I_f$	electrical current	10.4.3, 12.2.1a, 4.1.2, 6.4.1
$(i, i_1, i_2, \dots, i_k), (i_{ar}, i_{as}, i_{br}, i_{bs}), i_a, (i_a, i_b, i_c), (i_f, i_d), (i_r, i_\theta)$	electrical current	2.1, 4.1.3, 6.4.1, 4.1.7, 6.4.1, 4.1

Symbol	Meaning	Section
$\mathbf{i}_n$	unit vector perpendicular to area of integration	6.2.1
$\mathbf{i}_s$	unit vector normal to surface of integration	6.2.1
$(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z), (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$	unit vectors in coordinate directions	2.2.1c
$J, J_f$	current density	7.0, 1.1.1a
$J, J_r, (J_x, J_y, J_z)$	moment of inertia	5.1.2b, 4.1.1, 2.2.1c
$J_{xx}, J_{yz}$	products of inertia	2.2.1c
$j$	$\sqrt{-1}$	4.1.6a
$K$	loading factor	13.2.2
$K, \mathbf{K}_f$	surface current density	7.0, 1.1.1a
$K$	linear or torsional spring constant	2.2.1a
$K_i$	induced surface current density	7.0
$k, k_o, (k_r, k_i)$	wavenumber	7.1.3, 10.1.3, 10.0
$k$	summation index	2.1.1
$k$	maximum coefficient of coupling	4.1.6b
$k_n$	$n$ th eigenvalue	9.2
$(L, L_1, L_2), (L_a, L_f), L_m, (L_0, L_2), (L_T, L_s, L_{sr}), L_{ss}$	inductance	2.1.1, 6.4.1, 2.1.1, 4.2.1, 4.1.1, 4.2.4
$L$	length of incremental line segment	6.2.1
$l$	value of relative displacement for which spring force is zero	2.2.1a
$l, l_w, l_y$	length	
$M$	Hartmann number	14.2.2
$M$	mass of one mole of gas in kilograms	13.1.2
$M$	Mach number	13.2.1
$M$	mass	2.2.1c
$M$	number of mechanical terminal pairs	2.1.1
$M, M_s$	mutual inductance	4.1.1, 4.2.4
$M$	magnetization density	1.1.1a
$m$	mass/unit length of string	9.2
$N$	number of electrical terminal pairs	2.1.1
$N$	number of turns	5.2.2
$n$	number density of ions	12.3.1
$n$	integer	7.1.1
$\mathbf{n}$	unit normal vector	1.1.2
$\mathbf{P}$	polarization density	1.1.1a
$P$	power	12.2.1a
$p$	number of pole pairs in a machine	4.1.8
$p$	power per unit area	14.2.1
$p$	pressure	5.1.2d and 12.1.4
$P_o, P_o, P_m, P_r$	power	4.1.6a, 4.1.6b, 4.1.2, 4.1.6b
$Q$	electric charge	7.2.1a
$q, q_i, q_k$	electric charge	1.1.3 and 2.1.2, 8.1, 2.1.2
$R, R_i, R_o$	radius	

Symbol	Meaning	Section
$R, R_a, R_b, R_f, R_r, R_s$	resistance	
$(R, R_g)$	gas constant	13.1.2
$R_e$	electric Reynolds number	7.0
$R_m$	magnetic Reynolds number	7.0
$r$	radial coordinate	
$\mathbf{r}$	position vector of material	2.2.1c
$\mathbf{r}'$	position vector in moving reference frame	6.1
$r_m$	center of mass of rigid body	2.2.1c
$S$	reciprocal modulus of elasticity	11.5.2c
$S$	surface of integration	1.1.2a
$S$	normalized frequency	7.2.4
$S$	membrane tension	9.2
$S_z$	transverse force/unit length acting on string	9.2
$s$	complex frequency	5.1.1
$(s, s_{mT})$	slip	4.1.6b
$s_i$	$i$ th root of characteristic equation, a natural frequency	5.1.1
$T$	period of oscillation	5.2.1
$T$	temperature	13.1.2
$T, T, T^e, T_{em}, T_m, T_0, T_1$	torque	2.2.1c, 5.1.2b, 3.1.1, 4.1.6b, 4.1.1, 6.4.1, 6.4.1
$\mathbf{T}$	surface force	8.4
$T_{ij}^m$	mechanical stress tensor	13.1.2
$T_{m\pi}$	the component of the stress-tensor in the $m$ th-direction on a cartesian surface with a normal vector in the $\pi$ th-direction	8.1
$T_{or}$	constant of coulomb damping	4.1.1
$T_o$	initial stress distribution on thin rod	9.1.1
$T$	longitudinal stress on a thin rod	9.1.1
$T_z$	transverse force per unit area on membrane	9.2
$T_2$	transverse force per unit area acting on thin beam	11.4.2b
$t$	time	1.1.1
$t'$	time measured in moving reference frame	6.1
$U$	gravitational potential	12.1.3
$U$	longitudinal steady velocity of string or membrane	10.2
$u$	internal energy per unit mass	13.1.1
$u$	surface coordinate	11.3
$u_0(x - x_0)$	unit impulse at $x = x_0$	9.2.1
$u$	transverse deflection of wire in $x$ -direction	10.4.3
$u_{-1}(t)$	unit step occurring at $t = 0$	5.1.2b
$V, V_m$	velocity	7.0, 13.2.3
$V$	volume	1.1.2
$V, V_a, V_f, V_o, V_s$	voltage	
$V$	potential energy	5.2.1

Symbol	Meaning	Section
$v, \mathbf{v}$	velocity	
$(v, v_1, \dots, v_k)$	voltage	2.1.1
$v', (v_a, v_b, v_c),$ $v_f, v_{oc}, v_t$	voltage	
$v_n$	velocity of surface in normal direction	6.2.1
$v_o$	initial velocity distribution on thin rod	9.1.1
$v_p$	phase velocity	9.1.1 and 10.2
$\mathbf{v}^r$	relative velocity of inertial reference frames	6.1
$v_s$	$\sqrt{f/m}$ for a string under tension $f$ and having mass/unit length $m$	10.1.1
$v$	longitudinal material velocity on thin rod	9.1.1
$v$	transverse deflection of wire in $y$ -direction	10.4.3
$(W_e, W_m)$	energy stored in electromechanical coupling	3.1.1
$(W'_e, W'_m, W')$	coenergy stored in electromechanical coupling	3.1.2b
$W''$	hybrid energy function	5.2.1
$w$	width	5.2.2
$w$	energy density	11.5.2c
$w'$	coenergy density	8.5
$X$	equilibrium position	5.1.2a
$(x, x_1, x_2, \dots, x_k)$	displacement of mechanical node	2.1.1
$x$	dependent variable	5.1.1
$x_p$	particular solution of differential equation	5.1.1
$(x_1, x_2, x_3), (x, y, z)$	cartesian coordinates	8.1, 6.1
$(x', y', z')$	cartesian coordinates of moving frame	6.1
$(\alpha, \beta)$	constants along $C^+$ and $C^-$ characteristics, respectively	9.1.1
$(\alpha, \beta)$	see (10.2.20) or (10.2.27)	
$\alpha$	transverse wavenumber	11.4.3
$(\alpha, \beta)$	angles used to define shear strain	11.2
$(\alpha, \beta)$	constant angles	4.1.6b
$\alpha$	space decay parameter	7.1.4
$\alpha$	damping constant	5.1.2b
$\alpha$	equilibrium angle of torsional spring	2.2.1a
$\gamma$	ratio of specific heats	13.1.2
$\gamma$	piezoelectric constant	11.5.2c
$\gamma, \gamma_0, \gamma'$	angular position	
$\Delta_d(t)$	slope excitation of string	10.2.1b
$\Delta_0$	amplitude of sinusoidal slope excitation	10.2.1b
$\Delta r$	distance between unstressed material points	11.2.1a
$\Delta s$	distance between stressed positions of material points	11.2.1a
$\delta( )$	incremental change in ( )	8.5
$\delta, \delta_1, \delta_0$	displacement of elastic material	11.1, 9.1, 11.4.2a
$\delta$	thickness of incremental volume element	6.2.1
$\delta$	torque angle	4.1.6a

Symbol	Meaning	Section
$\delta_{ij}$	Kronecker delta	8.1
$(\delta_+, \delta_-)$	wave components traveling in the $\pm x$ -directions	9.1.1
$\epsilon$	linear permittivity	1.1.1b
$\epsilon_0$	permittivity of free space	1.1.1a
$\eta$	efficiency of an induction motor	4.1.6b
$\eta$	second coefficient of viscosity	14.1.1c
$\theta, \theta_i, \theta_m$	angular displacement	2.1.1, 3.1.1, 5.2.1
$\theta$	power factor angle; phase angle between current and voltage	4.1.6a
$\theta$	equilibrium angle	5.2.1
$\dot{\theta}$	angular velocity of armature	6.4.1
$\theta_m$	maximum angular deflection	5.2.1
$(\lambda, \lambda_1, \lambda_2, \dots, \lambda_k)$	magnetic flux linkage	2.1.1, 6.4.1, 4.1.7,
$\lambda_a$		4.1.3, 4.1
$(\lambda_a, \lambda_b, \lambda_c)$		
$(\lambda_{aT}, \lambda_{aS}, \lambda_{bT}, \lambda_{bS})$		
$(\lambda_T, \lambda_S)$		
$\lambda$	Lamé constant for elastic material	11.2.3
$\lambda$	wavelength	7.1.4
$\mu$	linear permeability	1.1.1a
$\mu, (\mu_+, \mu_-)$	mobility	12.3.1, 1.1.1b
$\mu$	coefficient of viscosity	14.1.1
$\mu_d$	coefficient of dynamic friction	2.2.1b
$\mu_0$	permeability of free space	1.1.1a
$\mu_s$	coefficient of static friction	2.2.1b
$\nu$	Poisson's ratio for elastic material	11.2.2
$\nu$	damping frequency	10.1.4
$(\xi, \xi)$	continuum displacement	8.5
$\xi_0$	initial deflection of string	9.2
$\xi_d$	amplitude of sinusoidal driving deflection	9.2
$(\xi_n(x), \hat{\xi}_n(x))$	$n$ th eigenfunctions	9.2.1b
$(\xi_+, \xi_-)$	amplitudes of forward and backward traveling waves	9.2
$\dot{\xi}_0(x)$	initial velocity of string	9.2
$\rho$	mass density	2.2.1c
$\rho_f$	free charge density	1.1.1a
$\rho_s$	surface mass density	11.3
$\Sigma$	surface of discontinuity	6.2
$\sigma$	conductivity	1.1.1a
$\sigma_f$	free surface charge density	1.1.1a
$\sigma_m$	surface mass density of membrane	9.2
$\sigma_0$	surface charge density	7.2.3
$\sigma_s$	surface conductivity	1.1.1a
$\sigma_u$	surface charge density	7.2.3
$\tau$	surface traction	8.2.1
$\tau, \tau_d$	diffusion time constant	7.1.1, 7.1.2a
$\tau$	relaxation time	7.2.1a

