MATH S390
Quantitative Models for Financial Risk
(Free Courseware)

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Contents

Chapter 1 Mathematical modelling of financial markets

1.1 About this module

Welcome to this free courseware module 'Mathematical modelling of financial markets'!

This module is taken from the OUHK course *MATH S390 Quantitative Models for Financial Risk (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&c=C_ETPU&cid=191154069600&lang=eng)*, a five-credit, Higher level course offered by the School of Science & Technology (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcSubWeb&l=C_ST&lid=191133000200&lang=eng) of the OUHK. This course is about the use of mathematics to solve real-life applications. You will learn how to represent real world financial problems through various types of mathematical methods (e.g. Black–Scholes models).

MATH S390 is mainly presented in printed format and comprises 5 study units. Each unit contains study content, self-tests, etc for students' self-learning. This module (The materials for this module, taken from the print-based course MATH S390, have been specially adapted to make them more suitable for studying online. In addition to this topic on 'How is money created?', which is an extract from Unit 1 of the course, the original Unit 1 also includes the topics 'The game theory and probabilistic methods' and 'Replicating portfolios'.) retains most of these elements, so you can have a taste of what an OUHK course is like. Please note that no credits can be earned on completion of this module. If you would like to pursue it further, you are welcome to enrol in *MATH S390 Quantitative Models for Financial Risk (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&c=C_ETPU&cid=191154069600&lang=eng)*.

This module will take you about **3 hours** to complete, including the time for completing the activities and self-tests (but not including the time for assigned readings).

Good luck, and enjoy your study!

1.2 Introduction

In recent decades, the fields of financial mathematics and financial risk evaluation have undergone explosive development. Financial mathematics, broadly defined, is a class of applied mathematics devoted to modelling and analysing financial markets, and to aiding the management of financial resources. We focus on the analysis of financial derivatives; in particular, options for pricing and hedging.

1.3 An introduction to financial markets and mathematical concepts

Almost everyone knows about the Hong Kong, London, New York and Tokyo stock exchanges. In fact, the trading activity in these markets frequently hits the front pages of newspapers and is featured on television daily news broadcasts. Of course, there are many other financial markets. Each of these has a character determined by the type of financial objects being exchanged.

Before getting involved with the details of financial risk (Exposure to adverse financial changes. More generally, risk is due to the uncertainty of future prices and values) and the quantitative process, it is important to set the scene by explaining the concepts related to three broad types of financial instruments that differ from basic equities:

- derivatives;
- forward and futures contracts; and
- options.

Basic equities

Basic equities include cash, foreign currency, stocks, bonds, and money market or bank accounts. Equities are characterized by the following three basic properties:

- they have value in and of themselves
- they are traded in well-regulated, transparent markets (e.g. stock exchanges)
- they are comparable to commodities; examples include physical goods, typically raw or partially processed agricultural or mining products that are commercially traded. In fact, they are all commercial goods.

1.3.1 What is a derivative?

In finance, a derivative is a financial instrument whose value is derived from one or more underlying assets. The derivative itself is just a contract between two parties for future transactions exchanging cash and/or assets. It depends on the value of the transaction, the gain or loss to either party, and depends on the value of other underlying variables at the transaction time. We need to clearly spell out the underlying variables in the contract. The most common underlying assets include stocks, bonds, commodities, currencies, interest rates and market indexes.

Examples of financial derivatives include futures, forwards, options and swaps. These financial instruments play an important and useful role in hedging (hedging means taking action to reduce or to protect against risk) and managing risk. For example:

- farmers can use derivatives to hedge against the risk of the price of their wheat falling before it has been harvested; and
- pension funds and insurance companies can use derivatives to hedge against large drops in the value of their portfolios.

Most derivatives are characterized by high leverage (a financial arrangement that has a multiplying effect on the profit (or loss) on an investment relative to change in value of the invested assets) — and this makes them risky. Thus, derivatives can pose dangers to the stability of financial markets and the overall economy.

1.3.2 Forward and futures contracts

In the world of finance, there are two common types of contracts between parties to buy or sell an asset at a specified future time. These are the *forward contract* and the *futures contract*. Both forward contracts and futures contracts are used to hedge investments. Although they have the same function, i.e. to buy or sell an asset at a specified future time, futures contracts and forward contracts also have distinct differences.