Symbol	Meaning	Section
$\tau_e$	electrical time constant	5.2.2
$\tau_m$	time for air gap to close	5.2.2
$\tau_o$	time constant	5.1.3
$\tau_t$	traversal time	7.1.2a
$\phi$	electric potential	7.2
$\phi$	magnetic flux	2.1.1
$\phi$	cylindrical coordinate	2.1.1
$\phi$	potential for H when $J_f = 0$	8.5.2
$\phi$	flow potential	12.2
$\chi_e$	electric susceptibility	1.1.1b
$\chi_m$	magnetic susceptibility	1.1.1a
$\psi$	the divergence of the material displacement	11.4
$\psi$	angle defined in Fig. 6.4.2	6.4.1
$\psi$	angular position in the air gap measured from stator winding (a) magnetic axis	4.1.4
$\psi$	electromagnetic force potential	12.2
$\psi$	angular deflection of wire	10.4.3
$\Omega$	equilibrium rotational speed	5.1.2b
$\Omega$	rotation vector in elastic material	11.2.1a
$\Omega_n$	real part of eigenfrequency (10.1.47)	10.1.4
$\omega, (\omega_r, \omega_s)$	radian frequency of electrical excitation	4.1.6a, 4.1.2
$\omega$	natural angular frequency (Im s)	5.1.2b
$\omega, \omega_m$	angular velocity	2.2.1c, 4.1.2
$\omega_c$	cutoff frequency for evanescent waves	10.1.2
$\omega_d$	driving frequency	9.2
$\omega_n$	nth eigenfrequency	9.2
$\omega_o$	natural angular frequency	5.1.3
$(\omega_r, \omega_i)$	real and imaginary parts of $\omega$	10.0
$\nabla$	nabla	6.1
$\nabla_\Sigma$	surface divergence	6.2.1

1.8. B/c-k

## Appendix G

# SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

### IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi \nabla\psi + \psi \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

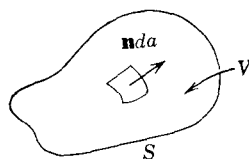
**THEOREMS**

$$\int_a^b \nabla \phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



Divergence theorem

$$\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV$$



Stokes's theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da$$

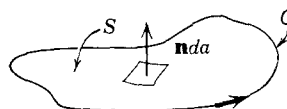
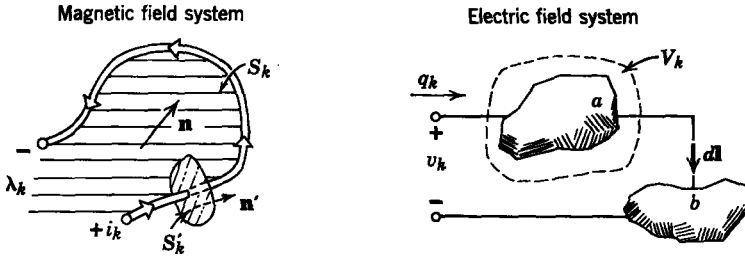




Table 1.2 Summary of Quasi-Static Electromagnetic Equations

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da$	(1.1.23)
			where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V \rho_f \, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da$	(1.1.27)
			where $\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}$ $\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$	

**Table 2.1 Summary of Terminal Variables and Terminal Relations**



**Definition of Terminal Variables**

Flux

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da$$

Current

$$i_k = \int_{S_k'} \mathbf{J}_f \cdot \mathbf{n}' \, da$$

Charge

$$q_k = \int_{V_k} \rho_f \, dV$$

Voltage

$$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

**Terminal Conditions**

$$v_k = \frac{d\lambda_k}{dt}$$

$$\lambda_k = \lambda_k(i_1 \cdots i_N; \text{geometry})$$

$$i_k = i_k(\lambda_1 \cdots \lambda_N; \text{geometry})$$

$$i_k = \frac{dq_k}{dt}$$

$$q_k = q_k(v_1 \cdots v_N; \text{geometry})$$

$$v_k = v_k(q_1 \cdots q_N; \text{geometry})$$

**Table 3.1 Energy Relations for an Electromechanical Coupling Network with N Electrical and M Mechanical Terminal Pairs\***

Magnetic Field Systems	Electric Field Systems
Conservation of Energy	
$dW_m = \sum_{j=1}^N i_j d\lambda_j - \sum_{j=1}^M f_j^e dx_j$	$(a) \quad dW_e = \sum_{j=1}^N v_j dq_j - \sum_{j=1}^M f_j^e dx_j$
$dW'_m = \sum_{j=1}^N \lambda_j di_j + \sum_{j=1}^M f_j^e dx_j$	$(c) \quad dW'_e = \sum_{j=1}^N q_j dv_j + \sum_{j=1}^M f_j^e dx_j$
Forces of Electric Origin, $j = 1, \dots, M$	
$f_j^e = - \frac{\partial W_m(\lambda_1, \dots, \lambda_N; x_1, \dots, x_M)}{\partial x_j}$	$(e) \quad f_j^e = - \frac{\partial W_e(q_1, \dots, q_N; x_1, \dots, x_M)}{\partial x_j}$
$f_j^e = \frac{\partial W'_m(i_1, \dots, i_N; x_1, \dots, x_M)}{\partial x_j}$	$(g) \quad f_j^e = \frac{\partial W'_e(v_1, \dots, v_N; x_1, \dots, x_M)}{\partial x_j}$
Relation of Energy to Coenergy	
$W_m + W'_m = \sum_{j=1}^N \lambda_j i_j$	$(i) \quad W_e + W'_e = \sum_{j=1}^N v_j q_j$
Energy and Coenergy from Electrical Terminal Relations	
$W_m = \sum_{j=1}^N \int_0^{\lambda_j} i_j(\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, 0, \dots, 0; x_1, \dots, x_M) d\lambda'_j$	$(k) \quad W_e = \sum_{j=1}^N \int_0^{q_j} v_j(q_1, \dots, q_{j-1}, q'_j, 0, \dots, 0; x_1, \dots, x_M) dq'_j$
$W'_m = \sum_{j=1}^N \int_0^{i_j} \lambda_j(i_1, \dots, i_{j-1}, i'_j, 0, \dots, 0; x_1, \dots, x_M) di'_j$	$(m) \quad W'_e = \sum_{j=1}^N \int_0^{v_j} q_j(v_1, \dots, v_{j-1}, v'_j, 0, \dots, 0; x_1, \dots, x_M) dv'_j$

\* The mechanical variables  $f_j$  and  $x_j$  can be regarded as the  $j$ th force and displacement or the  $j$ th torque  $T_j$  and angular displacement  $\theta_j$ .

**Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media**

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}'_f = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_{\Sigma} \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho'_f = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_{\Sigma} \cdot \mathbf{K}_f = v_n(\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)			

**From Chapter 8; The Stress Tensor and Related Tensor Concepts**

In what follows we assume a right-hand cartesian coordinate system  $x_1, x_2, x_3$ . The component of a vector in the direction of an axis carries the subscript of that axis. When we write  $F_m$  we mean the  $m$ th component of the vector  $F$ , where  $m$  can be 1, 2, or 3. When the index is repeated in a single term, it implies summation over the three values of the index

$$\frac{\partial H_n}{\partial x_n} = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = \nabla \cdot \mathbf{H}$$

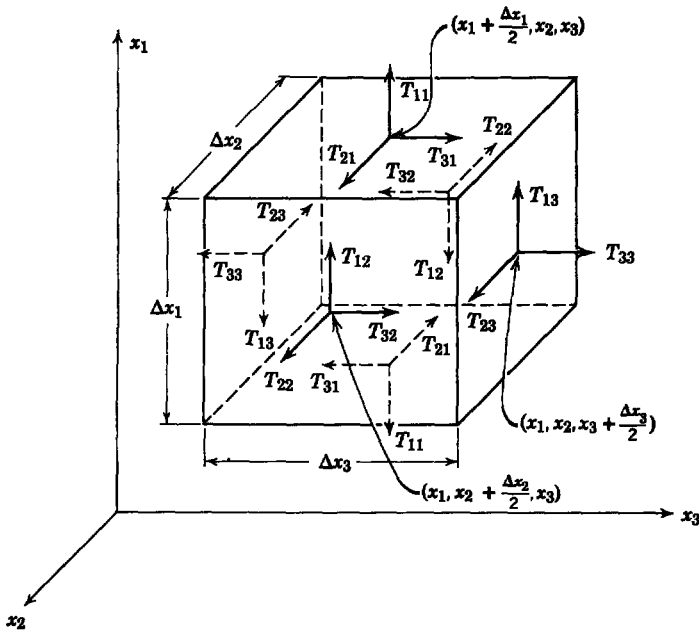
and

$$H_n \frac{\partial}{\partial x_n} = H_1 \frac{\partial}{\partial x_1} + H_2 \frac{\partial}{\partial x_2} + H_3 \frac{\partial}{\partial x_3} = \mathbf{H} \cdot \nabla.$$

This illustrates the *summation convention*. On the other hand,  $\partial H_m / \partial x_n$  represents any one of the nine possible derivatives of components of  $\mathbf{H}$  with respect to coordinates. We define the *Kronecker delta*  $\delta_{mn}$  which has the values

$$\delta_{mn} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } m \neq n. \end{cases} \quad (8.1.7)$$

The component  $T_{mn}$  of the stress tensor can be physically interpreted as the  $m$ th component of the traction (force per unit area) applied to a surface with a normal vector in the  $n$ -direction.



**Fig. 8.2.2** Rectangular volume with center at  $(x_1, x_2, x_3)$  showing the surfaces and directions of the stresses  $T_{mn}$ .

The  $x_1$ -component of the total force applied to the material within the volume of Fig. 8.2.2 is

$$\begin{aligned}
 f_1 = & T_{11} \left( x_1 + \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 - T_{11} \left( x_1 - \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 \\
 & + T_{12} \left( x_1, x_2 + \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 - T_{12} \left( x_1, x_2 - \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 \\
 & + T_{13} \left( x_1, x_2, x_3 + \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2 - T_{13} \left( x_1, x_2, x_3 - \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2.
 \end{aligned} \tag{8.2.3}$$

Here we have evaluated the components of the stress tensor at the centers of the surfaces on which they act; for example, the stress component  $T_{11}$  acting on the top surface is evaluated at a point having the same  $x_2$ - and  $x_3$ -coordinates as the center of the volume but an  $x_1$  coordinate  $\Delta x_1/2$  above the center.

The dimensions of the volume have already been specified as quite small. In fact, we are interested in the limit as the dimensions go to zero. Consequently, each component of the stress tensor is expanded in a Taylor series about the value at the volume center with only linear terms in each series retained to write (8.2.3) as

$$\begin{aligned}
 f_1 = & \left( T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} - T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} \right) \Delta x_2 \Delta x_3 \\
 & + \left( T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} - T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} \right) \Delta x_1 \Delta x_3 \\
 & + \left( T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} - T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2
 \end{aligned}$$

or

$$f_1 = \left( \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2 \Delta x_3. \tag{8.2.4}$$

All terms in this expression are to be evaluated at the center of the volume ( $x_1, x_2, x_3$ ). We have thus verified our physical intuition that space-varying stress tensor components are necessary to obtain a net force.

From (8.2.4) we can obtain the  $x_1$ -component of the force density  $\mathbf{F}$  at the point ( $x_1, x_2, x_3$ ) by writing

$$F_1 = \lim_{\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0} \frac{f_1}{\Delta x_1 \Delta x_2 \Delta x_3} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}. \tag{8.2.5}$$

The limiting process makes the expansion of (8.2.4) exact. The summation convention is used to write (8.2.5) as

$$F_1 = \frac{\partial T_{1n}}{\partial x_n}. \quad (8.2.6)$$

A similar process for the other two components of the force and force density yields the general result that the  $m$ th component of the force density at a point is

$$F_m = \frac{\partial T_{mn}}{\partial x_n}. \quad (8.2.7)$$

Now suppose we wish to find the  $m$ th component of the total force  $\mathbf{f}$  on material contained within the volume  $V$ . We can find it by performing the volume integration:

$$f_m = \int_V F_m dV = \int_V \frac{\partial T_{mn}}{\partial x_n} dV. \quad (8.1.13)$$

When we define the components of a vector  $\mathbf{A}$  as

$$A_1 = T_{m1}, \quad A_2 = T_{m2}, \quad A_3 = T_{m3}, \quad (8.1.14)$$

we can write (8.1.13) as

$$f_m = \int_V \frac{\partial A_n}{\partial x_n} dV = \int_V (\nabla \cdot \mathbf{A}) dV. \quad (8.1.15)$$

We now use the divergence theorem to change the volume integral to a surface integral,

$$f_m = \oint_S \mathbf{A} \cdot \mathbf{n} da = \oint_S A_n n_n da, \quad (8.1.16)$$

where  $n_n$  is the  $n$ th component of the outward-directed unit vector  $\mathbf{n}$  normal to the surface  $S$  and the surface  $S$  encloses the volume  $V$ . Substitution from (8.1.14) back into this expression yields

$$f_m = \oint_S T_{mn} n_n da. \quad (8.1.17)$$

where  $T_{mn} n_n$  is the  $m$ th component of the surface traction  $\boldsymbol{\tau}$ .

The traction  $\boldsymbol{\tau}$  is a vector. The components of this vector depend on the coordinate system in which  $\boldsymbol{\tau}$  is expressed; for example, the vector might be directed in one of the coordinate directions  $(x_1, x_2, x_3)$ , in which case there would be only one nonzero component of  $\boldsymbol{\tau}$ . In a second coordinate system  $(x'_1, x'_2, x'_3)$ , this same vector might have components in all of the coordinate directions. Analyzing a vector into orthogonal components along the coordinate axes is a familiar process. The components in a cartesian coordinate system  $(x'_1, x'_2, x'_3)$  are related to those in the cartesian coordinate system  $(x_1, x_2, x_3)$  by the three equations

$$\tau'_p = a_{pr} \tau_r, \quad (8.2.10)$$

where  $a_{pr}$  is the cosine of the angle between the  $x'_p$ -axis and the  $x_r$ -axis.

Similarly, the components of the stress tensor transform according to the equation

$$T'_{pq} = a_{pr}a_{qs}T_{rs}. \quad (8.2.17)$$

This relation provides the rule for finding the components of the stress in the primed coordinates, given the components in the unprimed coordinates. It serves the same purpose in dealing with tensors that (8.2.10) serves in dealing with vectors.

Equation 8.2.10 is the transformation of a vector  $\tau$  from an unprimed to a primed coordinate system. There is, in general, nothing to distinguish the two coordinate systems. We could just as well define a transformation from the primed to the unprimed coordinates by

$$\tau_s = b_{sp}\tau'_p, \quad (8.2.18)$$

where  $b_{sp}$  is the cosine of the angle between the  $x_s$ -axis and the  $x'_p$ -axis. But  $b_{sp}$ , from the definition following (8.2.10), is then also

$$b_{sp} \equiv a_{ps}; \quad (8.2.19)$$

that is, the transformation which reverses the transformation (8.2.10) is

$$\tau_s = a_{ps}\tau'_p. \quad (8.2.20)$$

Now we can establish an important property of the direction cosines  $a_{ps}$  by transforming the vector  $\tau$  to an arbitrary primed coordinate system and then transforming the components  $\tau'_m$  back to the unprimed system in which they must be the same as those we started with. Equation 8.2.10 provides the first transformation, whereas (8.2.20) provides the second; that is, we substitute (8.2.10) into (8.2.20) to obtain

$$\tau_s = a_{ps}a_{pr}\tau_r. \quad (8.2.21)$$

Remember that we are required to sum on both  $p$  and  $r$ ; for example, consider the case in which  $s = 1$ :

$$\begin{aligned} \tau_1 &= (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})\tau_1 \\ &\quad + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\tau_2 \\ &\quad + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\tau_3. \end{aligned} \quad (8.2.22)$$

This relation must hold in general. We have not specified either  $a_{ps}$  or  $\tau_m$ . Hence the second two bracketed quantities must vanish and the first must be unity. We can express this fact much more concisely by stating that in general

$$a_{ps}a_{pr} = \delta_{sr} \quad (8.2.23)$$



**Table 8.1 Electromagnetic Force Densities, Stress Tensors, and Surface Force Densities for Quasi-static Magnetic and Electric Field Systems\***

Description	Force Density $\mathbf{F}$	Stress Tensor $T_{mn}$ $F_m = \frac{\partial T_{mn}}{\partial x_n}$ (8.1.10)	Surface Force Density* $T_m = [T_{mn}]n_n$ (8.4.2)
Force on media carrying free current density $\mathbf{J}_f$ , $\mu$ constant	$\mathbf{J}_f \times \mathbf{B}$ (8.1.3)	$T_{mn} = \mu H_m H_n - \delta_{mn} \frac{1}{2} \mu H_k H_k$ (8.1.11)	$\mathbf{T} = \mathbf{K}_f \times \mu \langle \mathbf{H} \rangle$ $\mathbf{K}_f = \mathbf{n} \times [\mathbf{H}]$ (8.4.3)
Force on media supporting free charge density $\rho_f$ , $\epsilon$ constant	$\rho_f \mathbf{E}$ (8.3.3)	$T_{mn} = \epsilon E_m E_n - \delta_{mn} \frac{1}{2} \epsilon E_k E_k$ (8.3.10)	$\mathbf{T} = \sigma_f \langle \mathbf{E} \rangle$ $\sigma_f = \mathbf{n} \cdot [\epsilon \mathbf{E}]$ (8.4.8)
Force on free current plus magnetization force in which $\mathbf{B} = \mu \mathbf{H}$ both before and after media are deformed	$\mathbf{J}_f \times \mathbf{B} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla \mu$ $+ \frac{1}{2} \nabla \left( \mathbf{H} \cdot \mathbf{H} \rho \frac{\partial \mu}{\partial \rho} \right)$ (8.5.38)	$T_{mn} = \mu H_m H_n$ $- \frac{1}{2} \delta_{mn} \left( \mu - \rho \frac{\partial \mu}{\partial \rho} \right) H_k H_k$ (8.5.41)	
Force on free charge plus polarization force in which $\mathbf{D} = \epsilon \mathbf{E}$ both before and after media are deformed	$\rho_f \mathbf{E} - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \nabla \epsilon$ $+ \frac{1}{2} \nabla \left( \mathbf{E} \cdot \mathbf{E} \rho \frac{\partial \epsilon}{\partial \rho} \right)$ (8.5.45)	$T_{mn} = \epsilon E_m E_n$ $- \frac{1}{2} \delta_{mn} \left( \epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) E_k E_k$ (8.5.46)	

\*  $\langle \mathbf{A} \rangle \equiv \frac{\mathbf{A}^a + \mathbf{A}^b}{2}$

$[\mathbf{A}] \equiv \mathbf{A}^a - \mathbf{A}^b$

**Table 9.1 Modulus of Elasticity  $E$  and Density  $\rho$  for Representative Materials\***

Material	$E$ -units of $10^{11}$ N/m <sup>2</sup>	$\rho$ -units of $10^3$ kg/m <sup>3</sup>	$v_p$ -units† of m/sec
Aluminum (pure and alloy)	0.68–0.79	2.66–2.89	5100
Brass (60–70% Cu, 40–30% Zn)	1.0–1.1	8.36–8.51	3500
Copper	1.17–1.24	8.95–8.98	3700
Iron, cast (2.7–3.6% C)	0.89–1.45	6.96–7.35	4000
Steel (carbon and low alloy)	1.93–2.20	7.73–7.87	5100
Stainless steel (18% Cr, 8% Ni)	1.93–2.06	7.65–7.93	5100
Titanium (pure and alloy)	1.06–1.14	4.52	4900
Glass	0.49–0.79	2.38–3.88	4500
Methyl methacrylate	0.024–0.034	1.16	1600
Polyethylene	$1.38\text{--}3.8 \times 10^{-3}$	0.915	530
Rubber	$0.79\text{--}4.1 \times 10^{-5}$	0.99–1.245	46

\* See S. H. Crandall, and N. C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw-Hill, New York, 1959, for a list of references for these constants and a list of these constants in English units.

† Computed from average values of  $E$  and  $\rho$ .

**Table 9.2 Summary of One-Dimensional Mechanical Continua  
Introduced in Chapter 9**

Thin Elastic Rod
$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} + F_x$ $T = E \frac{\partial \delta}{\partial x}$ <p> <math>\delta</math>—longitudinal (<math>x</math>) displacement  <math>T</math>—normal stress  <math>\rho</math>—mass density  <math>E</math>—modulus of elasticity  <math>F_x</math>—longitudinal body force density                 </p>
Wire or "String"
$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S_z$ <p> <math>\xi</math>—transverse displacement  <math>m</math>—mass/unit length  <math>f</math>—tension (constant force)  <math>S_z</math>—transverse force/unit length                 </p>
Membrane
$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + T_z$ <p> <math>\xi</math>—transverse displacement  <math>\sigma_m</math>—surface mass density  <math>S</math>—tension in <math>y</math>- and <math>z</math>-directions                      (constant force per unit length)  <math>T_z</math>—<math>z</math>-directed force per unit area                 </p>

# INDEX

Numbers preceded by letters are Appendix references. Appendices A, B, and C are in Part One; Appendices D and E, Part Two; and Appendices F and G, Part Three.

- Acceleration, centrifugal fluid, 729
  - centripetal, 59
  - Coriolis, 59
  - Eulerian variable, 727
  - fluid, 727
  - instantaneous, 45
- Accelerator, electric field, 776
  - MHD, 825
  - particle, 608
- Acoustic delay lines, 480
- Acoustic waves, compressional in solid, 673
  - dilatational in solid, 673
  - elastic media, 671
  - fluid, 544
  - gases, 845
  - guided, 679, 683, 693
  - magnetic fields and, 846
  - membrane, 509
  - shear elastic, 675
  - string, 509
  - thin beam, 683
  - thin rod, 487, 681
- Acyclic machine, 286
- Air-gap magnetic fields, 114
- Alfvén velocity, 763
- Alfvén waves, 759
  - compressible fluids and, 841
  - cylindrical geometry, 767
  - effect of conductivity on, 772
  - mechanical analogue to, 766
  - nature of, 764
  - numerical example of, 771
  - resonances and, 771
  - standing, 771
  - torsional, 765
- Amortisseur winding, 164
- Ampère, 1
- Ampère's law, B6, C3, E3, G3
  - dynamic, B9
  - electromechanical, 304
  - example of, B7
  - integral form of, B36, C3, E3, G3
  - magnetization and, B26
- Amplifying wave, coupled system and, 608
  - electric field induced, 605
  - evanescent wave and, 607
  - space-time behavior of, 604, 606
- Angular frequency, 513
- Angular momentum, 248
- Angular velocity, 47
- Applications of electromechanics, 2
- Approximations, electromechanical, 206
- Armature, ac machine, 120
  - dc machine, 141, 293
- Armature reaction, 297
- Astrophysics and MHD, 552
- Attenuation, microwave, 561
- Average power converted, salient pole machine, 155
  - smooth-air-gap machine, 124
- Beats in space, 595
- Bernoulli's equation, 738
  - example of, 752
- Bessel functions, 408
  - roots of, 409
- Bias, linear transducer operation and, 201
- piezoelectricity and, 711
- Bode plot, 206
- Boundary, analytic description of, 269, 668
  - examples of moving, 269, 276, 279, 280, 364, 392, 397, 451, 460, 563, 574, 605, 627, 704, 783
  - moving, 267
  - well defined, 267
- Boundary condition, Alfvén waves, 769
- causality and, 491, 592, 607
- conservation of charge, 279, 374, 376, 394, 399
  - convection and, 267, 587, 598
  - dispersion and, 618
  - elastic media, 671, 676
  - electric displacement, 278
  - electric field intensity, 275, 278
  - electric field systems, 277, E6, G6
  - electromagnetic field, 267
  - electromechanical, 668
  - field transformations and, 275
  - geometric effect of, 280
  - initial condition and, 513
  - inviscid fluid, 752
  - inviscid fluid slip, 740
  - longitudinal and transverse, 680
  - magnetic field intensity, 273, 280
  - magnetic field systems, 270, E6, G6
  - magnetic field system current, 272
  - magnetic fluid, 774
  - magnetic flux density, 271
  - MHD, 769
  - motion and, 267, 491, 587, 592, 598, 607
  - string and membrane, 522
  - summary of electromagnetic, 268, E6, G6
  - thin rod, 493
  - viscous fluid, 873
- Boundary layer dynamics, 602
- Brake, induction, 134
  - MHD, 744
- Breakdown, electrical, 576, 782
- Breakdown strength of air, 576