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Forward contracts

A **forward contract** is a non-standardized contract between two parties. It is a private agreement to buy or sell a specific asset (such as wheat, oil, pigs, etc.) at a specific future date at a price agreed upon today. The two parties must bear each other's credit risk. A forward contract is not traded on an exchange, which means that it is not settled in cash daily.

Forward contracts are settled only at their maturity dates (the date on which the contract can be exercised). The profit or loss made from a forward contract depends on the difference between the forward price and the spot price of the asset on the day the forward contract matures. When entering into a forward transaction, the buyer of the forward contract is said to hold the long position (The party in the contract that is on the purchasing side of the contracted transaction. In forward or futures contracts, the party purchasing and taking delivery is long.). The seller is said to hold the short position (the counterpart to the long position) by agreeing to sell the asset on the same date for the same price.

The price that the underlying asset is bought/sold for is called the delivery price. This price is chosen so that the value of the contract to both sides is zero at the outset.

What this means will be explained shortly — roughly, it means that the price is fair, so neither party is taking advantage of the other.

The payout from a forward is based on the fixed price of the underlying asset at the maturity date. An example of a forward contract is provided in the following animated presentation.

Click this link to watch the video: http://www.opentextbooks.org.hk/system/files/resource/10/10585/10591/media/An%20example%20of%20a%20forward%20contract.mp4

Let's assume that I enter into a forward transaction in which I agree to sell 100 pigs at a fixed price of $3,000 per pig in a year's time when the contract matures. Assume that I don't have any pigs at the moment. If the price of a pig is $3,500 at the maturity date, then I make a loss on the forward transaction because I must buy 100 pigs at a price of $3,500 each in order to meet my obligation to deliver the pigs. Note that the buyer and I can opt not to perform the contract, since this is a non-standardized agreement between us.

On delivery, I only receive $3,000 per pig, so I incur a total loss of $50,000, that is 100 × $(3,500 – 3,000). But let's say the price of pigs has plummeted to $2,500 each. In this case, I profit to the tune of $50,000, that is 100 × $(3,000 – 2,500).

Futures contracts

A futures contract is a bit different from a forward contract. A futures contract requires delivery of a commodity, bond, currency, or stock index, at a specified price, on a specified future date, and it involves a set of standardized rules that need to be specified. The profit and loss made by each party on a futures contract are equal and opposite. In other words, a futures contract trade is a zero-sum game.

There is, however, always a chance that a party may fail to perform on a futures contract as required by its side of the contract agreement. Thus a futures contract relies on margin calls to guarantee performance, without considering the benefit of the individual parties. An exchange or clearinghouse acts as a counterpart on all future contracts and sets margins.

A futures contract is an example of a parametric contract, and is easily combined or traded as part of more complex financial derivatives deals (such as stock and currency deals). The benefit of parties will not be considered. An example of a futures contract is provided in the following animated presentation. Click on the image below to launch it.

Unlike forward contracts, a futures contract involves a set of standardized rules that need to be specified. Such rules might include:

- the grade of the deliverable, for example, the quality of the pig (in size or weight);
- the delivery date, for example, one year from the agreement of the contract; and
- the amount and units of the underlying asset are to be traded.

For example, let's say that 100 pigs are to be traded at the price of $3,000 per pig. Again, assume that I don't have any pigs. If the price of a pig is $3,500 at the end of the year, I definitely make a loss on the future transaction because I must buy 100 pigs at $3,500 each in order to meet my obligation to deliver the pigs.

Pricing a forward

In order to explain the pricing problem, we are going to turn a forward transaction into a simple mathematical expression. To formulate the expression, we reconsider forward contracts in more detail.

Recall that a forward contract is an agreement to buy or sell an asset on a specified future date for a fixed price. Let

K be the fixed price (also called the strike price), and

S_T be the spot price per pig at the maturity time T.

Supposing that the buyer has agreed to buy the pigs at the fixed price (known as a long position in pigs), then the payoff at time T from a long position (the buyer's position) is defined as

$S_T - K$

per pig, as illustrated in (a) in Figure 1.1. This payoff could be positive or negative, since the cost of entering into a forward contract is zero. The slope of ($S_T - K$) is positive, meaning that the payoff rise is proportional to the increase in the spot price S_T.