- Brush, dc machine, 292
  - liquid-metal, 316, 878
  - metal-graphite, 883
- Bullard's equation, 336
- Cables, charge relaxation in high voltage, 380
  - nonuniform conductivity in, 380
- Capability curve of synchronous generator, 170
- Capacitance, electrical linearity and, 30
  - example for calculation of, 32, 33
  - generalized, 28
  - quasi-static limit and, B18
- Causality, boundary conditions and, 592, 607
  - condition of, 491, 592, 607
- Center of mass, 46
- Channel, variable-area MHD, 751
- Characteristic dynamical times, excitation and, 332
  - material motion and, 332
- Characteristic equation, 181
- Characteristics, wave fronts and, 618
  - wave propagation and, 488, 490
  - waves with convection and, 586
- Charge, B1
  - conservation of, B5
  - net flow and flow of net, B6
  - test, 12
  - total, 29
- Charge-average velocity, B5
- Charge carriers, effect of motion on, 290
- Charge conservation, differential form of, B5
  - integral form of, B5
- Charge density, B1
  - effect of motion on, 290, 334, 382, 387, 388, 392, 397, 401
  - free, 7, B28
  - magnetic field system and, 288
- Charge distribution, effect of motion on, 334, 382, 387, 388, 392, 397, 401
- Charge relaxation, 330, 370
  - electrical transient, 372
  - examples of, 372, 375
  - excitation frequency and, 378, 400
  - frequency in frame of material and, 399
  - general equation for, 371
  - lumped-parameter models for, 331, 375
  - magnetic diffusion and, 401
  - motion sinusoidal excitation with, 392
  - moving frame and, 381
  - nonuniform properties and, 378
  - sources of charge and, 372
  - spatially and temporally periodic fields and, 397
  - steady motion and, 380
  - thunder storms, and, 388
  - traveling wave in a moving material and, 397
  - uniform properties and, 372
- Choking, constant area flow, 824
- Circuit breaker, transducer for a, 22
- Circuit theory, 16
- Coefficient, of sliding friction, 42
  - of static friction, 42
- Coefficients of viscosity, independence of, 870
- Coenergy, 73, E5, G5
  - electrical linearity and, 76
  - potential well motions and, 217
- Coenergy density, electric field system, 464, 714
  - magnetic field system, 456
- Collector rings, 120
- Commutation in dc machines, 296
- Commutator, 140
  - of dc machines, 292
- Commutator bars, 142
- Commutator machines, 140
  - ac generator, 329
  - brake operation of, 306
  - compound wound, 310
  - electrical power input of, 303
  - equation for armature of, 300
  - equation for field of, 297
  - equation of motion for, 297
  - generator operation of, 306
  - linear amplifier, 304
  - mechanical power output of, 303
  - motor operation of, 306
  - operation with alternating currents and, 312
  - properties of, 303
  - separately excited, 306
  - series excitation of, 309
  - shunt excitation of, 309
  - speed curves of, shunt excited, 310
  - speed regulation of, 307
  - summary of equations for, 303
  - torque-current curves of series excited, 311
  - torque-speed curves of shunt excited, 310
  - transient performance of, 306
- Compensating networks, 198
- Compensation in feedback loops, 198
- Compressibility constant, 845
- Compressibility of fluid, 725
- Compressible fluids, 813
  - electromechanical coupling to, 820
- Conduction, electrical, 7, B30
  - in electric field system, effect of motion on, 371
  - heat, 815
  - motion and electrical, 284, 289
- Conduction current, B6
  - absence of net free charge and, 374
- Conduction machine, MHD, 740
  - variable area, MHD, 753
  - see also* Commutator machine; DC machines
- Conductivity, air and water, 388
  - electrical, 7
  - electrical surface, 7
  - mechanical strength and, 698
  - nonuniform, 380
  - numerical values of, 345, 377
- Conductor, electric field perfect, 29, 213,

- 390, 400, 401
- magnetic field perfect, 18, 211, 223, 354, 401, 563
- Confinement, electromechanical, 4, 407
- Conservation, of charge, B5
  - displacement current and, B9
  - integral form of, B37
- of energy, 63, 66
  - continuum, 456, 464
  - continuum coupling and, 455
  - equation, steady state, 820
  - fluid, 814
  - incompressible fluid, 757
  - integral form of, 819
- of flux, lumped-parameter, 211, 220
- perfectly conducting fluid and, 761
- of mass, differential law of, 731
  - example of, 730
  - fluid, 729, 814
  - integral form of, 730
- of momentum, fluid, 731, 814
  - integral form of, 733, 734
  - interfacial, 671
  - stress and, 733
- Conservative systems, 213
- Constant charge dynamics, 205, 213
- Constant-current constant-flux dynamics, 220
- Constant-current constraint, continuum, 628
- Constant-current dynamics, 220
- Constant flux, lumped and continuum, 212
- Constant flux dynamics, fluid, 761
  - lumped-parameter, 211, 220
- Constant of the motion, fluid, 738
- Constant voltage dynamics, 204, 212, 226
- Constituent relations, electromagnetic, 283, B25
  - fluid, 815
  - fluid mechanical, 735
  - materials in motion and, 283
  - moving media in electric field systems and, 289
  - moving media in magnetic field systems and, 284
- Constitutive law, mobile ion, 778
  - piezoelectric slab, 712
- Contact resistance, MHD, 750
- Contacts, sliding, 42
- Continuity of space, 35
- Continuum and discrete dynamics, 553
- Continuum descriptions, 727
- Continuum electromechanical systems, 251
- Contour, deforming, 11, B32
- Control, dc machines and, 291
- Controlled thermonuclear reactions, 354
- Convection, dynamical effect of, 584
  - and instability, 593
- Convection current, B6
- Convective derivative, 259, 584
  - charge relaxation and, 381
  - example of, 729
  - magnetic diffusion and, 357
  - see also* Substantial derivative
- Convective second derivative, 585
- Coordinate system, inertial, 254
- Corona discharge, 776, 782
- Corona wind, demonstration of, 782
- Couette flow, plane, 876
- Coulomb's law, B1
  - point charge, B2
- Coupling, electromechanical, 15, 60
- Coupling to continuous media at terminal pairs, 498
- Coupling network, lossless and conservative, 63
- Creep, failure in solids by, 704
- Critical condition for instability, 568
- Crystals, electromechanics of, 651
  - piezoelectric materials as, 711
- Current, balanced two-phase, 113
  - conduction, B6
  - convection, B6
  - displacement, B9
  - electric field system, 29
  - free, B25
  - magnetization, B25
  - polarization, B29
- Current density, B5
  - diffusion of, 343
  - distribution of, 332
  - free, 7
- Current law, Kirchhoff's, 16
- Currents as functions of flux linkages, 26
- Current transformation, examples of, 226
- Cutoff condition, 559
- Cutoff frequency, 559
  - elastic shear waves, 695
  - membrane, 623
- Cutoff waves, 556
  - electromagnetic plasma, 638
  - membrane, 623
  - power flow and, 637
  - thin beam, 684
  - see also* Evanescent wave
- Cyclic energy conversion processes, 79
- Cylindrical coordinates, stress components in, 437
- Cylindrical modes, 648
- Damped waves, driven response of, 577
- Damper, linear ideal, 40
  - lumped element, 36, 40
  - square-law, 43, 229
- Damper winding in rotating machine, 164
- Damping, magnetic fluid, 750
  - negative, 198
  - spatial decay and, 560
  - wave dynamics with, 576
- Damping constant, 41
- Damping frequency, 577
- DC generator, magnetic saturation in, 310
  - self-excited, 310
- DC machines, 140; *see also* Commutator machines
- DC motor, self-excited, 308
  - series excited, 311

- starting torque of, 310
- torque-speed curves for, 306
- Definitions, electromagnetic, 7, B1
- Deforming contours of integration, 10, 18, 262, B32, 761
- Degree of freedom, 49
- Delay line, acoustic, 480
  - acoustic magnetostrictive, 708
  - fidelity in, 501
  - mechanical, 499
  - shear waves and, 696
- Delta function, B2
  - Kronecker, 421
- Derivative, convective, 259, 584, 726
  - individual, 728
  - particle, 728
  - Stokes, 728
  - substantial, 259, 584, 728
  - total, 728
- Dielectrophoresis, 783
- Difference equation, 620
- Differential equation, order of, 180
  - linear, 180
- Differential operators, moving coordinates and, 257
- Diffusion, magnetic, 576
  - magnetic analogous to mechanical, 580
  - of magnetic field and current, 335
- Diffusion equation, 337
- Diffusion time constant, 341
  - numerical values of, 344
- Diffusion wave, magnetic, 358
  - picture of, 581
  - space-time behavior of, 359
- Dilatational motion of fluid, 866
- Direction cosines, definition of, 435
  - relation among, 439
- Discrete systems, electromechanics of, 60
- Discrete variables, mechanical, 36
  - summary of electrical, 35
- Dispersion equation, absolutely unstable wire, 567
  - Alfvén wave, 769
  - amplifying wave, 602
  - convective instability, 602
  - damped waves, 577
  - elastic guided shear waves, 695
  - electron oscillations, 601
  - evanescent wave, 557
  - kink instability, 629
  - magnetic diffusion with motion, 357
  - membrane, 623
  - moving wire destabilized by magnetic field, 602
  - moving wire stabilized by magnetic field, 596
  - ordinary waves, 513
    - with convection, 594
    - on wire, 555
    - on wires and membranes, 513
  - resistive wall interactions, 609
  - sinusoidal steady-state, and 514
  - wire with convection and damping, 609
- Displacement, elastic materials, 486
  - elastic media, 652
  - lumped parameter systems, 36
  - one-dimensional, 483
  - relative, 657
  - and rotation, 657
  - and strain, 658
  - transformation of, 659
  - translational, 657
- Displacement current, B9
- Displacement current negligible, B19
- Distributed circuits, electromechanical, 651
- Divergence, surface, 272
  - tensor, 422, G9
  - theorem, B4, C2, E2, G2, G9
- Driven and transient response, unstable system, 569
- Driven response, one-dimensional continuum, 511
  - unstable wire, 568
- Driving function, differential equation, 180
  - sinusoidal, 181
- Dynamics, constant charge, 205, 213
  - constant current, 220
  - constant flux, 211, 220
  - constant voltage, 204, 212, 226
  - lumped-parameter, 179
  - reactance dominated, 138, 211, 220, 242, 336, 354, 368, 563
  - resistance dominated, 138, 209, 233, 242, 336, 354, 368, 503, 583, 611
  - two-dimensional, 621
- Dynamics of continua,  $x-t$  plane, 488, 586
  - $\omega-k$  plane, 511, 554
- Dynamo, electrohydrodynamic, 388
- Eddy currents, 342, 628
- Efficiency of induction machine, 134
- EHD, 3, 552, 776
- EHD pump, demonstration of, 783
- Eigenfrequencies, 518
  - electromechanical filter, 707
  - magnetic field, shift of, 562
  - not harmonic, 563, 684
  - wire stiffened by magnetic field, 562
- Eigenfunction, 518
- Eigenmode, 517
  - complex boundary conditions and, 533
  - orthogonality of, 341, 519, 520
- Eigenvalues, 518
  - dispersion and, 562
  - graphic solution for, 526
  - kink instability, 630
- Elastic beam, resonant circuit element, 688
- Elastic constants, independence of, 664
  - numerical values of, 486
- Elastic continua, 479
- Elastic failure, example of electromechanical, 701
- Elastic force density, 667
- Elastic guiding structures, 693
- Elasticity, summary of equations of, 666, 668
- Elasticity equations, steps in derivation of, 651

- Elastic material, ideal, 485
  - linear, 485
- Elastic media, 651
  - electromechanical design and, 697
  - electromechanics of, 696
  - equations of motion for, 653
  - quasi-statics of, 503
- Elastic model, membrane, 509
  - thin rod, 480
  - wire, 509
- Elastic waves, lumped mechanical elements and, 507
  - shear, 543
  - thin rod, 543
  - see also* Acoustic waves
- Electrical circuits, 16
- Electric displacement, 7, B28
- Electric field, effect of motion on, 334, 382, 387, 388, 392, 397, 401
- Electric field coupling to fluids, 776
- Electric field equations, periodic solution to, 281
- Electric field intensity, 7, B1
- Electric field system, B19
  - differential equations for, 8, E3, G3
  - integral equations for, 11, E3, G3
- Electric field transformation, example of, 262
  - Faraday's law and, 262
- Electric force, field description of, 440
  - fluids and, 776
  - stress tensor for, 441
- Electric force density, 418, 463
- Electric Reynolds number, 335, 370, 381, 383, 395, 399, 401, 575, 780
  - mobility model and, 780
- Electric shear, induced surface charge and, 400
- Electric surface force, 447
- Electrification, frictional, 552
- Electroelasticity, 553
- Electrogasdynamics generator, 782
- Electrohydrodynamic orientation, 785
- Electrohydrodynamic power generation, 782
- Electrohydrodynamics, 3, 552, 776
- Electrohydrodynamic stabilization, 786
- Electromagnetic equations, differential, 6, B12, B19, E3, G3
  - integral, 9, B32, E3, G3
  - quasi-static, 5, B19, B32, E3, G3
  - summary of quasi-static, 13, E3, G3
- Electromagnetic field equations, summary of, 268, E6, G6
- Electromagnetic fields, moving observer and, 254
- Electromagnetic theory, 5, B1
  - summary of, 5, E6, G6
- Electromagnetic waves, B13
  - absorption of, B25
- Electromechanical coupling, field description of, 251
- Electromechanics, continuum, 330
  - of elastic media, 651
  - incompressible fluids and, 737
  - lumped-parameter, 60
- Electron beam, 4, 552, 600, 608
  - magnetic field confinement of, 601
  - oscillations of, 600
- Electrostatic ac generator, 415
- Electrostatic self-excited generator, 388
- Electrostatic voltmeter, 94
- Electrostriction, incompressibility and, 784
- Electrostriction force density, 465
- Elements, lumped-parameter electrical, 16
  - lumped-parameter mechanical, 36
- Energy, conservation of fluid, 814
  - electrical linearity and, 76
  - electric field system conservation of, 66
  - internal or thermal gas, 813
  - internal per unit mass, 815
  - kinetic per unit mass, 815
  - magnetic field system conservation of, 63
  - magnetic stored, 64
  - potential and kinetic, 214
- Energy conversion, cyclic, 79, 110
  - electromechanical, 79
  - lumped-parameter systems, 79
- Energy density, B23
  - equal electric and magnetic, B24
- Energy dissipated, electromagnetic, B22
- Energy flux, B22
- Energy function, hybrid, 220
- Energy method, 60, 450, E5, G5
- Energy relations, summary of, 68, E5, G5
- Enthalpy, specific, 820
- Equation of motion, elastic media, 668
  - electromechanical, 84
  - examples of lumped-parameter, 84, 86
  - incompressible, irrotational inviscid flow, 738
  - linearized, 183
  - lumped mechanical, 49
- Equilibrium, of continuum, stability of, 574
  - dynamic or steady-state, 188
  - hydromagnetic, 561
  - kink instability of, 633
  - potential well stability of, 216
  - static, 182
- Equipotentials, fluid, 752
- Eulerian description, 727
- Evanescence with convection, 596
- Evanescence wave, 556
  - appearance of, 559
  - constant flux and, 563
  - dissipation and, 560
  - elastic shear, 695
  - equation for, 557
  - example of, 556
  - membrane, 560, 623
  - physical nature of, 560
  - signal transmission and, 639
  - sinusoidal steady-state, 558
  - thin beam, 684
- Evil, 697
- Failure in solids, fatigue and creep, 704
- Faraday, 1



- Faraday disk, 286  
 Faraday's law, B9  
   deforming contour of integration and, 262,  
     300, 315, 565, B32, E3, G3  
   differential form, 6, B10, E3, G3  
   example of integral, 262, 276, 286, 297,  
     315  
   integral form of, B10, B32  
   perfectly conducting fluid and, 761
- Fatigue, failure in solids by, 704
- Feedback, continuous media and, 548  
 stabilization by use of, 193
- Ferroelectrics, B29  
   piezoelectric materials and, 711
- Ferrohydrodynamics, 552, 772
- Field circuit of dc machine, 141
- Field equations, moving media, generalization  
 of, 252
- Fields and moving media, 251
- Field transformations, 268, E6, G6;  
*see also* Transformations
- Field winding, ac machine, 120  
 dc machine, 293
- Film, *Complex Waves I*, xi, 516, 559, 571,  
 634
- Film, *Complex Waves II*, xi, 573, 606
- Filter, electromechanical, 2, 200, 480, 704
- First law of thermodynamics, 63
- Flow, Hartmann, 884  
   irrotational fluid, 737  
   laminar, 725  
   turbulent, 725
- Flowmeter, liquid metal, 363
- Fluid, boundary condition for, 725  
   boundary condition on, inviscid, 752  
   compressibility of, 725  
   effect of temperature and pressure on, 724  
   electric field coupled, 776  
   electromechanics of, 724  
   ferromagnetic, 552, 772  
   highly conducting, 760  
   incompressible, 724, 735  
   inhomogeneous, 735  
   internal friction of, 724  
   inviscid, 724, 725  
   laminar and turbulent flow of, 725  
   magnetic field coupling to incompressible,  
     737  
   magnetizable, 772  
   Newtonian, 861  
   perfectly conducting, 563  
   solids and, 724  
   static, 735  
   viscous, 861
- Fluid dynamics, equations of inviscid com-  
 pressible, 813  
 equations of inviscid, incompressible, 726  
 equations of viscous, 871
- Fluid flow, accelerating but steady, 753  
 around a corner, 751  
 potential, 751  
 unsteady, 746  
 variable-area channel, 751
- Fluid-mechanical examples with viscosity,  
 875
- Fluid orientation systems, 785
- Fluid pendulum, electric-field coupled, 784  
   magnetic damping of, 750
- Fluid pump or accelerator, 776
- Fluid stagnation point, 752
- Fluid streamlines, 752
- Fluid transformer, variable-area channel  
 as, 756
- Flux amplification, plasmas and, 354
- Flux conservation, lumped-parameter, 211,  
 220  
   magnetic fields and, 352  
   perfectly conducting gas and, 849
- Flux density, mechanical amplification of,  
 354
- Flux linkage, 19, E4, G4  
 example of, 22, 23
- Force, charge, B1  
   derivative of inductance and, 453  
   electric origin, 67, E5, G5  
   electromagnetic, 12  
   field description of, 418  
   fluid electric, 776  
   Lorentz, 12, 255, 419  
   magnetic, B6  
   magnetization with one degree of free-  
     dom, 451  
   physical significance of electromagnetic,  
     420  
   polarized fluids, 463, 572, 784  
   single ion, 778  
   surface integral of stress and, 422
- Force-coenergy relations, 72, E5, G5
- Force density, 7  
   averaging of electric, 440  
   averaging of magnetic, 419  
   divergence of stress tensor and, 422,  
     427, G9  
   effect of permeability on, 455, 456  
   elastic medium, 667  
   electric, 12, B3, 440, G11  
   magnetic field systems, 419, 462, G11  
   electromagnetic fluid, 732  
   electrostriction, 465, G11  
   fluid mechanical, 732  
   fluid pressure, 736  
   free current, 419, G11  
   inviscid fluid mechanical, 737  
   lumped parameter model for, 455  
   magnetic, 12, 419, B9  
   magnetization, 448, 450, 462, G11  
   magnetostriction, 461, 462, G11  
   polarization, 450, 463, G11  
   summary of, 448, G11
- Forced related to variable capacitance, 75
- Force-energy relations, 67, E5, G5  
 examples of, 70
- Force equations, elastic media, 653
- Force of electric origin, 60, E5, G5
- Fourier series, 340
- Fourier transform, two-dimensional, 617
- Fourier transforms and series, diffusion