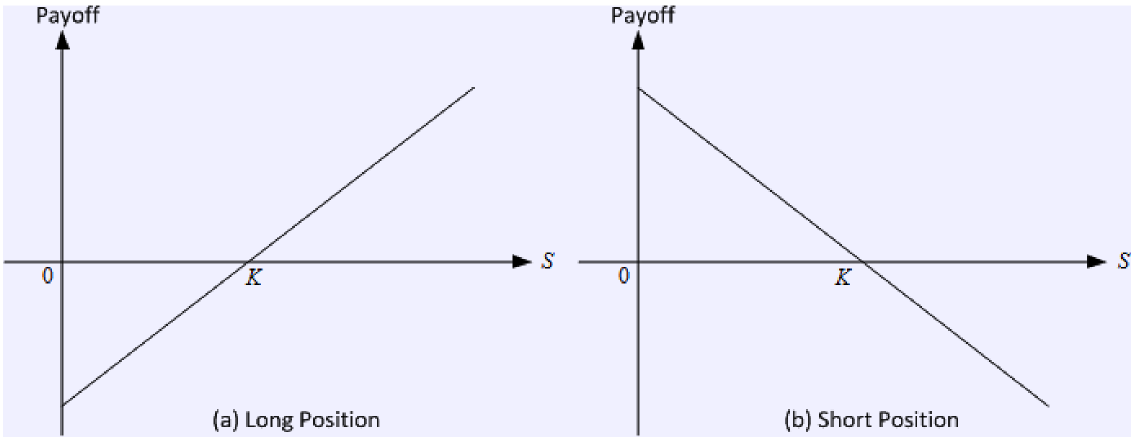

Fig. 1.1: Payoffs from forward contracts

Similarly, the payoff at time T from a short position (the seller's position) is defined as

$K - S_T$

per pig (or per unit of an asset) if I have agreed to sell pigs at a fixed price (known as a short position in pigs). The payoff has a downward slope, as shown in (b) in Figure 1.1.

Our next problem is how we can determine a fair delivery price, i.e. the fixed price in the forward contract. At the time a contract is prepared, neither buyer nor seller knows the forward price at the maturity date, so we can use a simple strategy to estimate a fair price . The following animated presentation shows you how this can work.

Determining a fair delivery price for a forward contract

Click this link to watch the video:
http://www.opentextbooks.org.hk/system/files/resource/10/10585/10591/media/Determining%20a%20fair%20delivery%20price%20for%20a%20forward%20contract.mp4

Let's assume that cash can be deposited at 5% interest per annum. If the fixed price is $3,000 per pig at the maturity date, and the forward price of a pig is $3,300, then we can borrow $3,000 at a 5% interest rate per year, so the annual interest repayment is $150; buy a pig on the market; enter into a short forward contract to sell a pig for $3,300 in one year's time; and, at the end of the year, sell the pig for $3,300, and repay the borrowed $3,000 plus $150 interest.

We have now made a risk-free profit of $150. In fact, any fixed price above $3,150 will result in a profit using this strategy.

But what if the fixed price is below $3,150? In this case, we can sell a pig for $3,000, assuming we have one; deposit $3,000 at a 5% interest rate per year; enter into a long forward contract to repurchase the pig in one year's time for the delivery price; and, at the end of the year, buy back the pig at the delivery price.

As long as the fixed price is under $3,150, a risk-free profit again emerges.

In theory, investors in the pig market will try to take advantage of any forward price that is not equal to $3,150. These trading activities are known as arbitrage, and will result in the forward price being exactly $3,150. This is known as an arbitrage free price because any other price results in an arbitrage opportunity.

What does all this tell us?

- First, the forward transaction costs nothing for either party to enter into. The arguments above show that either party could safely agree to the transaction without making a loss, through suitable borrowing/lending, etc.
- Second, the total return between the two parties will be the interest that can be earned on cash (such as the $150 in the example above). However, the payoff for one party is the opposite of that of the other, i.e. for every winner there is a loser. This illustrates the speculative nature of some of these instruments when they are traded in isolation. When future contracts are traded on their own for speculative purposes, e.g. on the Hang Seng Index, they can indeed be equivalent to betting on a racehorse.