- equation and, 340
- eigenmodes as, 517
- linear continuum systems and, 511, 554, 617
- linear lumped systems and, 200
- mutual inductance expansions and, 108, 153
- Frame of reference, laboratory, 254
- Free-body diagram, 49
- Free charge density, B28
- Free charge forces, avoidance of, 787
- Frequency, complex, 181, 554
  - complex angular, 554
  - natural, 181, 515
  - voltage tuning of, 704
- Frequency conditions for power conversion, 111, 155
- Frequency response of transducer, 204
- Friction, coulomb, 42
- Frozen fields, perfectly conducting gas and, 849
- Fusion machines, instability of, 571
- Galilean transformation, 584
- Gamma rays, B13
- Gas, perfect, 816
- Gas constant, 816
  - universal, 816
- Gases, definition of, 724
  - ionized, 813
- Gauss's law, differential form of, B5
  - example of, B4
  - integral form of, B3
  - magnetic field, B12
  - polarization and, B28
- Gauss's theorem, tensor form of, 423, G9
- Generators, electric field, 778
  - electrohydrodynamic applications of, 3
  - hydroelectric, 152
  - induction, 134
  - magnetohydrodynamic applications of, 3
  - MHD, 744
  - Van de Graaff, 3, 383, 385
- Geometrical compatibility, 53
- Geophysics and MHD, 552
- Gravitational potential, 733
- Gravity, artificial electric, 785
  - force density due to, 732
  - waves, 794
- Group velocity, 614
  - power flow and, 638
  - unstable media and, 617
- Guiding structures, evanescence in, 560
- Hartmann flow, 884
- Hartmann number, 887
- Heat transfer, EHD and, 552
- Homogeneity, B27
- Homogeneous differential equation, solution of, 180
- Homopolar machine, 286, 312
  - armature voltage for, 314
  - speed coefficient for, 314
  - summary of equations for, 316
  - torque for, 316
- Hunting transient of synchronous machine, 192
- Hydraulic turbine, 151
- Hydroelectric generator, 152
- Hydromagnetic equilibria, 561, 571
- Hysteresis, magnetic, B27
- Identities, C1, E1, G1
- Impedance, characteristic, 497
- Incompressibility, fluid, 735
- Incompressible fluids, MHD, 737
- Incompressible media, 380
- Incremental motions, *see* Linearization
- Independence of variables, 69, 97 (*see* Problem 3.16)
- Independent variables, change of, 72
- Index notation, 421, G7
- Inductance, calculation of, 22
  - electrical linearity and, 20
  - generalized, 17
  - quasi-static limit and, B18
- Induction, demonstration of motional, 253
  - law of, B9; *see also* Faraday's law
- Induction brake, 134
- Induction generator, 134
  - electric, 400
- Induction interaction, 367
- Induction law, integral, B32; *see also* Faraday's law
- Induction machine, 127
  - coefficient of coupling for, 135
  - distributed linear, 368
  - efficiency of, 134
  - equivalent circuit for, 131
  - loading of, 137
  - lumped-parameter, 368
  - MHD, 745
  - power flow in, 133
  - reactance and resistance dominated, 137
  - single phase, 138
  - squirrel-cage, 129
  - starting of, 137, 139
  - torque in, 132
  - torque-slip curve of, 135
  - variable resistance in, 136
  - wound rotor, 106
- Induction motor, 134
- Inductor, 17
- Inelastic behavior of solids, 699
- Influence coefficients, MHD, 822
  - variable-area MHD machine, 832
- Initial and boundary conditions, 513
- Initial conditions, convection and, 587
  - one-dimensional continuum, 488, 512
- Initial value problem, continuum, 488
- Instability, absolute, 566
  - and convective, 604
  - aeroelastic absolute, 793
  - convective, 601
  - dynamic, 192
  - electrohydrodynamic, 571
  - engineering limitations from convective, 604

- and equilibrium, example of, 185
- failure of a static argument to predict, 192
- fluid pendulum, 785
- fluid turbulence and, 725
- graphical determination of, 184
- heavy on light fluid, 571
- and initial conditions, 184
- kink, 627
- linear and nonlinear description of, 216
- nonconvective, 566
- nonlinearity and, 570
- omega- $k$  plot for, 569
- plasma, 553
- in presence of motion, 583
- Rayleigh-Taylor, 571
- resistive wall, 576, 608
- space-time dependence of absolute, 570
- static, 182
- in stationary media, 554
- Integral laws, electromagnetic, 9, B32, E3, G3
- Integrated electronics, electromechanics and, 688
- Integration contour, deforming, 11, B32
- Internal energy, perfect gas, 816
- Invariance of equations, 256
- Inviscid fluid, boundary condition for, 752
- Ion beams, 552
- Ion conduction, moving fluid and, 778
- Ion drag, efficiency of, 782
- Ion-drag phenomena, 776
- Ionized gases, acceleration of, 746
- Ion source, 776
- Isotropic elastic media, 660
- Isotropy, B27
- Kinetic energy, 214
- Kirchhoff's current law, 16
- Kirchhoff's laws, 15
  - electromechanical coupling and, 84
  - Kirchhoff's voltage law, 16
- Klystron, 601
- Kronecker delta function, 421, G7
- Lagrangian coordinates, 652
  - surface in, 669
- Lagrangian description, 727
- Lagrangian to Eulerian descriptions, 483
- Lamé constant, 667
  - numerical values of, 677
- Laplace's equation, fluid dynamics and, 737
  - two-dimensional flow and, 751
- Leakage resistance of capacitors, 377
- Legendre transformation, 73
- Length expander bar, equivalent circuit for, 716
  - piezoelectric, 712
- Levitating force, induction, 369
- Levitation, electromechanical, 4, 195, 365, 370
  - demonstration of magnetic, 370
  - and instability, 574
  - of liquids, EHD, 552
  - MHD, 552
  - solid and liquid magnetic, 365
- Light, velocity of, B14
- Linearity, electrical, 20, 30, B27
- Linearization, continuum, 483, 510, 556, 652, 842
  - error from, 224
  - lumped-parameter, 182
- Linear systems, 180
- Line integration in variable space, 64, 67
- Liquid drops, charge-carrying, 388
- Liquid level gauge, 416
- Liquid metal brush, 878
  - numerical example of, 883
- Liquid metal MHD, numerical example of 750
- Liquid metals, pumping of, 746
- Liquid orientation in fields, 785
- Liquids, definition of, 724
- Liquids and gases, comparison of, 724
- Loading factor, MHD machine, 833
- Lodestone, B25
- Long-wave limit, 283, 574
  - thin elastic rod and, 683
- Lord Kelvin, 389
- Lorentz force, 419
- Loss-dominated dynamics, continuum, 576
- Loss-dominated electromechanics, 229, 249
- Loss-dominated systems, 227
- Losses, fluid joule, 815
- Loudspeaker, model for, 527
- Lumped-parameter electromechanics, 60
- Lumped-parameter variables, summary of, 35, E4, G4
- Mach lines, 624
- Mach number, 624, 823
- Macroscopic models, electromagnetic, B25
- Magnet, permanent, 27
- Magnetic axes of rotating machines, 105
- Magnetic circuit, example of, 22, 23
- Magnetic diffusion, 330, 335
  - charge relaxation compared to, 401
  - competition between motion and, 351
  - cylindrical geometry and, 408
  - effect of motion on, 354
  - electrical transient, 338
  - induction machines and, 746
  - initial conditions for, 339
  - limit, of infinite conductivity in, 343
    - of small conductivity in, 343
  - liquid metals and, 354
  - lumped-parameter models for, 331, 334, 336
  - sinusoidal steady-state, 358
  - sinusoidal steady-state with motion, 355
  - steady-state, 337, 347
    - steady-state in the moving frame, 351
    - traveling-wave moving media, 364
- Magnetic diffusion time, 341, 772
- Magnetic field, air-gap, 114
  - induced and imposed, 212, 286, 332
  - origin of earths, 336, 552
- Magnetic field compression, 354
- Magnetic field equations, insulating me-

- dium, 773
- Magnetic field intensity, 7, B25
- Magnetic field system, 6, B19
- differential equations for, 6, B20, E6, G6
- integral equations for, 10, B32, E3, G3
- Magnetic field transformation, example of, 266; *see also* Transformations
- Magnetic fluid, demonstration of, 777
- Magnetic flux density, 7, B6
- Magnetic flux lines, frozen, 763
- Magnetic force, field description of, 418, G11
- stress tensor for, 422, G11
- Magnetic forces and mechanical design, 697
- Magnetic induction negligible, B19
- Magnetic piston, 354
- Magnetic pressure, 369
- Magnetic Reynolds numbers, 333, 349, 351, 353, 357, 401, 628, 741, 821
- MHD flow, 754
- numerical value of, 354
- Magnetic saturation in commutator machines, 297
- Magnetic surface force, 447
- Magnetic tension, 767
- Magnetization, B25
- effect of free current forces on, 455
- magnetization currents, B25
- magnetization density, 7, B25
- magnetization force, fluids and, 772
- one degree of freedom and, 451
- magnetization force density, changes in density and, 461
- example of, 460
- inhomogeneity and, 460
- in moving media, 285
- summary of, 448, G11
- Magnetoacoustic velocity, 850
- Magnetoacoustic wave, 846
- electrical losses and, 860
- flux and density in, 851
- numerical example, in gas, 852
- in liquid, 853
- Magnetoelasticity, 553
- Magnetofluid dynamics, 551
- Magnetogasdynamics, 551
- Magneto hydrodynamic conduction machine, 740
- Magneto hydrodynamic generator, constant-area, 821
- variable-area, 828
- Magneto hydrodynamics, 551
- constant-area channel, 740
- viscosity and, 725
- Magneto hydrodynamics of viscous fluids, 878
- Magnetostriction, 697
- one degree of freedom and, 452
- Magnetostriction force, incompressible fluid and, 776
- Magnetostrictive coupling, 707
- Magnetostrictive transducer, terminal representation of, 711
- Mass, conservation of fluid, 729
- elastic continua, quasi-static limit of, 507
- lumped-parameter, 36, 43
- total, 46
- Mass conservation, 731
- Mass density, 45
- elastic materials, numerical values of, 486 of solid, 486
- numerical values of, 486, G12
- Mass per unit area of membrane, 509
- Mass per unit length of wire, 511
- Matched termination, 497
- Material motion, waves and instabilities with, 583
- Matter, states of, 724
- Maxwell, 1
- Maxwell's equations, B12
- limiting forms of, B14
- Maxwell stress tensor, 420, 441, G7, G11
- Mechanical circuits, 36
- Mechanical continuum, 479
- Mechanical equations, lumped-parameter, 49
- Mechanical input power, fluid, 756
- variable-area channel, 756
- Mechanical lumped-parameter equations, examples of, 49, 51, 53
- Mechanics, lumped-parameter, 35
- rigid body, 35
- transformations and Newtonian, 254
- Membrane, elastic continua and, 509, 535, electric field and, 574
- equations of motion for, 511, 535, G13
- two-dimensional modes of, 622
- Membrane dynamics with convection, 584
- Mercury, density and conductivity of, 750
- properties of, 883
- Meteorology, EHD and, 552
- MFD, 551; *see also* MHD
- MGD, 551; *see also* MHD
- MHD, 551
- compressible fluids and, 813
- liquid metal numerical example of, 750
- magnetic damping in, 750
- transient effects in, 746, 759
- transient example of, 750
- variable-area channel in, 751
- of viscous fluids, 878
- MHD conduction machine, 821, 828
- equivalent circuit for, 742
- pressure drop in, 742
- terminal characteristics of, 742
- MHD constant-area channel, 740, 820
- MHD flows, dynamic effects in, 746
- MHD generator, comparison of, 839
- compressibility and, 820
- constant voltage constrained, 743
- distribution of properties in, 827
- end effects in, 797
- examples of, 840, 841
- Mach number in, 823
- numerical example of, 826
- temperature in, 823
- variable-area channel, 828
- viscosity and, 725, 884