(In fact, speculative trading in 'options' is actually far more dangerous than the kind of forward transaction we've considered here.)

Although this illustrates the potentially speculative nature of future transaction and the dangers of being able to enter into a large number of contracts at zero cost, there are a variety of ways in which these contracts can be used to reduce risk. This is known as hedging.

An example of hedging

Extending our pig case study, suppose a pig farmer has a good year in which his pigs breed 100 piglets. The current price is $3,000, and the farmer is concerned that this price may fall — or at least not grow in line with the one-year forward price of $3,150. He would be happy to receive $315,000 for selling the 100 pigs in a year's time. The farmer could therefore sell his 100 pigs in a forward contract. Alternatively, he could sell his 100 pigs when the piglets have grown up for a fixed amount of $315,000 in a year's time. This allows him to prevent a reduction in value due to the price of pigs falling, as well as enabling him to fix his future cash-flow in order to reduce the uncertainty of his business. The farmer has 'hedged' his exposure to pig prices — he has hedged his bets.

1.3.3 Options

A more interesting type of financial derivative is an option that represents a contract sold by one party (called the option seller or writer (the party in the short position)) to another party (called the option buyer or holder (the party in the long position)). This type of contract offers the holder the right — but not the obligation — to buy (i.e. a 'call option') or sell (i.e. a 'put option') a security or another underlying financial asset such as stock, commodity, or currency at a specific date T in the future for a fixed price (i.e. the strike price) *K*.

There are many different type of options in the financial market. Here we only consider the basic types, namely:

- call options
  a call option gives the holder the right, but not the obligation, to buy the underlying asset at a fixed price on a specific date; and
- put options
  a put option gives the holder the right, but not the obligation, to sell the underlying asset at a fixed price on a specific date.

The fact that the holder of an option contract has the option to exercise, rather than an obligation to do so, leads to the one-sidedness of the investment.

We can then go on to talk about 'vanilla options (vanilla options are common, widely available options, often exchange traded, with simple payoff functions based on a single underlying asset)' and 'exotic options (exotic options are more complex than vanilla options, and are traded over the counter)'. Again, it is important to realize that derivatives can be risky when used in isolation to speculate on the performance of underlying assets in the future. Equally, though, derivatives offer a flexible and efficient method of reducing risk when used in connection with the underlying assets. Effectively, they can be used as insurance against losses.

This example illustrates a case involving a call option.

An example of a call option

Suppose a property investor, Mr Li, is looking to buy some real estate. During his search, he notices someone is selling 10 houses for $1,000,000 each ($10,000,000 total). During this time, he also comes across a very interesting rumour. A friend of his who works in a government office tells him that there is talk of reducing the supply of land in the near future. Mr Li knows that if the land supply is reduced, the price of houses will skyrocket. But if the government provides more land than expected, he knows he'll be stuck with some property he didn't want.

Mr Li presents the seller of the houses with an interesting proposition: he will give her $1,000,000 to let him buy the houses within the next three months. If he decides to purchase, he'll pay the full $10,000,000. If he decides not to purchase, the seller can keep the money.

In other words, Mr Li's not buying the houses; he's buying the right to purchase the houses.

Solution

This would be referred to as a call option, because it allows Mr Li to buy within a given period of time. (A put option would allow him to sell within a given of time.)

The underlying asset would be the 10 houses, because that's what Mr Li would receive if the purchase took place. The other details of the option are as follows:

- The *premium* would be $1,000,000 because this is the amount he must pay to enter into the agreement.

- The *strike* price would be $10,000,000 because that's the price he'd pay the seller for the underlying asset.
- The *expiration* would be three months, which is the total period of time in which the agreement is valid.

If the rumour Mr Li heard proves true, he could exercise his option at the maturity date, and buy the houses for the agreed-upon price of $10,000,000.

Having worked through these examples, we are now going to generalize the difference between the European calls and puts and American calls and puts.

European calls and puts

European calls and puts are call and put options on a single underlying asset. The designation European means the holder may exercise the option only at the maturity time T.

- A European call option allows the holder the right to buy the stock at time T for a specified price X. The payoff to the long position is

  $\max\{S_T - X - ce^{rT}, 0 - ce^{rT}\}(1.8)$
- A European put option allows the holder the right to sell the stock at the maturity time for a specified price . The payoff to the long position is

  $\max\{X - S_T - ce^{rT}, 0 - ce^{rT}\}(1.9)$

American calls and puts

American calls and puts are also call and put options on a single underlying asset. The designation 'American' means the holder may exercise the option at any time on or before the maturity time T.

- An American call option allows the holder the right to buy the stock at time

  T^* £ T

  for price X. The payoff to the long position is

  $\max\{S_{T^*} - X - ce^{rT^*}, 0 - ce^{rT^*}\}(1.10)$
- An American put option allows the holder the right to sell the stock at time

  T^* £ T

  for price X. The payoff to the long position is

  $\max\{X - S_{T^*} - ce^{rT^*}, 0 - ce^{rT^*}\}(1.11)$

As you've learned in this section, investors and financial officers use derivatives to reduce the risk on transactions they need or want to make in the future. They pay a price for this, either in terms or options premiums, or in reduced potential gain. Speculators can use derivatives to speculate on future price movements of an asset without investing in that asset itself. Derivatives can also be used as leverage instruments.

At the end of this section, it should be pointed out that we have been computing prices using an extremely simple stock model. Fortunately, our answers and our methods can be generalized to more realistic stock models in further studies.

1.3.3.1 Self-test 1

1. A call option on Illinois stock specifies an exercise price of $38. Today's price for the stock is $40. The premium on the call option is $5. Assume that the option will not be exercised until the maturity date, if at all. Complete the following table.

Assumed stock price at the time the call option is about to expire	Net profit or loss per share to be earned by the writer (seller) of the call option
$37	
$39	
$41	
$43	
$45	
$49	

2. A call option on Michigan stock specifies an exercise price of $55. Today's price for the stock is $54 per share. The premium on the call option is $3. Assume that the option will not be exercised until maturity, if at all. Complete the following table for a speculator who purchases the call option.

Assumed stock price at the time the call option is about to expire	Net profit or loss per share to be earned by the speculator
$50	
$52	
$54	
$56	
$58	
$60	
$62	

1.4 Mathematical modelling of financial markets

This section introduces the techniques used to model the financial market with uncertainties. We will start from the very basic model. To start, we consider a market with *M* assets. There are two things we need to define:

1. For all possible outcomes of the future market, we define

   Ω = the set of events coming from the market histories. (2.1)

2. For each asset 1 £ *i* £ *M* with future market ω outcome and future time $t \geq 0$, we define

   $S_t^{(i)}(\omega)$ = the price of asset *i* at time *t*. (2.2)

We demonstrate the use of the modelling technique by considering two different forms of models:

1. the single-period market model; and
2. the two-period market model.

The single-period market model

This model is only concerned with one asset, one period and two possible market outcomes. This model is commonly called the *binomial model*.
We assume $t = 0$ at the beginning (today), and $t = 1$ at the end of the period.
There are two possible outcomes at the first period, we need to define the following variables for the model:

1. market upswing, denoted by *u*, and the asset return denoted by *g*, or
2. market downswing, denoted by *d*, and the asset return denoted by *l*.

   Intuitively, we have 1 < g. Mathematically, we define the event space of a single-period model as

   $$\Omega \triangleq \{u, d\},$$

   and the price of the assets at *t*

   $$S_1(u) \triangleq gS_0$$
   $$S_1(d) \triangleq lS_0,$$

   where S_0 is the price of the assets at $t = 0$.

If we work this one model period further, we derive the second model.

The two-period market model

This model extends the one period mode that is concerned with one asset over two periods.

Starting with the first period, we have another two possible outcomes at the second period, i.e. a market upswing, denoted by *u*, and with the asset return denoted by *g*; or a market downswing , denoted by *d*, and with the asset return denoted by *l*.