- MHD machine, compressible and incompressible, 825  
 constant velocity, loading factor and aspect ratio, 834  
 dynamic operation of, 746  
 equivalent circuit for variable area, 756  
 loading factor of, 833  
 operation of brake, pump, generator, 744  
 quasi-one-dimensional, 828  
 steady-state operation of, 740  
 velocity profile of, 891
- MHD plane Couette flow, 884  
 MHD plane Poiseuille flow, 878  
 MHD pressure driven flow, 884  
 MHD pump or accelerator, 824  
 MHD transient phenomena, 759  
 MHD variable-area channel equations, conservation of energy and, 831, 833  
 conservation of mass and, 831, 833  
 conservation of momentum and, 831, 833  
 local Mach number and, 823, 833  
 local sound velocity and, 822, 833  
 mechanical equation of state and, 816, 833  
 Ohm's law and, 830, 833  
 thermal equations of state and, 820, 833
- MHD variable-area machine, equations for, 833
- MHD variable-area pumps, generators and brakes, 751
- Microphone, capacitor, 201  
 fidelity of, 204  
 Microphones, 200  
 Microwave magnetics, 553  
 Microwave power generation, 552  
 Mobility, 289, B31  
 ion, 778
- Model, engineering, 206  
 Modulus of elasticity, 485  
 numerical values of, 486, G12
- Molecular weight of gas, 816  
 Moment of inertia, 36, 48  
 Momentum, conservation of, *see* Conservation of momentum  
 Momentum density, fluid, 734  
 Motor, commutator, 140, 291  
 induction, 134  
 reluctance, 156  
 synchronous, 119
- Moving media, electromagnetic fields and, 251
- Mutual inductance, calculation of, 22
- Natural frequencies, 515  
 dispersion equation and, 517
- Natural modes, dispersion and, 561  
 kink instability, 635  
 of membrane, 624, 625  
 overdamped and underdamped, 583  
 of unstable wire, 569
- Navier-Stokes equation, 872  
 Negative sequence currents, 144  
 Networks, compensating, 198  
 Newtonian fluids, 861  
 Newton's laws, 15, 35  
 elastic media and, 653  
 Newton's second law, 44, 50  
 electromechanical coupling and, 84  
 fluid and, 729, 731  
 Node, mechanical, 36, 49  
 Nonlinear systems, 206, 213  
 Nonuniform magnetic field, motion of conductor through, 367
- Normal modes, 511  
 boundary conditions and, 524  
 Normal strain and shear stress, 662  
 Normal stress and normal strain, 661  
 Normal vector, analytic description of, 269
- Oerstad, 1, B25  
 Ohm's law, 7, B30  
 for moving media, 284, 298
- Omega- $k$  plot, absolutely unstable wire, 567  
 amplifying wave, 603  
 convective instability, 603  
 damped waves, complex  $k$  for real omega, 579  
 elastic guided shear waves, 695  
 electron oscillations, 601  
 evanescent wave, 557, 559, 597, 615, 695  
 moving wire, with destabilizing magnetic force, 603  
 with resistive wall, complex  $k$  for real omega, 611  
 with resistive wall, complex omega for real  $k$ , 610  
 ordinary wave, with convection, 594  
 on wires and membranes, 514  
 ordinary waves, 514, 555  
 unstable eigenfrequencies and, 569  
 waves with damping showing eigenfrequencies, 582  
 wire stabilized by magnetic field, 557
- Orientation, electrohydrodynamic, 571  
 electromechanical, 4  
 of liquids, dielectrophoretic, 785  
 EHD, 552
- Orthogonality, eigenfunctions and, 341, 519, 520
- Oscillations, nonlinear, 226  
 with convection, 596
- Oscillators in motion, 599
- Overstability, 192
- Particles, charge carriers and, 782
- Particular solution of differential equation, 180
- Pendulum, hydrodynamic, 746  
 simple mechanical, 214
- Perfect conductor, no slip condition on, 769
- Perfect gas law, 816
- Perfectly conducting gas, dynamics of, 846
- Perfectly conducting media, *see* Conductor
- Permanent magnet, in electromechanics, 27  
 example of, 28  
 as rotor for machine, 127
- Permanent set, solids and, 700

- Permeability, 7, B27  
   deformation and, 459  
   density dependence of, 454  
   free space, 7, B7  
 Permittivity, 9, B30  
   free space, 7, 9, B2  
 Perturbations, 183  
 Phase sequence, 144  
 Phase velocity, 613  
   diffusion wave, 358  
   dispersive wave, 598  
   membrane wave, 512  
   numerical elastic compressional wave, 677  
   numerical elastic shear wave, 677  
   numerical thin rod, 486, G12  
   ordinary wave, 487  
   thin rod, 487  
   wire wave, 512  
 Physical acoustics, 553, 651  
 Piezoelectric coupling, 711  
   reciprocity in, 712  
 Piezoelectric devices, example of, 717  
 Piezoelectricity, 553, 711  
 Piezoelectric length expander bar, 712  
 Piezoelectric resonator, equivalent circuit for, 716  
 Piezoelectric transducer, admittance of, 714  
 Piezomagnetism, 553  
 Plane motion, 44  
 Plasma, confinement of, 552  
   electromechanics and, 4  
   evanescent waves in, 561, 638  
   heating of, 552  
   lumped-parameter model for, 223  
   magnetic bottle for, 563  
   magnetic diffusion and, 408  
   MHD and, 553  
   solid state, 553  
 Plasma dynamics, 553  
 Plasma frequency, 600  
 Poiseuille flow, plane, 878  
 Poisson's ratio, 662  
   numerical values of, 666  
 Polarization, effect of motion on, 290  
   current, B29  
   density, 7, B28  
   electric, B27  
   force, 463, 571, G11  
 Polarization force, one degree of freedom, 464  
 Polarization interactions, liquids and, 783  
 Polarization stress tensor, 463, G11  
 Pole pairs, 148  
 Poles in a machine, 146  
 Polyphase machines, 142  
 Position vector, 45  
 Positive sequence currents, 144  
 Potential, electric, B9  
   electromagnetic force, 738  
   gravitational, 733  
   mechanical, 214  
   velocity, 737  
   Potential difference, B10  
   Potential energy, 214  
   Potential flow, irrotational electrical forces and, 738  
   Potential fluid flow, two-dimensional, 751  
   Potential plot, 214  
   Potential well, electrical constraints and, 217  
     electromechanical system and, 217  
     temporal behavior from, 224  
   Power, conservation of, 64  
   Power density input to fluid, 818  
   Power factor, 126  
   Power flow, group velocity and, 638  
     ordinary and evanescent waves and, 638  
     rotating machines and, 110  
   Power generation, ionized gases and, 552  
   microwave, 552, 553  
   Power input, electrical, 64  
     fluid electrical, 818  
     mechanical, 64  
     mechanical MHD, 743  
   Power input to fluid, electric forces and, 819  
     electrical losses and, 818, 819  
     magnetic forces and, 818  
     pressure forces and, 818  
   Power output, electric MHD, 743  
   Power theorem, wire in magnetic field, 637, 644  
   Poynting's theorem, B22  
   Pressure, density and temperature dependence of, 816  
     hydrostatic, 735  
     hydrostatic example of, 736  
     incompressible fluids and significance of, 753  
     isotropic, 735  
     magnetic, 369  
     normal compressive stress and, 735  
     significance of negative, 753  
     velocity and, 753  
   Principal axes, 49  
   Principal modes, 681  
     elastic structure, 679  
     shear wave, 695  
   Principle of virtual work, *see* Conservation, of energy  
   Products of inertia, 48  
   Propagation, 613  
   Propulsion, electromagnetic, 552  
     electromechanical, 4  
     MHD space, 746  
   Pulling out of step for synchronous machine, 125  
   Pump, electric field, 776  
     electrostatic, 778  
     liquid metal induction, 365  
     MHD, 744, 746  
     variation of parameters in MHD, 825  
   Pumping, EHD, 552  
   MHD, 552  
   Quasi-one-dimensional model, charge relaxa-

- tion, 392, 394
- electron beam, 600
- gravity wave, 794
- magnetic diffusion, 347
- membrane, 509, 648
  - and fluid, 793
- MHD generator, 828
- thin bar, 712
- thin beam, 683
- thin rod, 480, 681
- wire or string, 509
  - in field, 556, 563, 574, 605, 627
- Quasi-static approximations, 6, B17
- Quasi-static limit, sinusoidal steady-state and, 515, 534
  - wavelength and, B17
  - wire and, 534
- Quasi-statics, conditions for, B21
  - correction fields for, B21
  - elastic media and, 503
  - electromagnetic, B19
- Quasi-static systems, electric, 8
  - magnetic, 6
- Radiation, heat, 815
- Rate of strain, 864
- Reactance-dominated dynamics, 138, 211, 220, 242, 336, 354, 368, 563, 759
- Reciprocity, electromechanical coupling and, 77
  - piezoelectric coupling and, 713
- Reference system, inertial, 44
- Regulation, transformer design and, 699
- Relative displacement, rotation, strain and, 658
- Relativity, Einstein, 254
  - Galilean, 255
  - postulate of special, 261
  - theory of, 44
- Relaxation time, free charge, 372
  - numerical values of, 377
- Relay, damped time-delay, 229
- Reluctance motor, 156
- Resistance-dominated dynamics, 138, 209, 233, 242, 336, 354, 368, 503, 583, 611
  - MHD, 750
- Resistive wall damping, continuum, 583
- Resistive wall instability, nature of, 612
- Resistive wall wave amplification, 608
- Resonance, electromechanically driven continuum and, 533
  - response of continua and, 515
- Resonance frequencies, magnetic field shift of, 563
  - membrane, 624
  - natural frequencies and, 515
- Resonant gate transistor, 688
- Response, sinusoidal steady-state, 181, 200, 514
- Rigid body, 44
- Rigid-body mechanics, 35
- Rotating machines, 103
  - air-gap magnetic fields in, 114
  - applications of, 3
  - balanced two-phase, 113
  - classification of, 119
  - commutator type, 140, 255, 292
  - computation of mutual inductance in, 22
  - dc, 140, 291
  - differential equations for, 106
  - effect of poles on speed of, 149
  - electric field type, 177
  - energy conversion conditions for, 110
  - energy conversion in salient pole, 154
  - equations for salient pole, 151
  - hunting transient of synchronous, 192
  - induction, 127
  - losses in, 109
  - magnetic saturation in, 106
  - mutual inductance in, 108
  - number of poles in, 146
  - polyphase, 142
  - power flow in, 110
  - salient pole, 103, 150
  - single-phase, 106
  - single-phase salient-pole, 79
  - smooth-air-gap, 103, 104
  - stresses in rotor of, 697
  - superconducting rotor in, 92
  - synchronous, 119
  - two-phase, smooth-air-gap, 111
  - winding distribution of, 108
- Rotating machines, physical structure,
  - acyclic generator, 287
  - commutator type, 292
  - dc motor, 293
  - development of dc, 295
  - distribution of currents and, 166, 169
  - four-pole, salient pole, 164
  - four-pole, single phase, 147
  - homopolar, 313
  - hydroelectric generator, 152
  - multiple-pole rotor, 146
  - rotor of induction motor, 107
  - rotor of salient-pole synchronous, 151
  - synchronous, salient-pole, 152
  - salient-pole, two phase, 158
  - salient-pole, single phase, 150
  - smooth-air-gap, single phase, 104
  - stator for induction motor, 106
  - three-phase stator, 145
  - turboalternator, 120
  - two-pole commutator, 294
- Rotation, fluid, 865
- Rotation vector, 658
- Rotor of rotating machines, 104, 107, 112, 120, 146, 147, 150, 151, 152, 158, 164, 166, 169
- Rotor teeth, shield effect of, 301
- Saliency in different machines, 156
- Salient-pole rotating machines, 103, 150
- Salient poles and dc machines, 293
- Servomotor, 140
- Shading coils in machines, 139
- Shear flow, 862, 864, 875