In this situation, we have $2^2 = 4$ possible events, which is given as

$$\Omega \triangleq \{(u,\ u),\ (u,\ d),\ (d,\ u),\ (d,\ d)\}.$$

As usual, the price of the assets at $t = 1$ and $t = 2$ is given by

$$\begin{aligned} S_1(u,\ u) &= S_1(u,\ d) \triangleq gS_0 \\ S_1(d,\ u) &= S_1(d,\ d) \triangleq lS_0 \\ S_2(u,\ u) &\triangleq g^2S_0 \\ S_2(u,\ d) &= S_2(d,\ u) \triangleq glS_0 \\ S_2(d,\ d) &\triangleq l^2S_0, \end{aligned}$$

where S_0 is the price of the assets $t = 0$.

Having discussed the single-period and the two-period models, we can now use the same approach to generalize a standard form of the equation. We define

$$\Omega = \{\omega_1,\ \omega_2,\ \cdots,\ \omega_N\}. \tag{2.3}$$

For each ω_i Î Ω, we have

$$Pr = (\{\omega_i\}) = p_i = the\ probability\ that\ \omega_i\ is\ the\ outcome \tag{2.4}$$

and

$$\overset{N}{\underset{i\ =\ 1}{\text{S}}} \Pr(\{\omega_i\}) = 1. \tag{2.5}$$

The following example demonstrates how the probabilities are included in the two-period binomial market model. The set of market histories is given as

$$\Omega = \{(u,u)\,,\ (u,d)\,,\ (d,u)\,,\ (d,d)\}\,.$$

The following observation is made:

$$\Pr((u, u)) = \frac{1}{2}$$

$$\Pr((u, d)) = \frac{1}{4}$$

$$\Pr((d, u)) = \frac{1}{8}$$

$$\Pr((d, d)) = \frac{1}{8}.$$

Let *A* be the event of downswing in the second period. Find the probability that the event of downswing occurs in the second period.

Solution

The event of downswing in the second period is A = {(u, d),(d, d)}; hence:

$$\Pr(A) = \Pr((u, d)) + \Pr((d, d)) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}.$$

1.4.1 Self-test 2

An example of a bull market

Assume the probability of an upswing in each period is

$$\frac{3}{4}$$

and that market movements in different periods are independent. Then, the following observation is made over the two-period model:

$$\Pr((u, u)) = \left(\frac{3}{4}\right)^2$$

$$\Pr((u, d)) = \frac{3}{4} \times \frac{1}{4}$$

$$\Pr((d, u)) = \frac{1}{4} \times \frac{3}{4}$$

$$\Pr((d, d)) = \left(\frac{1}{4}\right)^2.$$

Find the probability that there is at least one upswing.

Solution

The event that contains at least one upswing is *A* = {(*u*, *d*), (*d*, *u*), (*u*, *u*)}. The only event that has all downswings is {(*d*, *d*)}; hence:

$$\begin{aligned} Pr(at\ least\ one\ upswing) &= 1 - Pr(all\ downswing) \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16} \end{aligned}$$

This example illustrates the situation of a bull market, i.e. the probability of an upswing in each period is greater than $\frac{1}{2}$

Now assume that the probability of an upswing in each period is $\frac{1}{4}$, and that market movements in different periods are independent. This is known as a bear market.

Find the probability that there is at least one downswing over periods of time.

1.4.1.1 Feedback

The following observation is made over the two periods:

$\Pr((u,\ u)) = \left(\frac{1}{4}\right)^2,$

$\Pr((u,\ d)) = \frac{1}{4} \times \frac{3}{4},$

$\Pr((d,\ u)) = \frac{3}{4} \times \frac{1}{4}$ and

$\Pr((d,\ d)) = \left(\frac{3}{4}\right)^2.$

Note that

$$A = \{(u, d),\ (d, u),\ (u, u)\}$$

and hence

$$\begin{aligned} Pr(at\ least\ one\ downswing) &= 1 - Pr(all\ upswing) \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16} \end{aligned}$$

1.5 Conclusion

This module introduced the basic terminologies used in the field of financial mathematics, as well as the concepts used in the financial industries; we have explored various financial instruments, such as options, and their associated risk.

If you would like to learn more on this subject, you are welcome to enrol in *MATH S390 Quantitative Models for Financial Risk (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcGenericPage2010&c=C_ETPU&cid=191154069600&lang=eng)* offered by the School of Science & Technology (http://www.ouhk.edu.hk/wcsprd/Satellite?pagename=OUHK/tcSubWeb&l=C_ST&lid=191133000200&lang=eng) of the OUHK.