- magnetic coupling, 878
- Shear modulus, 664
  - numerical values of, 666
- Shear rate, 866
- Shear strain, 543, 655
  - normal strain and, 663
  - shear stress and, 664
- Shear stress, 543
- Shear waves, elastic slab and, 693
- Shearing modes, beam principal, 683
- Shock tube, example related to, 276
- Shock waves, supersonic flow and, 592
- Sinusoidal steady-state, 181, 200, 514
  - convection and establishing, 592
- Sinusoidal steady-state response, elastic continua, 514
- Skin depth, 357
  - numerical values of, 361
- Skin effect, 358
  - effect of motion on, 361
- Slip of induction machine, 131
- Slip rings, 120
  - ac machines and, 120
- Slots of dc machine, 296
- Sodium liquid, density of, 771
- Solids, definition of, 724
- Sound speed, gases, 844
  - liquids, 845
- Sound velocity, *see* Velocity
- Sound waves, *see* Acoustic waves
- Source, force, 37
  - position, 36
  - velocity, 37
- Space charge, fluid and, 780
- Space-charge oscillations, 601
- Speakers, 200
- Specific heat capacity, constant pressure, 817
  - constant volume, 816
  - ratio of, 817
- Speed coefficient, of commutator machine, 300
  - torque on dc machine and, 302
- Speed control of rotating machines, 149
- Speedometer transducer, 170
- Speed voltage in commutator machine, 299
- Spring, linear ideal, 38
  - lumped element, 36, 38
- quasi-static limit of elastic continua and, 505
- torsional, 40
- Spring constant, 39
- Stability, 182, 566, 583
- Stagnation point, fluid, 752
- Standing waves, electromagnetic, B16
  - electromechanical, 516, 559, 596, 771
- State, coupling network, 61, 65
  - thermal, 816
- Stator, of rotating machines, 104, 106, 120, 145, 147, 150, 152, 158, 164, 166, 169
  - smooth-air-gap, 103
- Stinger, magnetic, 193
- Strain, formal derivation of, 656
  - geometric significance of, 654
  - normal, 654
  - permanent, 700
  - shear, 543, 654
    - as a tensor, 659
    - thin rod, normal, 484
- Strain components, 656
- Strain-displacement relation, 653
  - thin-rod, 485
- Strain rate, 724, 864
  - dilatational, 869
- Strain-rate tensor, 864
- Streaming electron oscillations, 600
- Streamline, fluid, 752
- Stress, fluid isotropy and, 868
  - fluid mechanical, 872
  - hydrostatic, 724
  - limiting, 700
  - normal, 432
  - shear, 432, 543
  - and traction, 424, G9
- Stress components, 425
- Stress-strain, nonlinear, 700
- Stress-strain rate relations, 868
- Stress-strain relation, 660, 668
  - thin-rod, 485
- Stress-tensor, elastic media and, 667
  - example of magnetic, 428
  - magnetization, 462, G11
  - Maxwell, 420
  - physical interpretation of, 425, G7
  - polarization, 463, G11
  - pressure as, 735
  - properties of, 423, G7
  - surface force density and, 446, G9
  - symmetry of, 422
  - total force and, 444, G9
- Stress tensors, summary of, 448, G11
- String, convection and, 584
  - equation of motion for, 511, 535
  - and membrane, electromechanical coupling to, 522
- see also* Wire
- Subsonic steady MHD flow, 823
- Subsonic velocity, 587
- Substantial derivative, 259, 584, 726; *see also* Convective derivative
- Summation convention, 421, G7
- Superconductors, flux amplification in, 354
- Supersonic steady MHD flow, 823
- Supersonic steady-state dynamics, 524
- Supersonic velocity, 587
- Surface charge density, free, 7
- Surface conduction in moving media, 285
- Surface current density, free, 7
- Surface force, example of, 449
  - magnetization, 775
- Surface force densities, summary of, 448, G11
- Surface force density, 445, G11
  - free surface charge and, 447, G11
  - free surface currents and, 447, G11
- Surface tension, 605
- Susceptance, electromechanical driving, 531
- Susceptibility, dielectric, 9, B30
  - electric, 9, B30
  - magnetic, 7, B27
- Suspension, magnetic, 193



- Symbols, A1, D1, F1
- Symbols for electromagnetic quantities, 7
- Synchronous condenser, 127
- Synchronous machine, 119
  - equivalent circuit of, 123
  - hunting transient of, 192
  - phasor diagram for, 124, 162
  - polyphase, salient-pole, 157
  - torque in, 122, 123, 125
  - torque of salient-pole, two-phase, 160, 162
- Synchronous motor, performance of, 126
- Synchronous reactance, 123
- Synchronous traveling-wave energy conversion, 117
  
- Tachometer, drag-cup, 363
- Taylor series, evaluation of displacement
  - with, 483
  - multivariable, 187
  - single variable, 183
- Teeth of dc machine, 296
- Temperature, electrical conductivity and, 380
- Tension, of membrane, 509
  - of wire, 511
- Tensor, first and second order, 437
  - one-dimensional divergence of, 482
  - surface integration of, 428, 441, 444, G9
  - transformation law, 437, G10
  - transformation of, 434, G9
- Tensor strain, 659
- Tensor transformation, example of, 437
- Terminal pairs, mechanical, 36
- Terminal variables, summary of, 35, E4, G4
- Terminal voltage, definition of, 18
- Theorems, C2, E2, G2
- Thermonuclear devices, electromechanics
  - and, 4
- Thermonuclear fusion, 552
- Theta-pinch machine, 408
- Thin beam, 683
  - boundary conditions for, 687
  - cantilevered, 688
  - deflections of, 691, 692
  - eigenvalues of, 692
  - electromechanical elements and, 688, 691, 701, 704
  - equation for, 687
  - resonance frequencies of, 692
  - static loading of, 701
- Thin rod, 681
  - boundary conditions for, 494
  - conditions for, 683
  - equations of motion for, 485, G13
  - force equation for, 484
  - longitudinal motion of, 480
  - transverse motions of, 682
- Three-phase currents, 143
- Time constant, charge relaxation, 372
  - magnetic diffusion, 341
- Time delay, acoustic and electromagnetic, 499
- Time-delay relay, electrically damped, 249
- Time derivative, moving coordinates and, 258
- Time rate of change, moving grain and, 727
- Torque, dc machine, 302
  - electrical, 66
  - Lorentz force density and, 301
  - pull-out, 124
- Torque-angle, 123
- Torque-angle characteristic of synchronous machine, 125
- Torque-angle curve, salient-pole synchronous machine, 163
- Torque-slip curve for induction machine, 135
- Torque-speed curve, single phase induction machine, 139
- Torsional vibrations of thin rod, 543
- Traction, 424, 432
  - pressure and, 735
  - stress and, 432, G9
- Traction drives, 310
- Transducer, applications of, 2
  - continuum, 704
  - example of equations for, 84, 86
  - fidelity of, 203
  - incremental motion, 180, 193, 200
  - Magnetostrictive, 708
- Transfer function capacitor microphone, 204
- electromechanical filter, 706
- Transformations, electric field system, 264
  - Galilean coordinate, 254, 256
  - integral laws and, 11, 276, 300, 315, B32
  - Lorentz, 254
  - Lorentz force and, 262
  - magnetic field system, 260
  - primed to unprimed frame, 439
  - summary of field, 268, E6, G6
  - vector and tensor, 434, G9
- Transformer, electromechanical effects in, 697
  - step-down, 698
  - tested to failure, 698
- Transformer efficiency, mechanical design
  - and, 699
- Transformer talk, 697
- Transient response, convective instability, 621
  - elastic continua, 517
  - MHD system, 751
  - one-dimensional continua, 511
  - superposition of eigenmodes in, 518
  - supersonic media, 593
- Transient waves, convection and, 587
- Transmission line, electromagnetic, B16
  - parallel plate, B15
  - thin rod and, 488
- Transmission without distortion in elastic structures, 696
- Traveling wave, 487
  - convection and, 586
  - magnetic diffusion in terms of, 357
  - single-phase excitation of, 118
  - standing wave and, 116
  - two-dimensional, 622
  - two-dimensional elastic, 694
  - two-phase current excitation of, 116
- Traveling-wave induction interaction, 368
- Traveling-wave MHD interaction, 746
- Traveling-wave solutions, 554
- Traveling-wave tube, 602
- Turboalternator, 120
- Turbulence in fluids, 725
- Turbulent flow, 43

- Ultrasonic amplification, 602
- Ultrasonics in integrated electronics, 688
- Units of electromagnetic quantities, 7
- Van de Graaff generator, example of, 383, 385
  - gaseous, 778
- Variable, dependent, 180
  - independent, differential equation, 180
  - thermodynamic independent, 64
- Variable capacitance continuum coupling, 704
- V curve for synchronous machine, 125
- Vector, transformation of, 434, 659
- Vector transformation, example of, 435
- Velocity, absolute, 44
  - acoustic elastic wave, 673, 677
  - acoustic fluid wave, 844, 846
  - Alfvén wave, 763, 772
  - charge-average, B5
  - charge relaxation used to measure, 396
  - charge relaxation wave, 395
  - compressional elastic wave, 673, 677
  - dilatational elastic wave, 673, 677
  - elastic distortion wave, 675, 677
  - fast and slow wave, 586
  - light wave, B14
  - magnetic diffusion wave, 358
  - magnetic flux wave, 114
  - magnetoacoustic wave, 850, 852
  - measurement of material, 356, 362
  - membrane wave, 512
  - phase, 488
  - shear elastic wave, 675, 677
  - thin rod wave, 486, 487, 682
  - wavefront, 618
    - with dispersion, 598
  - wire or string wave, 512
- Velocity potential, 737
- Viscosity, 862
  - coefficient of, 863
  - examples of, 875
  - fluid, 724
  - mathematical description of, 862
  - second coefficient of, 871
- Viscous flow, pressure driven, 877
- Viscous fluids, 861
- Viscous losses, turbulent flow, 725
- Voltage, definition of, B10
  - speed, 20, 21
  - terminal, 18
  - transformer, 20, 21
- Voltage equation, Kirchhoff, 16
- Ward-Leonard system, 307
- Water waves, 794
- Wave amplification, 601
- Wave equation, 487
- Wavenumber, 357, 513
  - complex, 554, 607
- Wave propagation, 487
  - characteristics and, 487, 586, 618
  - group velocity and, 616
  - phase velocity and, 613
- Wave reflection at a boundary, 493
- Waves, acoustic elastic, 673
  - acoustic in fluid, 544, 841, 842, 845
  - Alfvén, 759
  - compressional elastic, 673
  - convection and, 586
  - cutoff, *see* Cutoff waves
  - damping and, 576
  - diffusion, 355, 576
  - dilatational, 672
  - dispersionless, 555
  - dispersion of, 488
  - of distortion, 675
  - elastic shear, 678
  - electromagnetic, B13, 488
  - electromechanical in fluids, 759
  - evanescent, *see* Evanescent waves
  - fast and slow, 586
  - fast and slow circularly polarized, 631
  - fluid convection and, 860
  - fluid shear, 760
  - fluid sound, 813
  - incident and reflected at a boundary, 494
  - light, B13
  - longitudinal elastic, 673
  - magnetoacoustic, 841, 846
  - motion and, 583
  - plasma, 553, 600, 638
  - radio, B13
  - rotational, 671
  - shear elastic, 675
  - stationary media and, 554
  - surface gravity, 794
  - thin rod and infinite media, 673
- Wave transients, solution for, 490
- Wind tunnel, magnetic stinger in, 193
- Windings, balanced two-phase, 113
  - dc machine, 292
  - lap, 296
  - wave, 296
- Wire, continuum elastic, 509, 535
  - convection and dynamics of, 584
  - dynamics of, 554
  - equations of motion for, 511, G13
  - magnetic field and, 556, 566, 627
  - two-dimensional motions of, 627
- Yield strength, elastic, 700
- Young's modulus, 485, G12
- Zero-gravity experiments, KC-135 trajectory and, 787